

Isovector Axial Form Factor of the Nucleon from Lattice QCD

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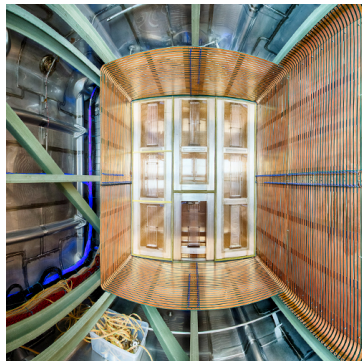
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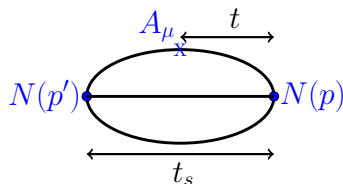
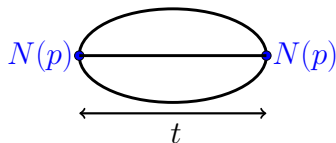
Motivation

- The axial form factor of the nucleon G_A plays a central role in understanding the quasi-elastic part of GeV-scale neutrino-nucleus cross sections.
- Particularly for the long-baseline neutrino oscillation experiment DUNE, these cross-sections must be known with few-percent uncertainties to enable a reliable reconstruction of the incident neutrino energy.
- In the absence of modern, high-quality experimental measurements of $G_A(Q^2)$, calculations for the axial form factor from lattice QCD are of crucial importance in order to maximize the scientific output of neutrino-oscillation experiments.



This talk is based on [arXiv:2207.03440](https://arxiv.org/abs/2207.03440), the Mainz group's recently published calculation of $G_A(Q^2)$.

- The matrix elements of the local iso-vector axial current $A_\mu^a(x)$ are parameterized by $G_A(Q^2)$ and $G_P(Q^2)$.
- Choose the current component transverse to the momentum transfer to project out the axial form factor.
- Calculate two- and three-point correlation functions $C_2(\vec{p}, t)$ and $C_3(\vec{q}, t, t_s)$ (here $\vec{q} = \vec{p}' - \vec{p}$ and $Q^2 = \vec{q}^2 - (E_{\vec{p}'} - E_{\vec{p}})^2$).



- We use the ratio $R(\vec{q}, t, t_s) \equiv \frac{C_3(\vec{q}, t, t_s)}{C_2(0, t_s)} \sqrt{\frac{C_2(\vec{q}, t_s - t) C_2(\vec{0}, t) C_2(\vec{0}, t_s)}{C_2(\vec{0}, t_s - t) C_2(\vec{q}, t) C_2(\vec{q}, t_s)}}$ to build the summed insertion

$$S(\vec{q}, t_s) \equiv a \sqrt{\frac{2E_{\vec{q}}}{m + E_{\vec{q}}}} \sum_{t=a}^{t_s-a} R(\vec{q}, t, t_s) \stackrel{t_s \rightarrow \infty}{\Rightarrow} b_0(\vec{q}) + t_s G_A(Q^2) + O(t_s e^{-\Delta t_s}).$$

- Parameterize the FF from the outset via the z -expansion,

$$G_A(Q^2) = \sum_{n=0}^{n_{\max}} a_n z^n(Q^2), \quad z(Q^2) = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}}}}, \quad t_{\text{cut}} = 9(M_{\pi}^{\text{phys}})^2.$$

- The sums $S(\vec{q}, t_s)$ are fitted simultaneously for different \vec{q} and t_s .
- Free fit parameters: coefficients a_n plus the offsets $b_0(\vec{q})$.
- Summation method suppresses excited state contributions.
- Get the coefficients a_n directly, and extrapolate them to the continuum, infinite volume, and physical mass limit.
- Note that a_0 is the axial charge g_A .

Choice of source-sink separation

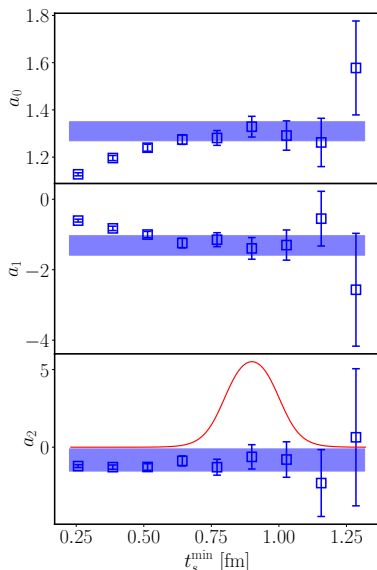
Choice of source-sink separations t_s in the summation method is tricky:

- at small values of t_s contributions from excited state are significant, and
- at large t_s the signal-to-noise ratio becomes poor.

Rather than choosing a single t_s^{\min} , we average the fit results using a weight

$$\frac{1}{\mathcal{N}_w} \left[\tanh\left(\frac{t_s^{\min} - t_w^l}{dt_w}\right) - \tanh\left(\frac{t_s^{\min} - t_w^u}{dt_w}\right) \right]$$

with $t_w^l = 0.8$ fm, $t_w^u = 1.0$ fm and $dt_w = 0.08$ fm. The weights are normalised by \mathcal{N}_w so that the sum is 1.



CLS $N_f = 2 + 1$ ensembles

- non-perturbatively $O(a)$ -improved Wilson fermions
- tree-level improved Lüscher-Weisz gauge action
- four lattice spacings: $a \sim 0.086, 0.076, 0.064$, and 0.050 fm
- multiple pion masses, one slightly below the physical value
- large volumes: $M_\pi L \geq 4$

ID	β	T/a	L/a	M_π/MeV	$M_\pi L$	M_N/GeV	N_{conf}	N_{meas}	t_s/fm	N_{t_s}
H102	3.40	96	32	354	4.96	1.103	2005	32080	0.35..1.47	14
H105	3.40	96	32	280	3.93	1.045	1027	49296	0.35..1.47	14
C101	3.40	96	48	225	4.73	0.980	2000	64000	0.35..1.47	14
N101	3.40	128	48	281	5.91	1.030	1596	51072	0.35..1.47	14
S400	3.46	128	32	350	4.33	1.130	2873	45968	0.31..1.53	9
N451	3.46	128	48	286	5.31	1.045	1011	129408	0.31..1.53	9
D450	3.46	128	64	216	5.35	0.978	500	64000	0.31..1.53	17
N203	3.55	128	48	346	5.41	1.112	1543	24688	0.26..1.41	10
N200	3.55	128	48	281	4.39	1.063	1712	20544	0.26..1.41	10
D200	3.55	128	64	203	4.22	0.966	2000	64000	0.26..1.41	10
E250	3.55	192	96	129	4.04	0.928	400	102400	0.26..1.41	10
N302	3.70	128	48	348	4.22	1.146	2201	35216	0.20..1.40	13
J303	3.70	192	64	260	4.19	1.048	1073	17168	0.20..1.40	13
E300	3.70	192	96	174	4.21	0.962	570	18240	0.20..1.40	13

Chiral extrapolation

Three ansätze for the chiral extrapolation:

- 1 Linear in M_π^2 for all coefficients a_i .
- 2 An ansatz linear in M_π^2 for a_1 and a_2 , and with a chiral log for a_0 :

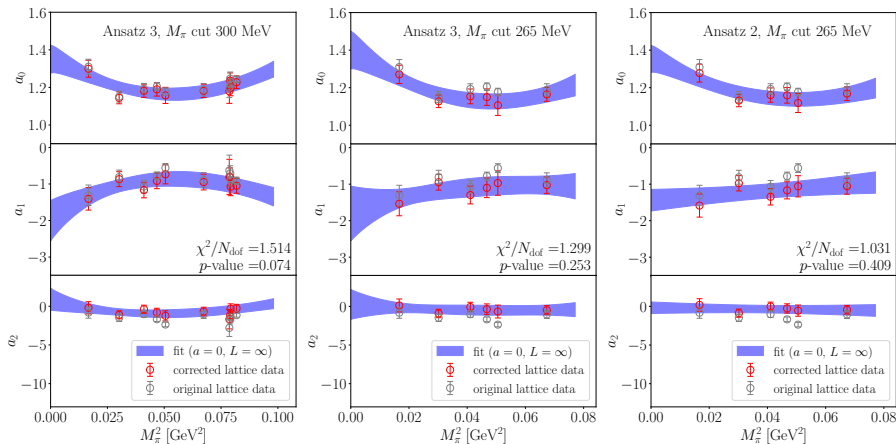
$$a_0 = g_a^{(0)} + g_a^{(1)} M_\pi^2 + g_a^{(3)} M_\pi^3 - g_a^{(2)} M_\pi^2 \ln \frac{M_\pi}{M_n},$$
$$g_a^{(1)} = 4d_{16} - \frac{(g_a^{(0)})^3}{16\pi^2 F_p^2}, \quad g_a^{(2)} = \frac{g_a^{(0)}}{8\pi^2 F_p^2} \left(1 + (2g_a^{(0)})^2\right),$$
$$g_a^{(3)} = \frac{g_a^{(0)}}{8\pi F_p^2 M_n} \left(1 + (g_a^{(0)})^2\right) - \frac{g_a^{(0)}}{6\pi F_p^2} \Delta_{c_3, c_4}.$$

$\Delta_{c_3, c_4} = c_3 - 2c_4$ is a combination of LECs c_3 and c_4 . The free fit parameters for a_0 chiral extrapolation are $g_a^{(0)}$, d_{16} and Δ_{c_3, c_4} .

- 3 Same as ansatz 2, but including M_π^3 terms for coefficients a_1 and a_2 (chiral log only for a_0).

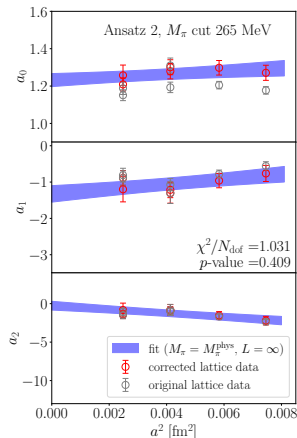
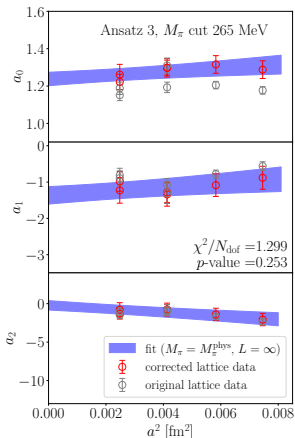
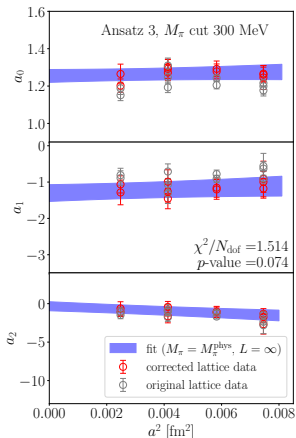
Global fit: chiral extrapolation

- The coefficients a_i do not have common fit parameters, but they are correlated within an ensemble. These correlations are included in fits.
- We use multiple pion mass cuts to estimate the systematic effect.



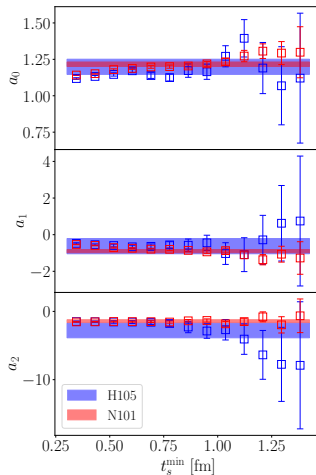
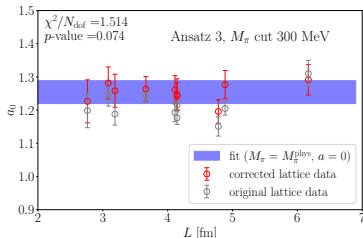
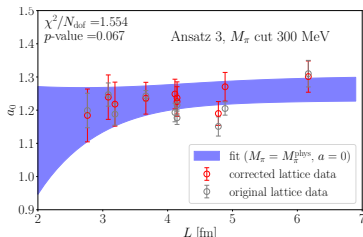
Global fit: continuum extrapolation

- The continuum extrapolation is linear in a^2 for all coefficients a_i .
- Include a cut in lattice spacing to estimate the systematic effect.



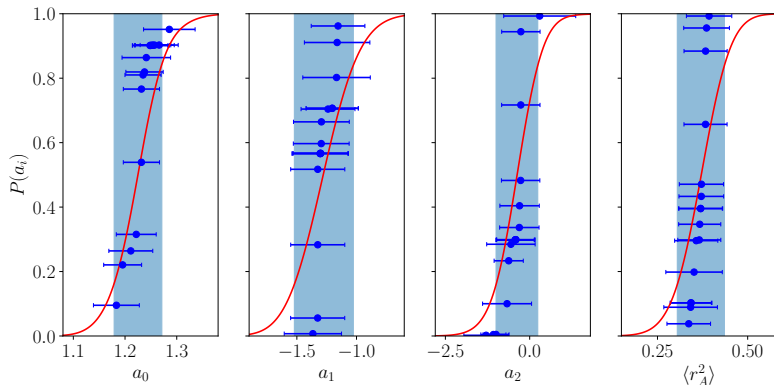
Finite size effects

Although we do not observe large finite volume effects, we include a term $\frac{M_\pi^2}{\sqrt{M_\pi L}} e^{-M_\pi L}$ in some of the global fits.

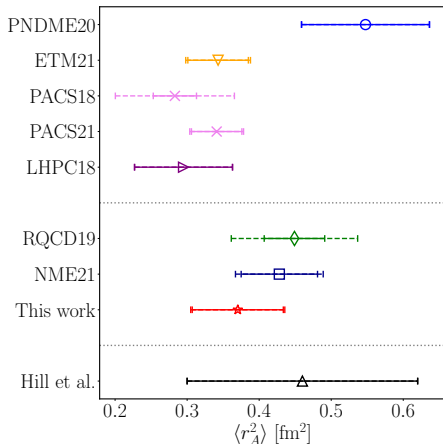


Model average (AIC)

- Different fit ansätze and cuts can be equally well motivated.
- Perform a weighted average using Akaike Information Criterion (AIC).
- Provides an estimate of systematic uncertainties.
- We choose $w_i^{\text{AIC}} = N e^{-\frac{1}{2}(\chi_i^2 + 2n_i^{\text{par}} - n_i^{\text{data}})}$, where χ_i^2 , n_i^{par} and n_i^{data} characterize the i th fit. N normalises the weights so that $\sum_i w_i^{\text{AIC}} = 1$.



Axial square radius $\langle r_A^2 \rangle$

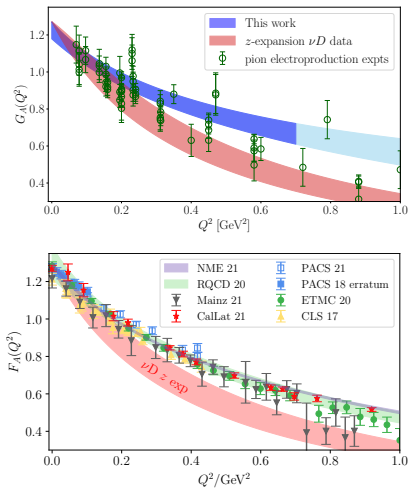


- Three lattice QCD results with full error budget (chiral and continuum extrapolation), others at physical pion mass on a single ensemble
- Hill et al. is an average of the values obtained from z -expansion fits to neutrino scattering and muon capture

Our result: $\langle r_A^2 \rangle = 0.370 \pm 0.063$ (stat) ± 0.016 (syst) fm².

PNDME20 [arXiv:1905.06470], ETM21 [arXiv:2011.13342], PACS18 [arXiv:1811.07292], PACS21 [arXiv:2107.07085], LHPC18 [arXiv:1711.11385], RQCD19 [arXiv:1911.13150], NME21 [arXiv:2103.05599], Hill et al. [arXiv:1708.08462]

Shape of the form factor $G_A(Q^2)$



- Our result for the axial form factor agrees well with other lattice QCD calculations.
- Comparison to data from pion electroproduction expts. and to a z -expansion fit to νD data shows a clear tension at large Q^2 .
- Our result for the axial charge $g_A = 1.225 \pm 0.039_{\text{(stat)}} \pm 0.025_{\text{(syst)}}$ agrees with the PDG value at $\sim 1\sigma$.
- The Mainz group has a dedicated project for precise determination of the charges, including g_A .

See Konstantin Ottnad's talk on Tuesday at 16:30!

Result: $G_A(Q^2)$ in the range $0 \leq Q^2 \leq 0.7 \text{ GeV}^2$

Our results for the coefficients of the z -expansion of the nucleon axial form factor in the continuum and at the physical pion mass are

$$a_0 = 1.225 \pm 0.039 \text{ (stat)} \pm 0.025 \text{ (syst)},$$

$$a_1 = -1.274 \pm 0.237 \text{ (stat)} \pm 0.070 \text{ (syst)},$$

$$a_2 = -0.379 \pm 0.592 \text{ (stat)} \pm 0.179 \text{ (syst)},$$

with a correlation matrix

$$M_{\text{corr}} = \begin{pmatrix} 1.00000 & -0.67758 & 0.61681 \\ -0.67758 & 1.00000 & -0.91219 \\ 0.61681 & -0.91219 & 1.00000 \end{pmatrix}.$$

Let us recall the z -expansion formula

$$G_A(Q^2) = \sum_{n=0}^{n_{\text{max}}} a_n z^n(Q^2), \quad z(Q^2) = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}}}}, \quad t_{\text{cut}} = (3M_{\pi^0})^2.$$

- A new method to extract the axial form factor of the nucleon, combining two well-known methods into one analysis step:
 - the summation method ensures that excited-state effects are sufficiently suppressed,
 - and the z -expansion readily provides the parameterization of the Q^2 dependence of the form factor.
- We observe good agreement with other lattice QCD determinations of the axial FF, which strengthens the tension with the shape of the FF extracted from νD data.
- Previously good agreement was found for the isovector vector form factors between lattice QCD and phenomenological determinations, which are far more precise than in the axial-vector case. This supports the finding that the axial FF falls off less steeply than thought.
- Our main results are the coefficients of the z -expansion — please use!
- Significant improvements are straightforwardly possible, though computationally costly, as our results are statistics-limited.

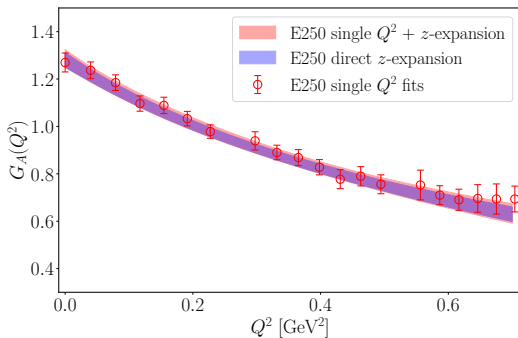
Thank you!

Any questions?

Backup slides

Testing the method

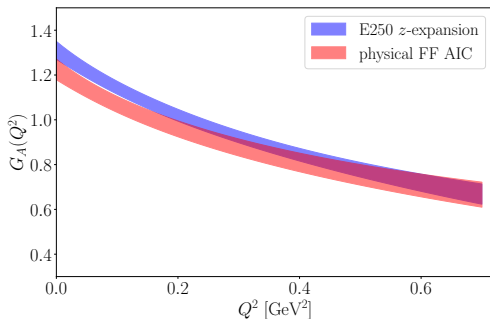
- Doing the analysis in two steps (first extract the value of the FF at each Q^2 , then parameterize the Q^2 dependence using z -expansion) or in one step would be exactly equivalent in ideal world.



- We test that this also works in practice, and that large correlation matrices or numerical instabilities do not cause any problems. In fact, the fits in the one-step analysis tend to be very stable.

E250 vs the extrapolation to physical point

- Ensemble E250 has a small lattice spacing, large volume and the pion mass is close to (slightly below) the physical pion mass.



- We expect our final result for the form factor after continuum, infinite volume and chiral extrapolations to be close to the result we obtain on E250.
- The agreement is found to be good – the small difference shows how large the corrections are due to finite lattice spacing, finite volume, and tuning of the pion mass, at different Q^2 .

Akaike Information Criterion

- The weights w_i^{AIC} are interpreted as probabilities, and the analyses follow a normal (Gaussian) distribution $N(a_i; m_k, \sigma_k)$ for the quantity a_i . m_k and σ_k are the jackknife average and error in the k -th analysis.
- A joint distribution function can then be defined as

$$\sum_k w_k^{\text{AIC}} N(a_i; m_k, \sigma_k),$$

which includes both statistical and systematic uncertainties.

- The corresponding cumulative distribution function reads

$$P(a_i) = \int_{-\infty}^{a_i} da'_i \sum_k w_k^{\text{AIC}} N(a'_i; m_k, \sigma_k).$$

The median of the CDF gives the central value of a_i and its total error is given by the 16% and 84% percentiles of the CDF.

- Noticing that scaling σ_k by a factor of $\sqrt{\lambda}$ scales the statistical error by $\sqrt{\lambda}$, but does not scale the systematic error, using $\lambda = 1$ and $\lambda = 2$ allows us to calculate the break-up into stat. and syst. parts.