Pion polarizability from four-point functions in lattice QCD

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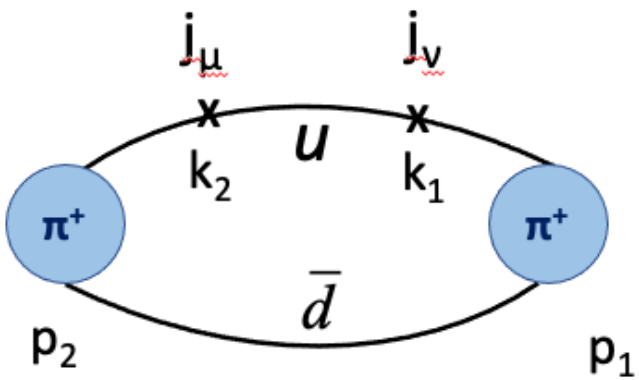

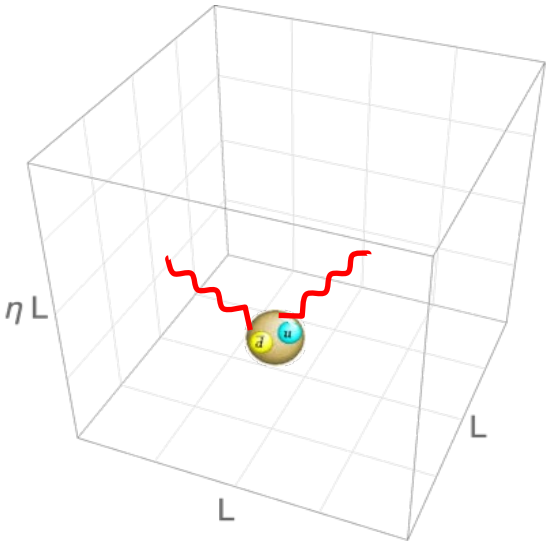

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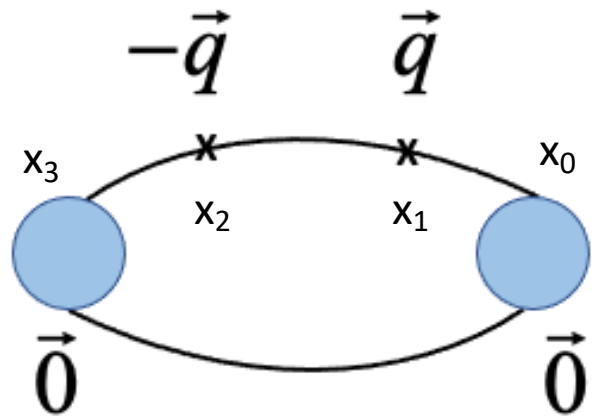
Four-point function

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$

$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

Euclidean spacetime

Kinematics
(zero-momentum Breit frame)



Zero-momentum pions are approximated by wall sources without gauge-fixing.

At large time separations,

$$Q_{\mu\mu}(\vec{q}, t) = N_s^2 \sum_n |\langle\pi(\vec{0})|j^L_\mu(0)|n(\vec{q})\rangle|^2 e^{-a(E_n - m_\pi)t} - N_s^2 \sum_n |\langle 0|j^L_\mu(0)|n(\vec{q})\rangle|^2 e^{-aE_n t}.$$

Elastic contribution is from n=1 term in the first sum.

Charged pion polarizabilities

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_E^\pi = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

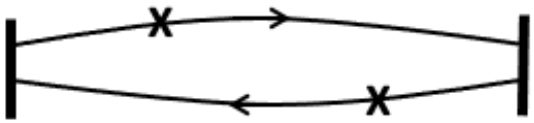
PRD104, 034506 (2021), Wilcox, Lee

Charge radius can be extracted from elastic part of the same Q_{44},

$$Q_{44}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg 1]{} \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\boldsymbol{q}^2)\, e^{-a(E_\pi(\boldsymbol{q})-m_\pi)t}$$

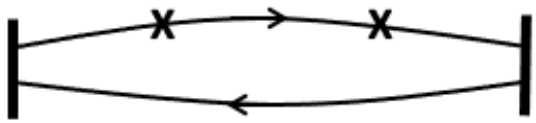
Wick contractions

(a)



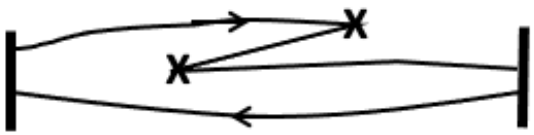
connected insertion: different flavor

(b)



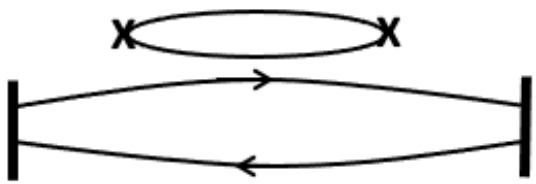
connected insertion: same flavor

(c)



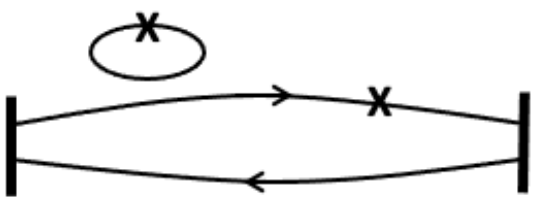
connected insertion: same flavor Z-graph

(d)



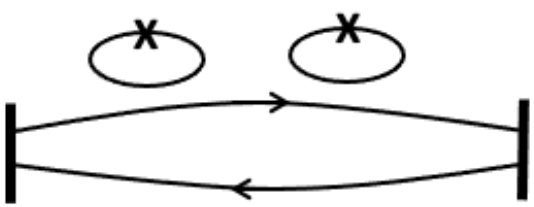
disconnected insertion: single loop, double current

(e)



disconnected insertion: single loop

(f)



disconnected insertion: double loop

Local vs. Conserved current

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j_\mu^L(x_2) j_\nu^L(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$

$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

Pion operator $\psi_{\pi^+}(x) = \bar{d}(x)\gamma_5 u(x)$

Local current $j_\mu^{(PC)} = Z_V \left(q_u \bar{u}\gamma_\mu u + q_d \bar{d}\gamma_\mu d\right)$

Requires renormalization, but simpler correlation functions:
20 diagrams if u and d distinct, 12 diagrams if isospin symmetry (m_u=m_d)

Conserved-current (point-split) for Wilson-like fermions,

$$\boxed{\begin{aligned} j_\mu^{(PS)}(x) = {} & q_u\kappa[-\bar{u}(x)(1-\gamma_\mu)U_\mu(x)u(x+a\hat{\mu}) + \bar{u}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)u(x)] \\ & + q_d\kappa[-\bar{d}(x)(1-\gamma_\mu)U_\mu(x)d(x+a\hat{\mu}) + \bar{d}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)d(x)] \end{aligned}}$$

No renormalization (Z_V=1) but more complicated functions:
80 diagrams if u and d distinct, 48 diagrams if isospin symmetry.
Involves gauge links explicitly.
Q_{44}(**q**=0,t1,t2) = charge factor (independent of t1 and t2)

Lattice setup

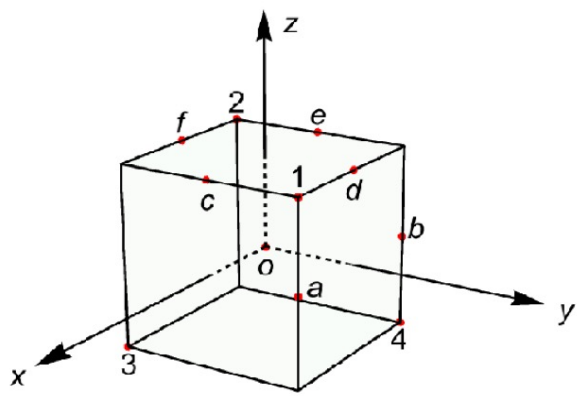
Proof-of-concept:

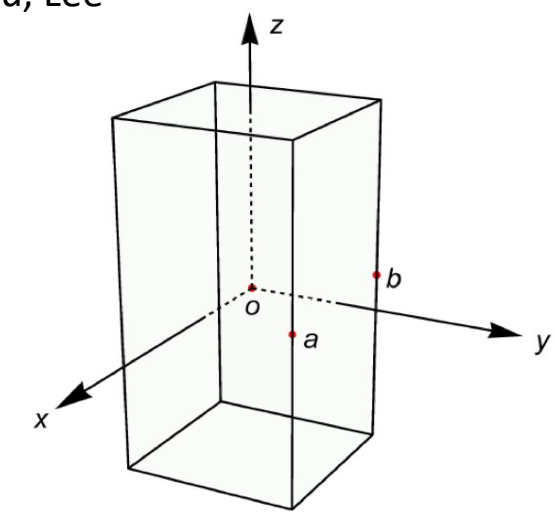
Quenched Wilson action at β=6.0 and κ=0.152 and 0.1555, 24^4 lattice, Dirichlet bc, 100 configs.

Production runs:

Two-flavor nHYP clover fermion + Lüscher-Weisz gauge action.

Scale for this action set in PRD102, 094506 (2020), Niyazi, Alexandru, Lee

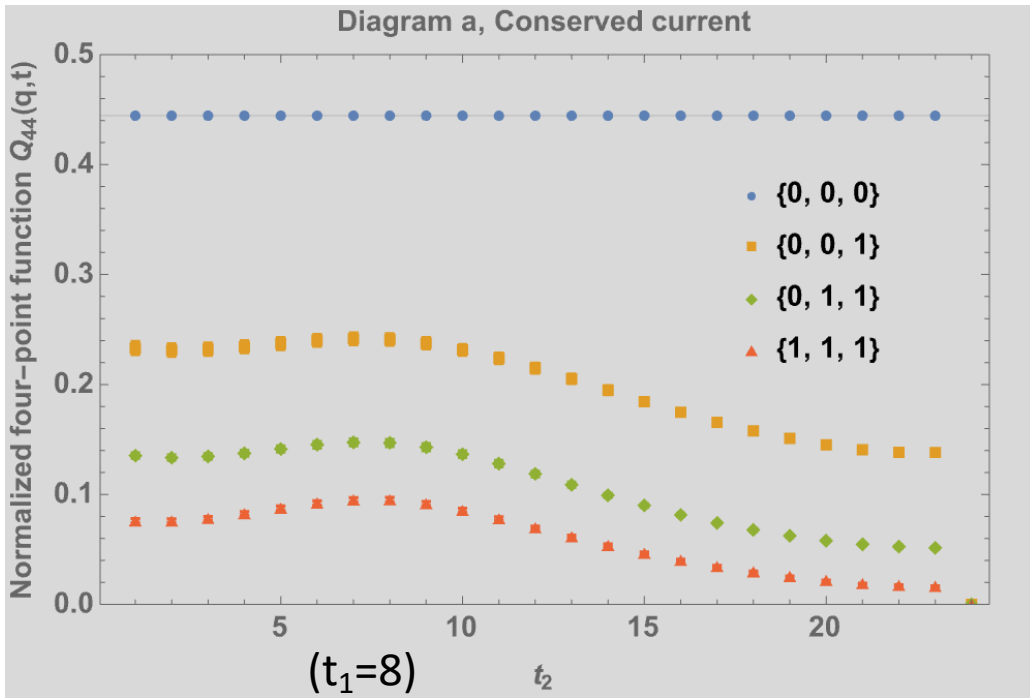




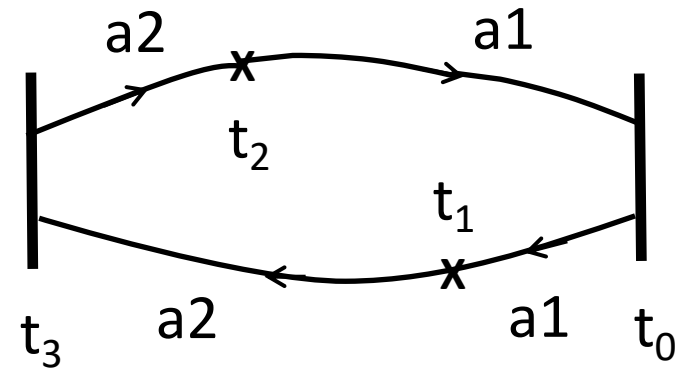
	ensemble	$N_t \times N_{x,y}^2 \times N_z$	η	a[fm]	$N_{\rm cfg}$	aM_π	am_N	$am_{u/d}^{pcac}$	af_π
m_π=315 MeV	$\mathcal{E}_1$	$48 \times 24^2 \times 24$	1.0	0.1210(2)(24)	300	0.1934(5)	0.644(6)	0.01237(9)	0.0648(8)
	$\mathcal{E}_2$	$48 \times 24^2 \times 30$	1.25	–	–	–	–	–	–
	$\mathcal{E}_3$	$48 \times 24^2 \times 48$	2.0	–	–	–	–	–	–
m_π=227 MeV	$\mathcal{E}_4$	$64 \times 24^2 \times 24$	1.0	0.1215(3)(24)	400	0.1390(5)	0.62(1)	0.00617(9)	0.060(1)
	$\mathcal{E}_5$	$64 \times 24^2 \times 28$	1.17	–	–	–	–	–	–
	$\mathcal{E}_6$	$64 \times 24^2 \times 32$	1.33	–	–	–	–	–	–

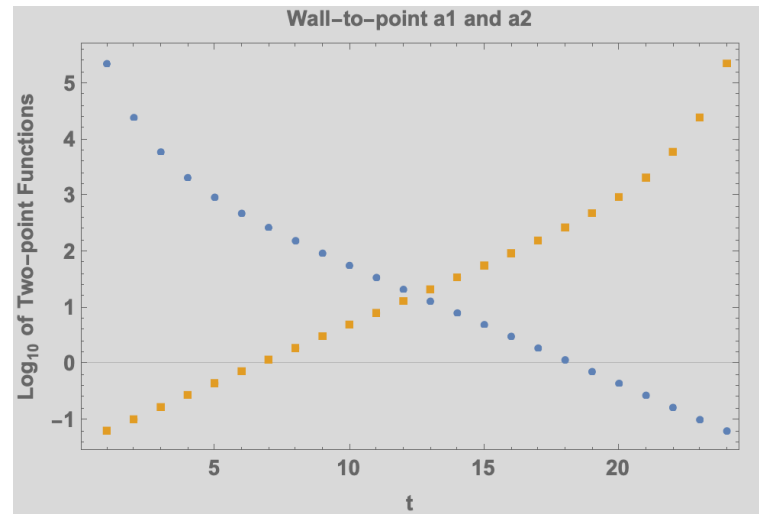
Elongated lattices to study volume dependence

Diagram (a)



(t_1=8)





$$\tilde{Q}_{44}^{(a,PS)}(\boldsymbol{q},t_1,t_2) = \frac{4}{9}\kappa^2 \operatorname*{Tr}_{s,c}\Bigg[$$

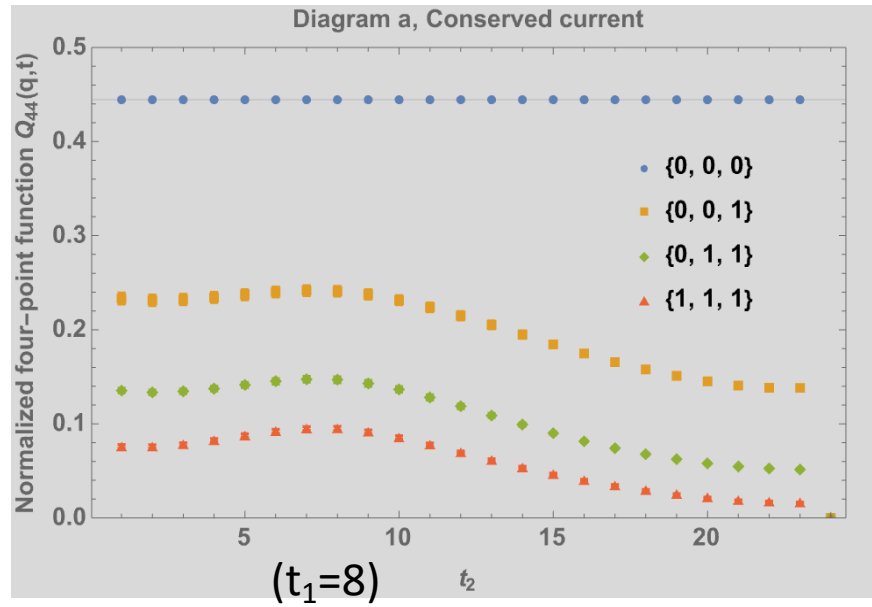
$$\boldsymbol{q} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right), \quad n_x, n_y, n_z = 0, \pm 1, \pm 2, \cdots,$$

$$\Big([P(t_2)V_{a2}]^\dagger \gamma_5(1-\gamma_4)e^{i\boldsymbol{q}}U_4(t_2,t_2+1)P(t_2+1)V_{a1} - [P(t_2+1)V_{a2}]^\dagger \gamma_5(1+\gamma_4)U_4^\dagger(t_2+1,t_2)e^{i\boldsymbol{q}}P(t_2)V_{a1}\Big)^\dagger$$

$$\Big([P(t_1)V_{a2}]^\dagger \gamma_5(1-\gamma_4)e^{i\boldsymbol{q}}U_4(t_1,t_1+1)P(t_1+1)V_{a1} - [P(t_1+1)V_{a2}]^\dagger \gamma_5(1+\gamma_4)U_4^\dagger(t_1+1,t_1)e^{i\boldsymbol{q}}P(t_1)V_{a1}\Big)\Bigg]$$

Renormalization constant from Diagram (a)

$$\frac{Q_{44}^{(\mathrm{Conserved})}(\mathbf{q}=0)}{Q_{44}^{(\mathrm{Local})}(\mathbf{q}=0)} = Z_V^2$$



(t_1=8)

$Z_V \approx 1.50$

Wilson action at β=6.0 and κ=0.152

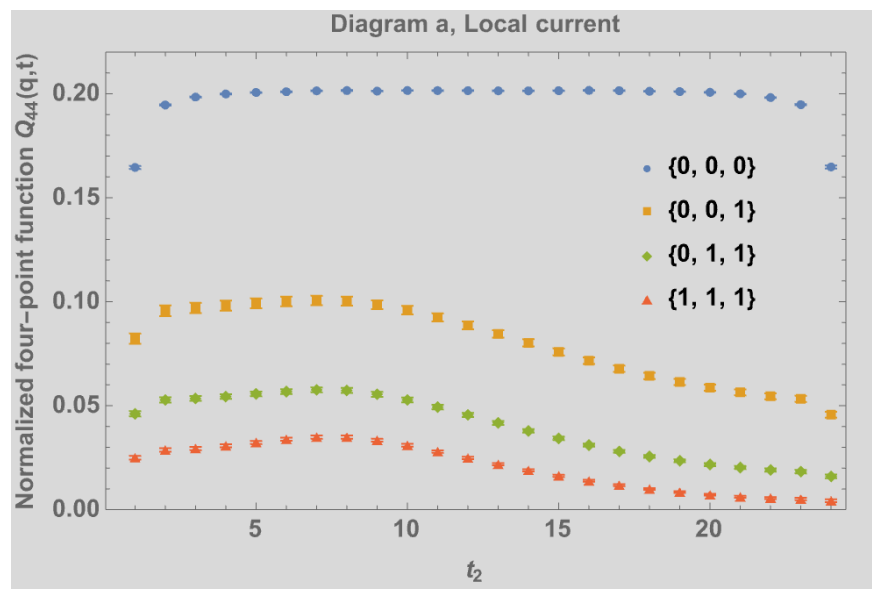
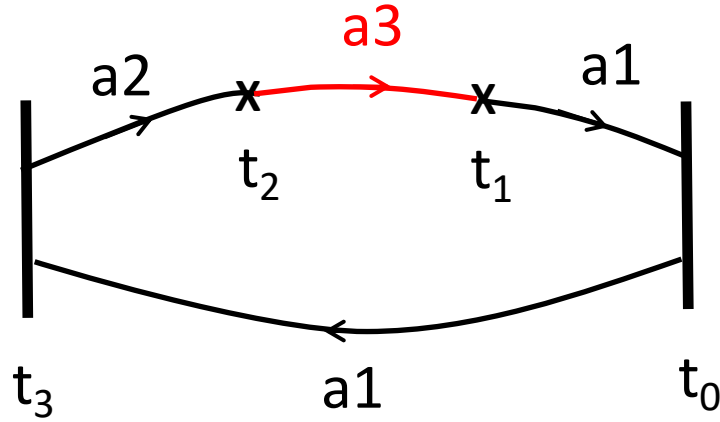
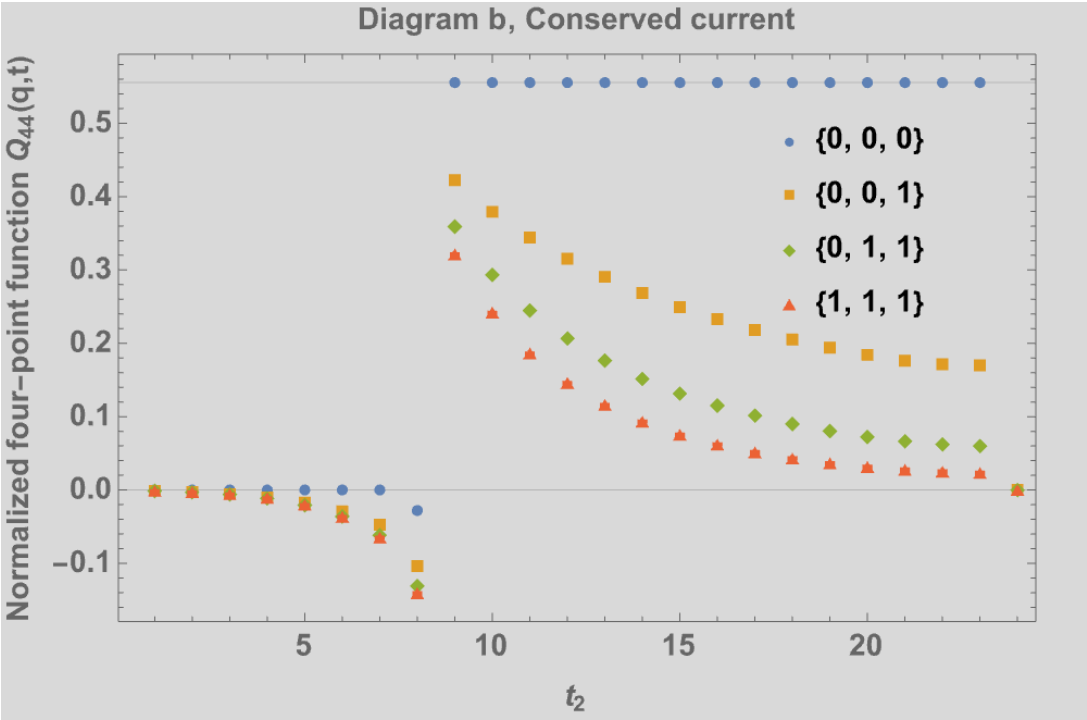


Diagram (b)





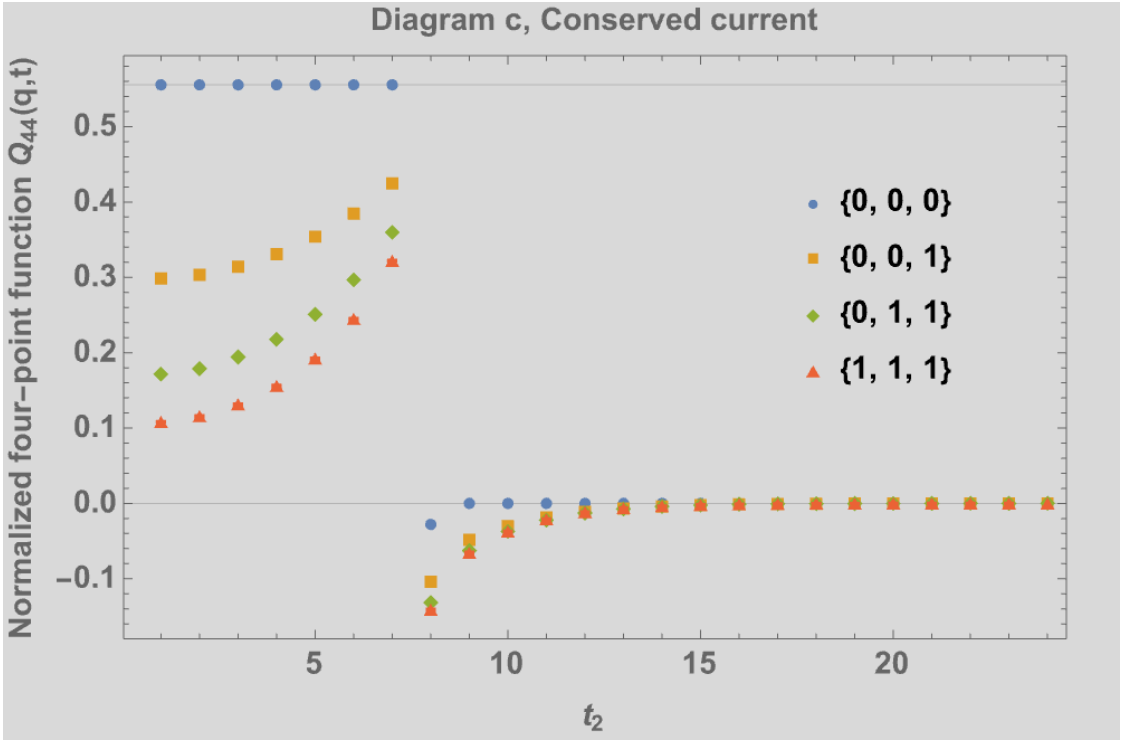
$$\tilde{Q}_{44}^{(b,PS)}(\boldsymbol{q},t_2) = -\frac{5}{9}\kappa^2 \operatorname*{Tr}_{s,c}\Big[[P(t_2)\gamma_5 V_{a3}^{(4,PS)}(\mathbf{q})]^\dagger(1-\gamma_4)e^{-i\boldsymbol{q}}U_4(t_2,t_2+1)P(t_2+1)V_{a2}\mathcal{W}^T P(t_3)\gamma_5 V_{a1}$$

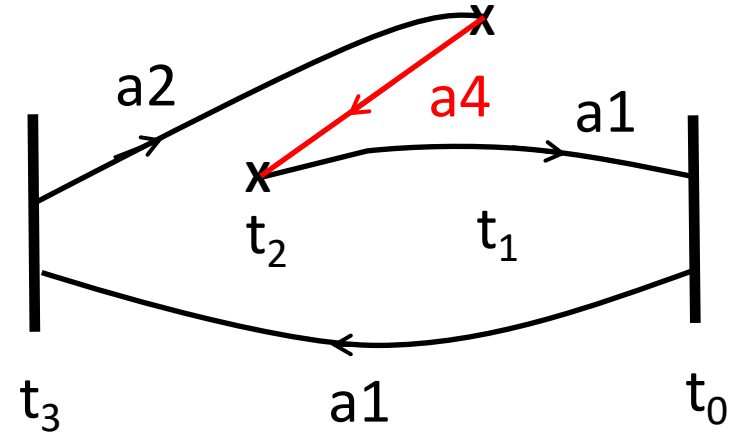
$$-[P(t_2+1)\gamma_5 V_{a3}^{(4,PS)}(\mathbf{q})]^\dagger(1+\gamma_4)U_4^\dagger(t_2+1,t_2)e^{-i\boldsymbol{q}}P(t_2)V_{a2}\mathcal{W}^T P(t_3)\gamma_5 V_{a1}\Big]$$

SST propagator

$$V_{a3}^{(4,PS)}(\boldsymbol{q}) \equiv M_q^{-1}\Big[P^T(t_1)(1-\gamma_4)e^{-i\boldsymbol{q}}U_4(t_1,t_1+1)P(t_1+1)V_{a1} - P^T(t_1+1)(1+\gamma_4)U_4^\dagger(t_1+1,t_1)e^{-i\boldsymbol{q}}P(t_1)V_{a1}\Big]$$

Diagram (c)





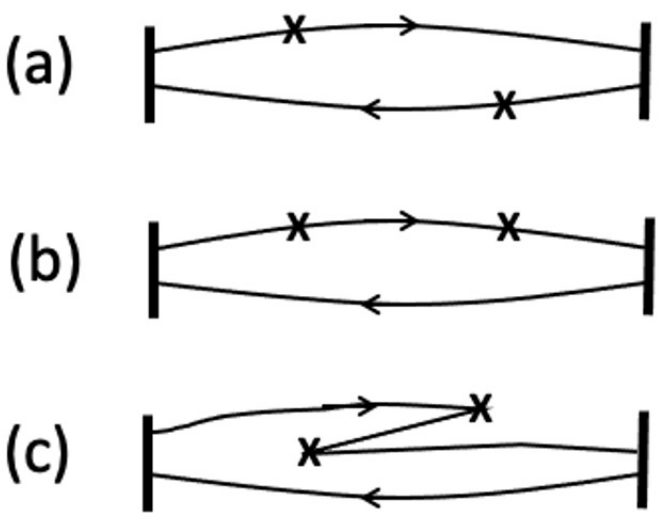
$$\tilde{Q}_{44}^{(c,PS)}(\boldsymbol{q},t_2) = \frac{5}{9}\kappa^2 \operatorname*{Tr}_{s,c}\Big[[P(t_2)\gamma_5 V_{a1}]^\dagger(1-\gamma_4)e^{-i\mathbf{q}}U_4(t_2,t_2+1)P(t_2+1)V_{a4}^{(4,PS)}(\mathbf{q})\mathcal{W}^T P(t_3)\gamma_5 V_{a1}$$

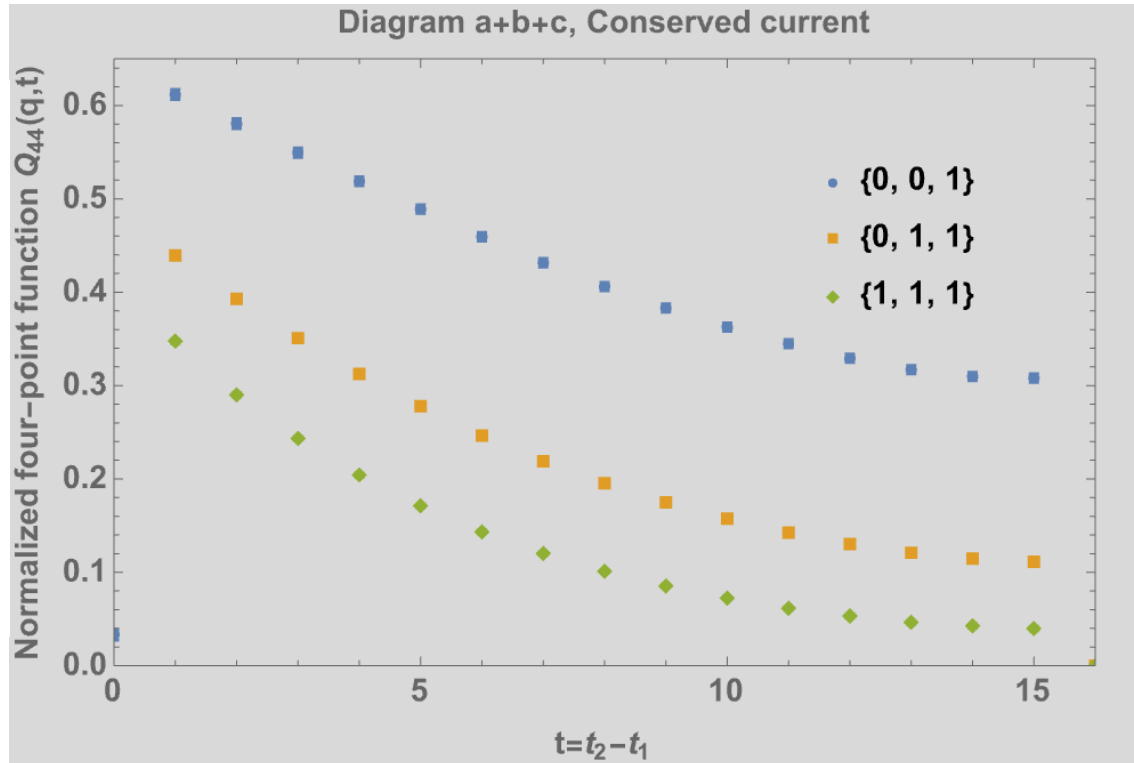
$$-[P(t_2+1)\gamma_5 V_{a1}]^\dagger(1+\gamma_4)U_4^\dagger(t_2+1,t_2)e^{-i\mathbf{q}}P(t_2)V_{a4}^{(4,PS)}(\mathbf{q})\mathcal{W}^T P(t_3)\gamma_5 V_{a1}\Big]$$

SST propagator

$$V_{a4}^{(4,PS)}(\boldsymbol{q}) \equiv M_q^{-1}\Big[P^T(t_1)(1-\gamma_4)e^{i\boldsymbol{q}}U_4(t_1,t_1+1)P(t_1+1)V_{a2} - P(t_1+1)^T(1+\gamma_4)U_4^\dagger(t_1+1,t_1)e^{i\boldsymbol{q}}P(t_1)V_{a2}\Big]$$

Diagrams (a+b+c)



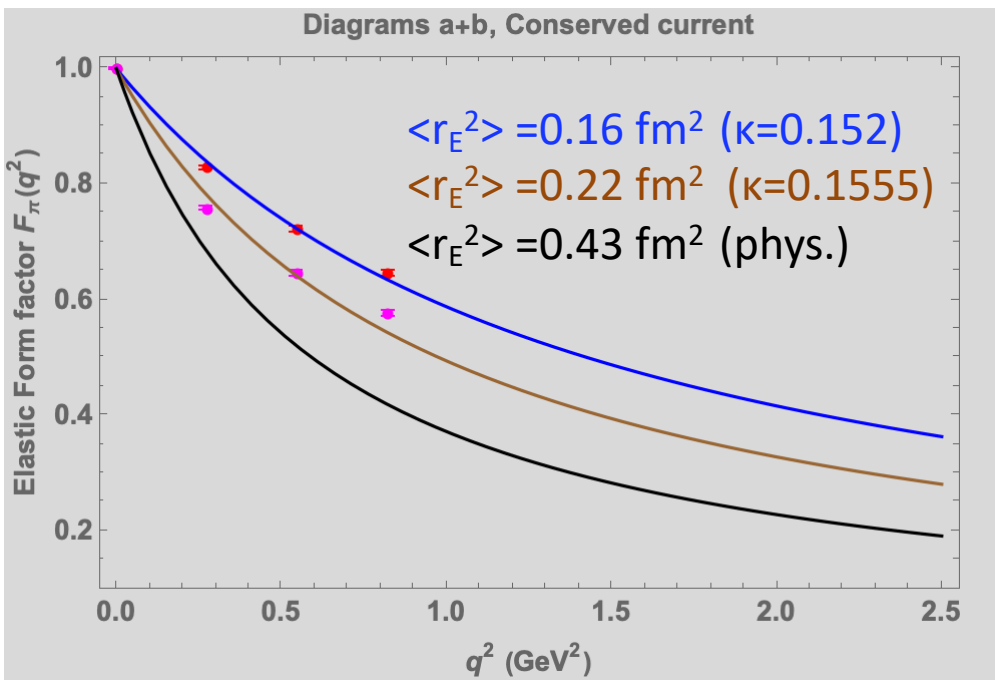


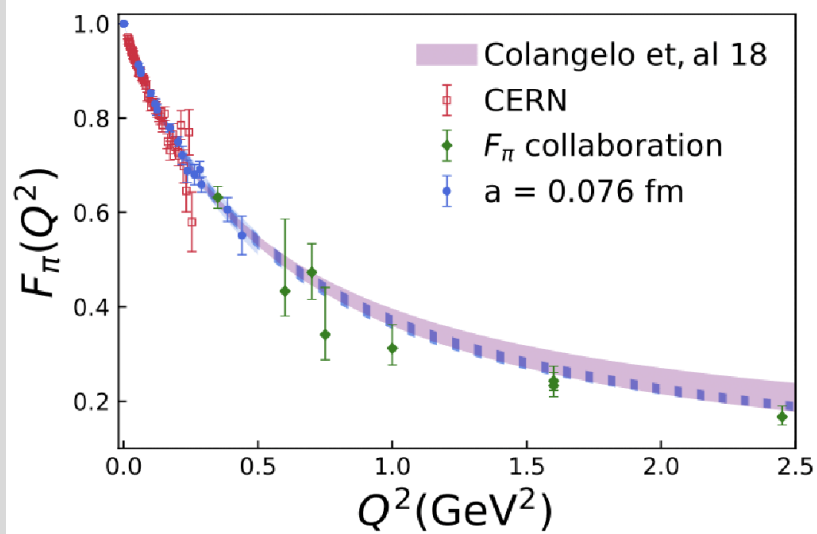
$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

Pion form factor

$$Q_{44}^{elas}(\mathbf{q},t) = \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\mathbf{q}^2)\, e^{-a(E_\pi - m_\pi)t}$$

$$F_\pi(\mathbf{q}^2) = 1/(1+\frac{\mathbf{q}^2}{m^2}) \qquad \langle r_E^2 \rangle = -6\frac{dF_\pi(\mathbf{q}^2)}{d\mathbf{q}^2} \qquad E_\pi = \sqrt{\mathbf{q}^2 + m_\pi^2}$$

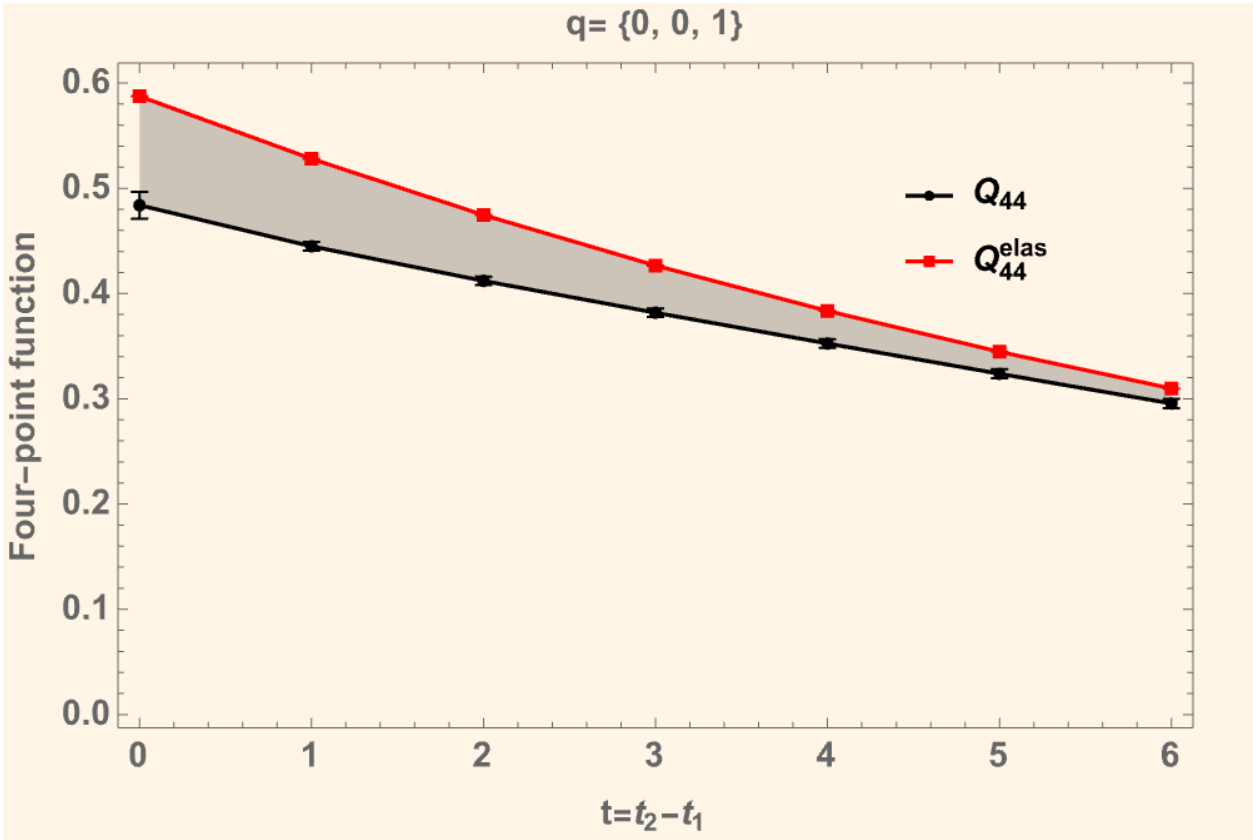




(arXiv:2102.06047, Gao et al)

Q integral

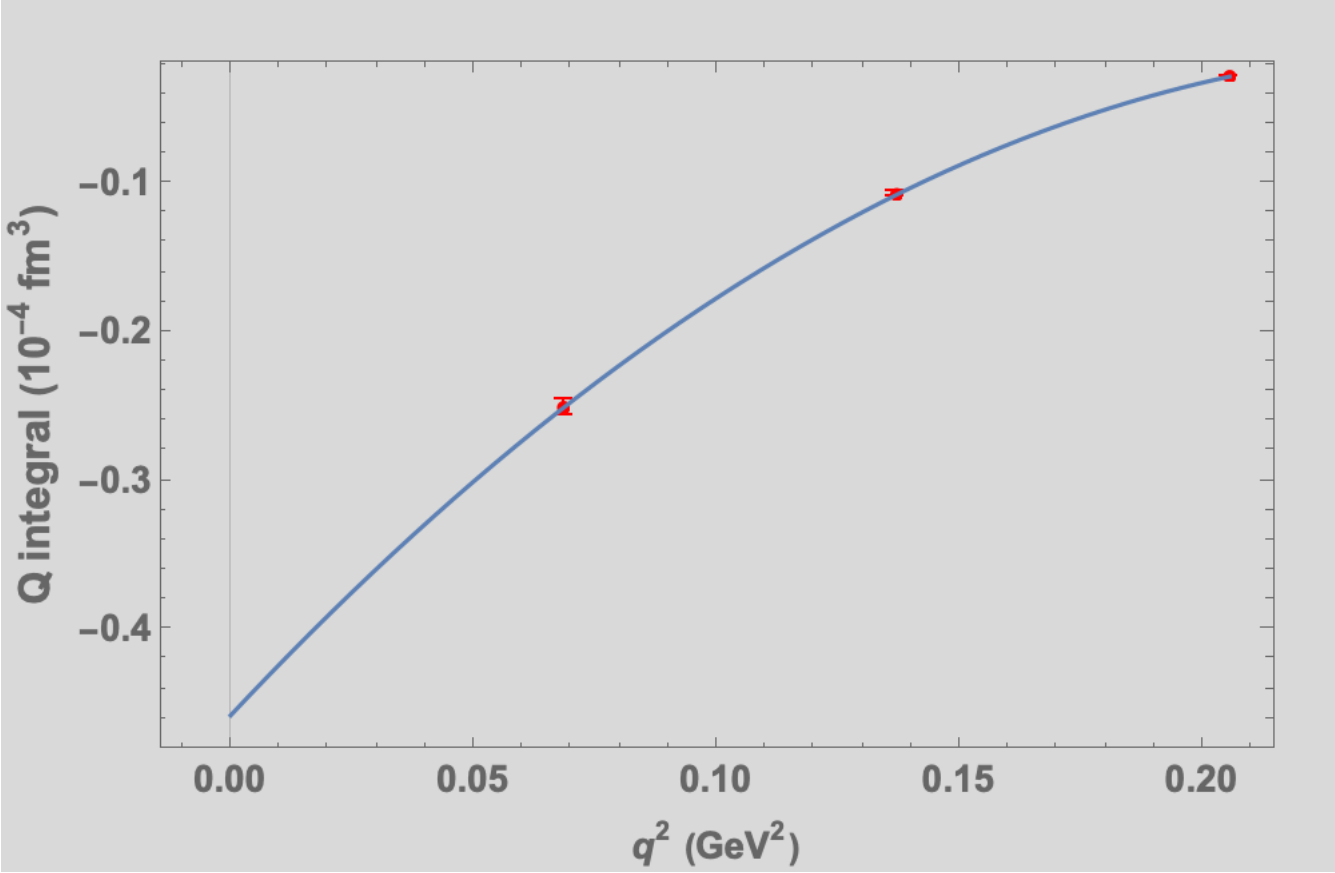
$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$



Negative of the shaded area

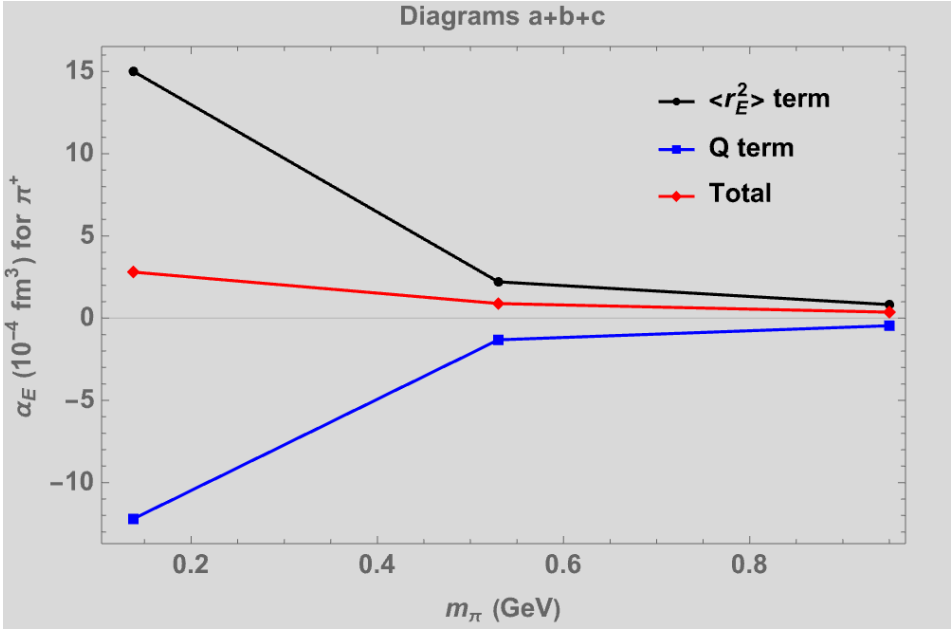
Extrapolation to $q^2=0$

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$



Pion mass dependence

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$



α_E (10^{-4} fm^3)	m_π=960 MeV	m_π=530 MeV	m_π=138 MeV
$\langle r_E^2\rangle$ term	0.82	2.21	15.0
Q term	-0.46	-1.32	-12.2
Total	0.58	1.71	2.8

Strong m_π dependence of individual terms but opposite sign. Gentle dependence of the total.

Conclusion

- Four-point functions offer good physics payout
  - form factors (charge radius)
  - polarizabilities
  - renormalization constant Z_V (local vs. conserved current)
- Alternative to background field method for charged hadrons that avoids the issues of
  - acceleration in electric field
  - Landau levels in magnetic field
- Proof-of-concept simulations for π^+ with quenched Wilson fermions on 24^4 lattices so far
  - validate the connected diagrams for conserved currents
  - show promise for extraction of $\langle r_E^2 \rangle$ and α_E
  - Issues to study: t=0 and q^2=0 extrapolations, Dirichlet bc
- Outlook
  - two-flavor nhyp fermions (smaller m_π, infinite-volume extrapolation)
  - magnetic polarizability (β_M) from Q_{11} components
  - disconnected insertions
  - proton ($\langle r_E^2 \rangle$, α_E, μ_p, β_M)

“Pion electric polarizabilities from lattice QCD”

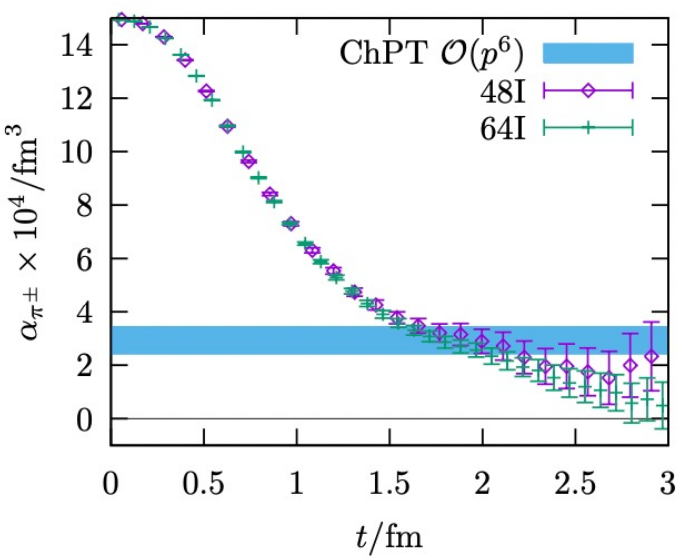
X. Feng, T. Izubuchi, L. Jin, M. Golterman

arXiv:2201.01396 (Lattice 2021)

Domain-wall ensembles
at physical pion mass

	Volume	a^{-1} (GeV)	L (fm)	M_π (MeV)	$t_{\rm sep}$ (a)
48I	$48^3 \times 96$	1.730(4)	5.5	135	12
64I	$64^3 \times 128$	2.359(7)	5.4	135	18
24D	$24^3 \times 64$	1.0158(40)	4.7	142	8
32D	$32^3 \times 64$	1.0158(40)	6.2	142	8

$$\alpha_\pi(t) = -\int_{-t<t_x<t} \int_{\vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_\pi} \langle \pi | T\vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi \rangle - \alpha_\pi^{\rm Born}$$



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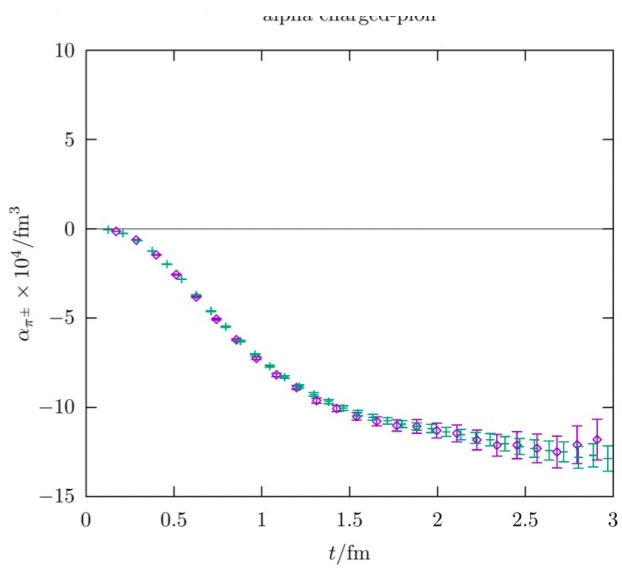

$$+ \frac{\alpha \langle r_E^2 \rangle}{3m_\pi}$$