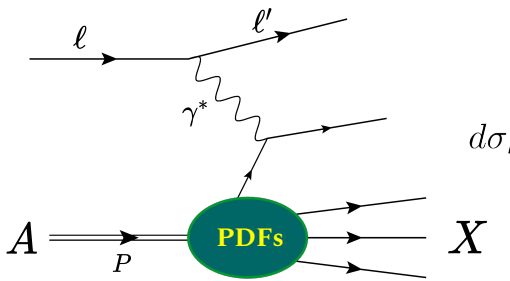
Leading-twist Quark PDFs of the Nucleon from Ioffe-time Pseudo-distributions

August 10, 2022

Colin Egerer
For the **HadStruc** Collaboration

Jefferson Lab

Parton Distributions & Hadron Structure



$$d\sigma_h \sim f_{a/h} \otimes d\sigma_q$$

$$F_i\left(x, Q^2\right) = \sum_{a=q,\bar{q},g} f_{a/h}\left(x, \mu^2\right) \otimes H_i^a\left(x, \frac{Q^2}{\mu^2}, \alpha_s\left(\mu^2\right)\right) + h.t.$$

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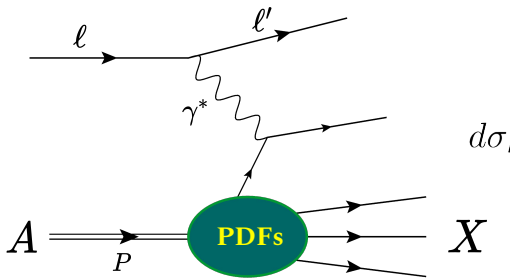
Parton Distributions & Hadron Structure



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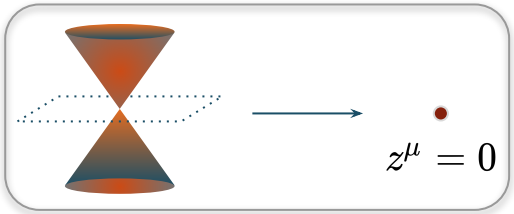
$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle h(p)| \overline{\psi}\left(\tfrac{z}{2}\right) \gamma^+ \Phi^{(f)}_{\hat{z}^-}\left(\{\tfrac{z}{2}, -\tfrac{z}{2}\}\right) \psi\left(-\tfrac{z}{2}\right) |h(p)\rangle$$



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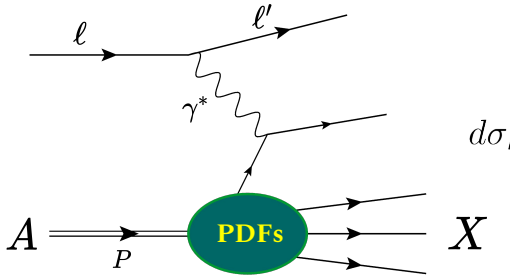

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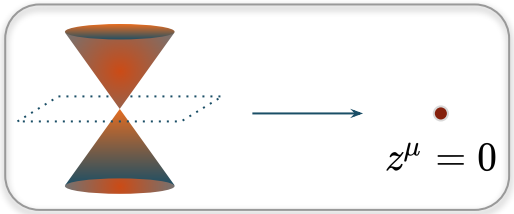
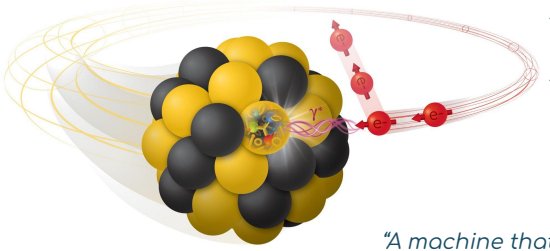


EIC will probe hadron structure at unprecedented precision



- ex. spin-dependent PDFs
- hadron tomography/ confinement/saturation
  - some dists. will remain hard to quantify in experiment

"A machine that will unlock the secrets of the strongest force in Nature"

Roadmap - From Matrix Elements to PDFs…

Short-distance Factorization

- space-like matrix elements

Obtaining Unknown PDF

- orthogonal polynomials
- leading-twist & corrections

PHYSICAL REVIEW D **105**, 034507 (2022)

Transversity parton distribution function of the nucleon using the pseudodistribution approach

Colin Egerer,[1,2] Christos Kallidonis,[2] Joseph Karpie,[3] Nikhil Karthik,[1,2] Christopher J. Monahan,[1,2] Wayne Morris,[4,2] Kostas Orginos,[1,2] Anatoly Radyushkin,[4,2] Eloy Romero,[2] Raza Sabbir Sufian,[1,2] and Savvas Zafeiropoulos[5]

Transversity Quark PDF

Model Selection

- model choice and data cuts
- biased result

Unpolarized Quark PDF

JHEP — Published for SISSA by Springer

Towards high-precision parton distributions from lattice QCD via distillation

Colin Egerer,[a,b] Robert G. Edwards,[b] Christos Kallidonis,[b] Kostas Orginos,[a,b] Anatoly V. Radyushkin,[b,c] David G. Richards,[b] Eloy Romero[b] and Savvas Zafeiropoulos[d] on behalf of the HadStruc collaboration

Quark Helicity

- numerical methods - control of systematic contamination

Off-forward Regime…

On Short-Distance Factorization (SDF)

This talk: matrix elements of space-like separated parton bilinears

$$M^{[\Gamma]}(p,z) = \langle h(p)|\,\overline{\psi}(z)\,\Gamma\Phi^{(f)}_{\hat{z}}(\{z,0\})\,\psi(0)\,|h(p)\rangle$$

Additional UV singularities for space-like Wilson line

LaMET & Quasi-Distributions

X. Gao [Wed. 2:20pm] L. Walter [Wed. 2:40pm]
J. Dodson [Wed. 5:10pm] J. Holligan [Thu. 10:20am]

X. Ji, Phys. Rev. Lett. 110 (2013) 262002

SDF & Pseudo-Distributions

J. Karpie [Wed. 9:50am] J. Delmar [Wed. 4:30pm]
M. Bhat [Wed. 4:50pm] R. Sufian [Wed. 5:50pm]

V. Braun and D. Müller, Eur.Phys.J.C 55 (2008) 349-361
A. Radyushkin, PRD 96 (2017) 3, 034025

Matrix elements admit a Lorentz decomposition

$$M^{[\Gamma]}(p,z) = \sum_i \mathcal{K}_i(p^\mu, z^\mu, \cdots)\,\mathcal{M}_i(\nu, z^2)$$

$\nu \equiv p\cdot z$

Short-Distance Factorization →

Ioffe-time Pseudo-Distribution (pseudo-ITD) — Perturbatively calculable Wilson coefficients — Ioffe-time Distribution (ITD)

$$\mathcal{M}_i(\nu, z^2) = \sum_a C_a(z^2\mu^2, \alpha_s) \otimes \mathcal{Q}_a(\nu, \mu^2) + \mathcal{O}(z^2\Lambda^2_{\rm QCD})$$

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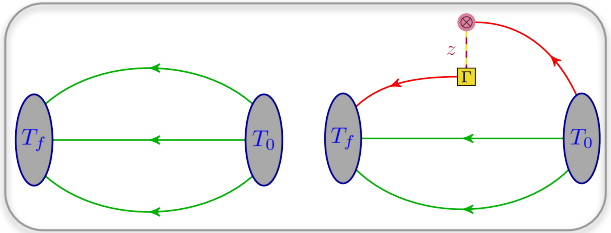
	Lorentz Decomposition	Defining Kinematics	Leading-twist Information	Accessing Leading-Amplitude in LQCD
Unpol.	$2p^\alpha\mathcal{M}(\nu,z^2) + 2z^\alpha\mathcal{N}(\nu,z^2)$	$p^\alpha = \left(p^+, \frac{m_h^2}{2p^+}, \mathbf{0}_\perp\right)$ $z^\alpha = (0, z^-, \mathbf{0}_\perp)$ $\alpha = +$	$\mathcal{M}(p^+z^-,0)_{\mu^2}$	$p^\alpha = (\mathbf{0}_\perp, p_z, E)$ $z^\alpha = (\mathbf{0}_\perp, z_3, 0)$ $\alpha = 4$
Trans.	$2[p^\alpha, S_\perp^\beta]\mathcal{M}(\nu,z^2) + 2im_N^2[z^\alpha, S_\perp^\beta]\mathcal{N}(\nu,z^2) + 2m_N^2[z^\alpha,p^\beta](z\cdot S_\perp)\mathcal{R}(\nu,z^2)$	▲ As Above ▲	$\mathcal{M}(p^+z^-,0)_{\mu^2}$	▲ As Above ▲ $\alpha = 4$ $\beta = \perp$
Helicity	$-2m_N S^\alpha\mathcal{M}(\nu,z^2) - 2im_N p^\alpha(z\cdot S)\mathcal{N}(\nu,z^2) + 2m_N^3 z^\alpha(z\cdot S)\mathcal{R}(\nu,z^2)$	▲ As Above ▲	$[\mathcal{M}(p^+z^-,0) + ip^+z^-\mathcal{N}(p^+z^-,0)]_{\mu^2}$	▲ As Above ▲ $\alpha = 3$

From Matrix Elements to Pseudo-Distributions

Needed correlation functions:

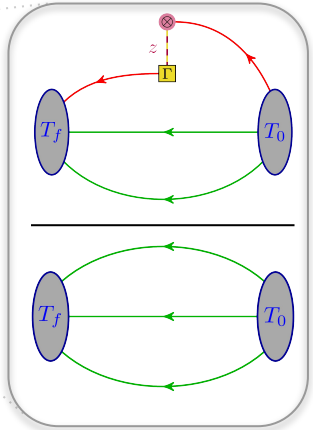


➢ Leverage *Distillation* M. Peardon et al., Phys. Rev. D80, 054506 (2009)
w/ momentum smearing algorithm: **CE** et al., PRD 103 (2021) 3, 034502

Summation method - further excited-state suppression

L. Maiani et al., Nucl. Phys. B293 (1987)
C. Bouchard et al., PRD 96, no. 1, 014504 (2017)

$$R(p_z, z_3; T) = \sum_{\tau/a=1}^{T-1} \frac{C_3(p_z, T, \tau; z_3)}{C_2(p_z, T)}$$



$$R_{\text{fit}}(p_z, z_3; T) = \mathcal{A} + M_4(p_z, z_3)\, T + \mathcal{O}\left(e^{-\Delta E T}\right)$$

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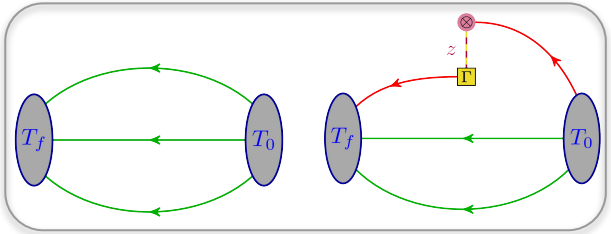

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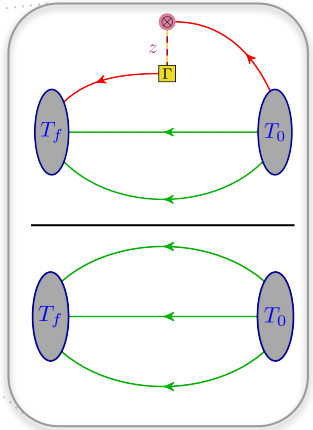

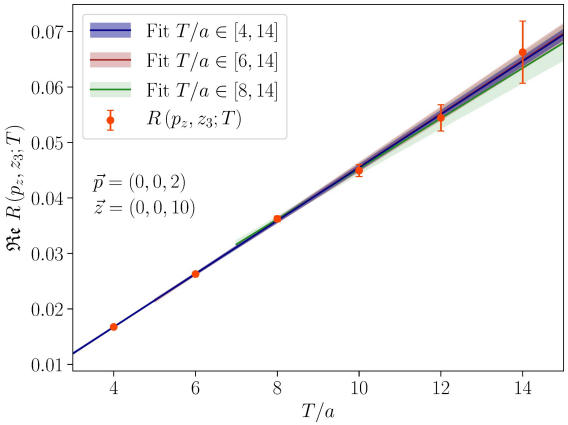

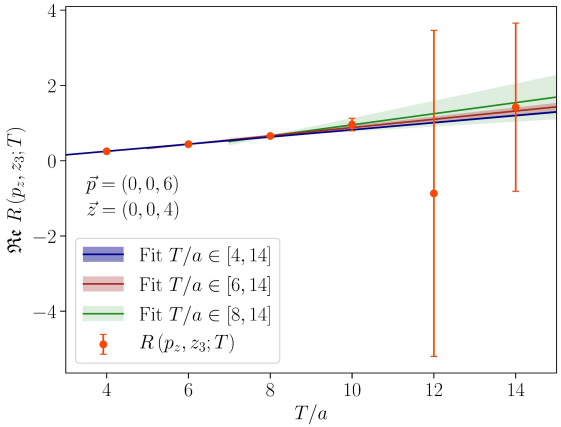

CE, R. Edwards, C. Kallidonis et al., JHEP 11 (2021) 148

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Reduced pseudo-ITD: remove divergences associated with Wilson-line

$$\mathfrak{M}(\nu, z^2) = \frac{M_4(p, z)/M_4(p, 0)}{M_4(0, z)/M_4(0, 0)}$$

K. Orginos et al., Phys. Rev. D 96, 094503 (2017)

Determining the Unknown PDFs

Ill-posed (pseudo-)ITD/PDF matching relation:

$$\mathfrak{M}\left(\nu, z^2\right) = \int_{-1}^{1} \mathrm{d}x\ \mathcal{K}\left(x\nu, z^2\mu^2\right) f_{q/h}\left(x, \mu^2\right) + \sum_{k=1}^{\infty} \mathcal{B}\left(\nu\right)\left(z^2\right)^k$$

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One choice: model parameterization

$$\int_{-1}^{1} dz\,(1-z)^{\alpha}\,(1+z)^{\beta}\,J_n^{(\alpha,\beta)}\left(z\right) J_m^{(\alpha,\beta)}\left(z\right) = \delta_{n,m} h_n\left(\alpha,\beta\right)$$

Change of variables

➢ polynomials span support interval of PDFs

$$f_{q/h}\left(x\right) = x^{\alpha}\left(1-x\right)^{\beta} \sum_{n=0}^{\infty} C_{q,n}^{(\alpha,\beta)} \boxed{\Omega_n^{(\alpha,\beta)}\left(x\right)}$$

J. Karpie, K. Orginos, A. Radyushkin et al., JHEP 11 (2021) 024

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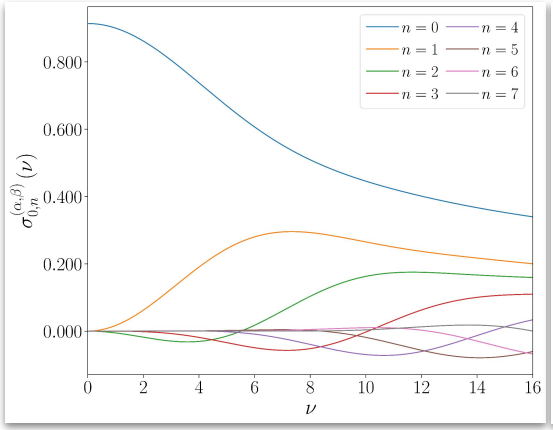
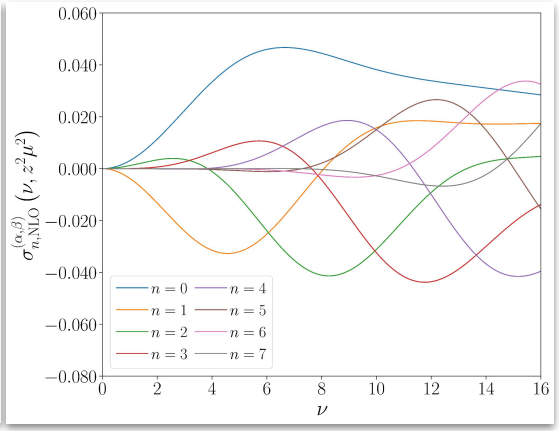
$$\mathfrak{Re}\,\mathfrak{M}_{\mathrm{fit}}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{\mathrm{v},n}^{lt\,(\alpha,\beta)}$$

$$\mathfrak{Im}\,\mathfrak{M}_{\mathrm{fit}}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \eta_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{+,n}^{lt\,(\alpha,\beta)}$$

$$\sigma_{0,n}^{(\alpha,\beta)}(\nu) = \sigma_n^{(\alpha,\beta)}(\nu, z^2\mu^2)|_{\alpha_s=0}$$

$$\sigma_{n,\mathrm{NLO}}^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) = \sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) - \sigma_{0,n}^{(\alpha,\beta)}(\nu)$$





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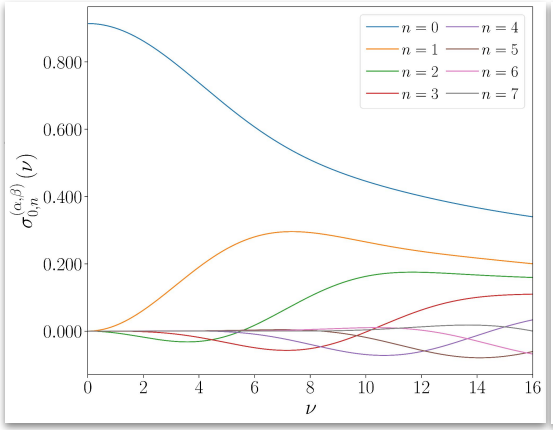
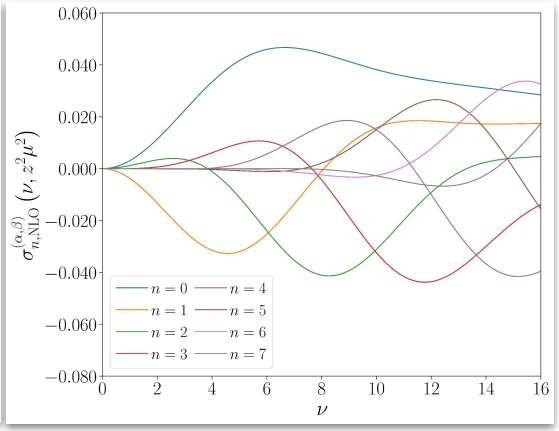




Objective: determine most likely parameters given data & prior information

- Bayes' Theorem
- maximize posterior distribution
- stabilize optimization: `Variable Projection`

G. Golub and V. Pereyra, SIAM Journal on Numerical Analysis 10, 413 (1973)

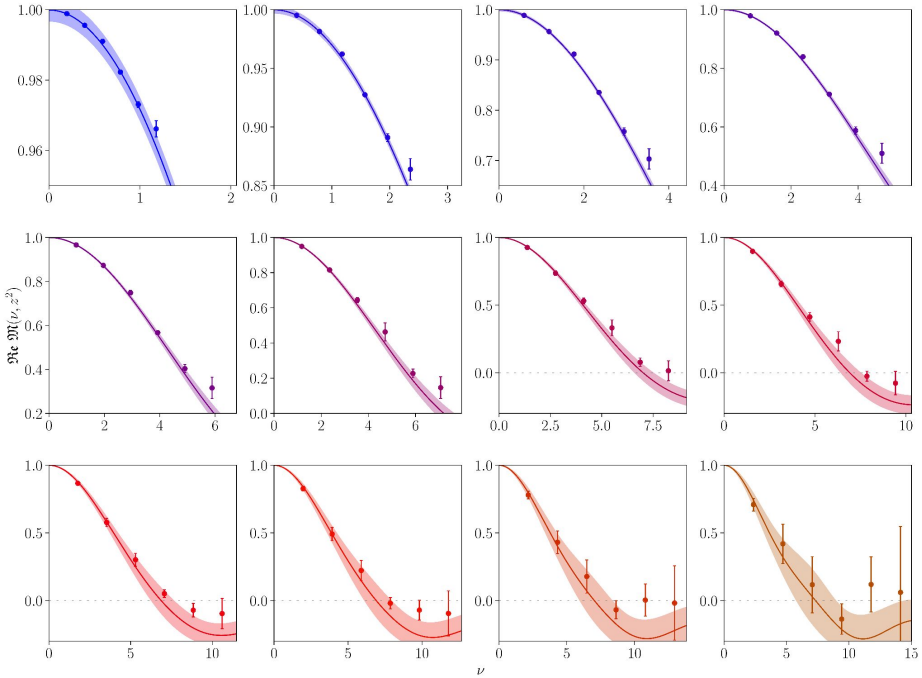
Optimal Fit for Unpolarized Valence Quark PDF

(isovector combination only herein)

ID	a (fm)	m_π (MeV)	β	$c_{\rm SW}$	$L^3 \times T$	$N_{\rm cfg}$
E1	0.094(1)	358(3)	6.3	1.205	$32^3 \times 64$	349

Parameters/Statistics

ID	$N_{\rm vec}$	$N_{\rm srcs}$	T/a	$p_z \times \left(\frac{2\pi}{L}\right)$	z/a
E1	64	4	$4, 6, \cdots, 14$ $0.38, \cdots, 1.32$ fm	$0, \pm1, \cdots, \pm6$ $0, 0.41, \cdots, 2.47$ GeV	$0, \pm1, \cdots, \pm12, \cdots$ $0, 0.094, \cdots, 1.13$ fm



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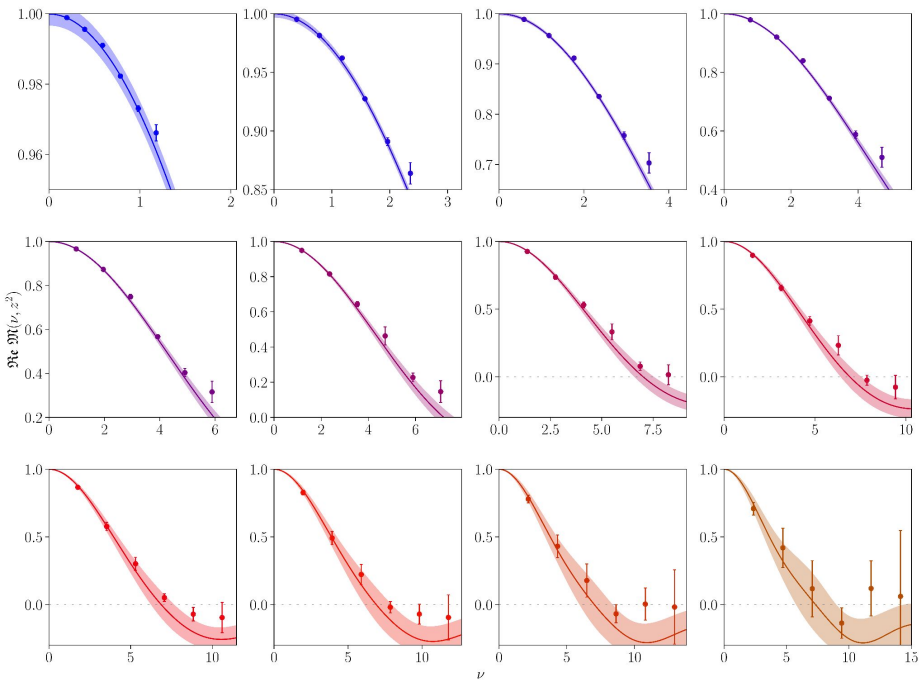
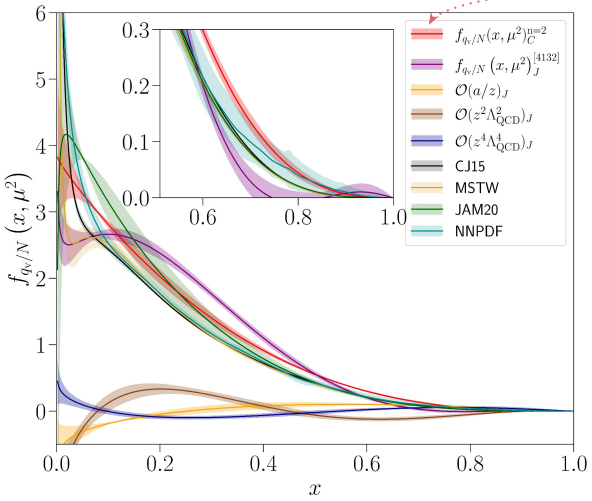


[Uncorrelated]
2-param ansatz



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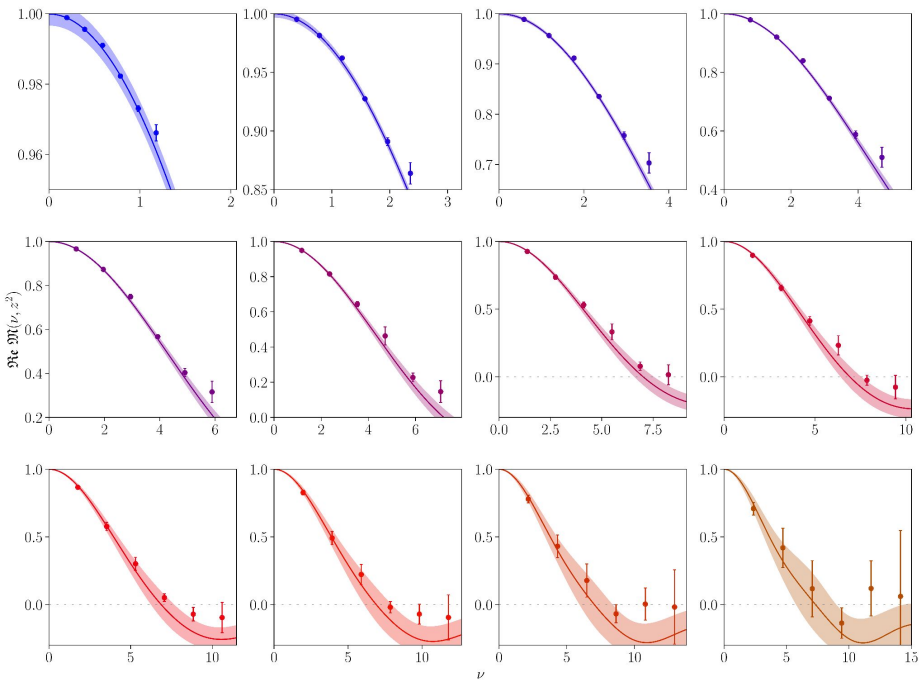
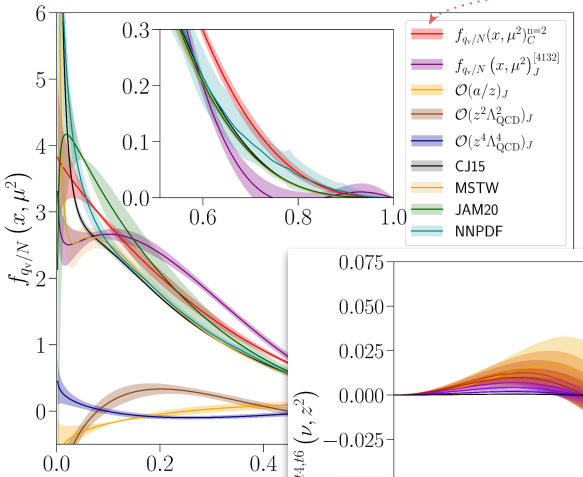
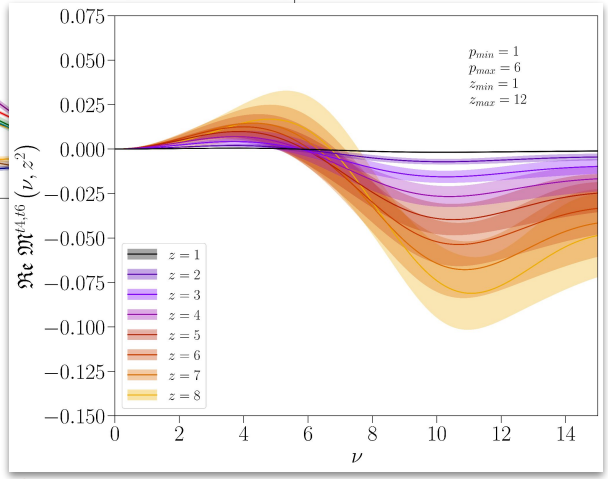


[Uncorrelated]
2-param ansatz



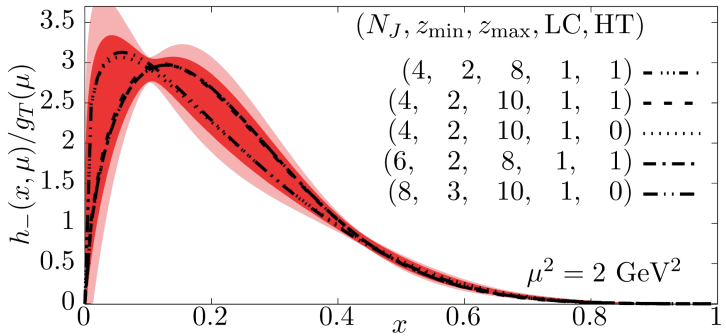


CE, R. Edwards, C. Kallidonis et al., JHEP 11 (2021) 148

Model Averaging & Data Cuts

Despite parameterization via Jacobi polynomials...

➢ bias introduced with any choice of truncation/cuts on data/inclusion of systematic effect



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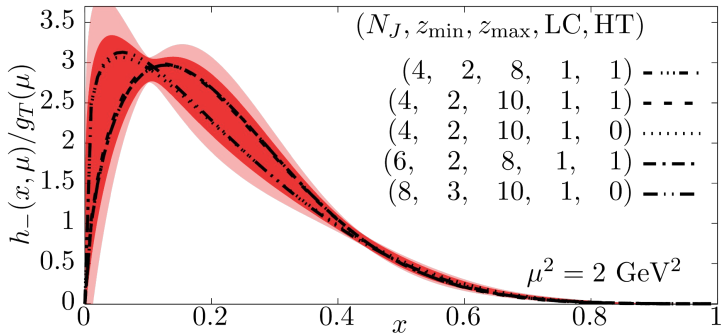

Akaike Information Criterion (AIC)

H. Akaike, IEEE Transactions on Automatic Control, vol.19, no.6, 716-723 (1974)

- ➢ weights assigned based on quality of fit, number of datapoints and parameters

$$w^{(m)} = \frac{e^{-\frac{1}{2}\mathrm{AIC}(m)}}{\sum_{n\in\mathrm{fit}} e^{-\frac{1}{2}\mathrm{AIC}(n)}}$$

- ➢ ideally, averages away model biases for large number of models

$$\mathrm{AIC}(n) = \mathcal{L}_n + 2p_n + \frac{2p_n\,(p_n+1)}{(d_n - p_n - 1)}$$

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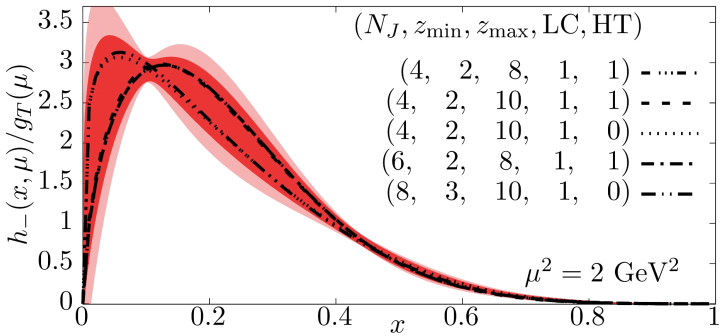

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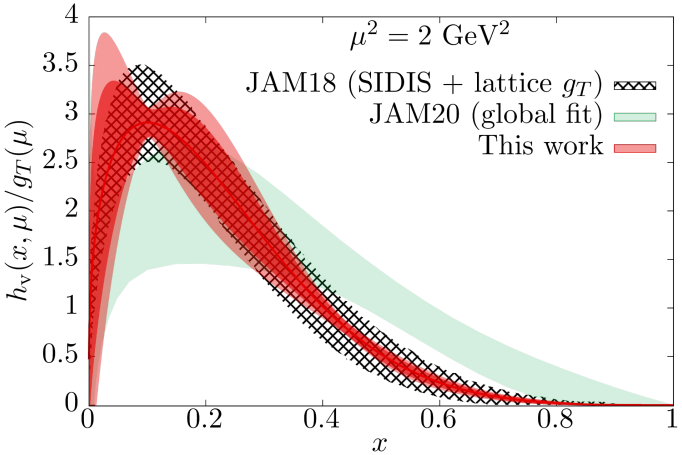

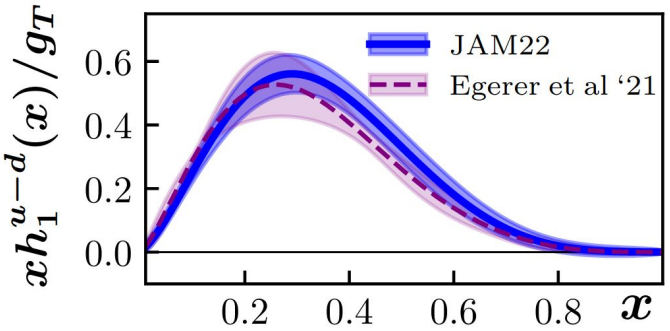

[HadStruc] **CE**, C. Kallidonis, J. Karpie, N. Karthik et al., Phys. Rev. D 105 (2022) 3, 034507

L. Gamberg et al., arXiv:2205.00999 [hep-ph]

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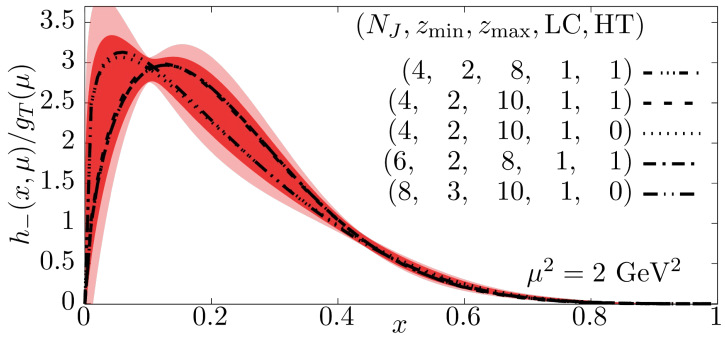

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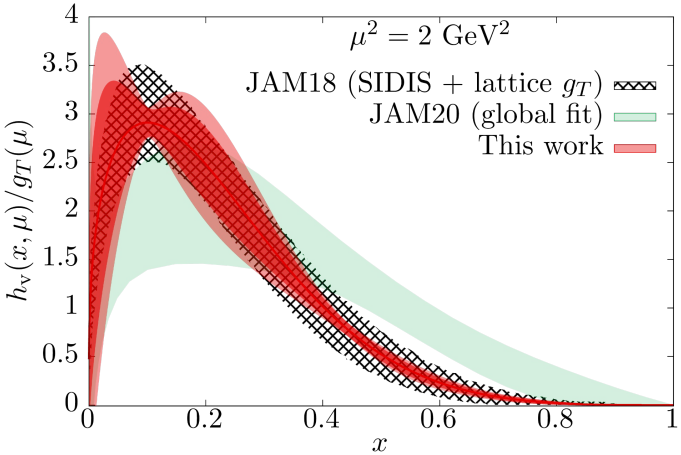
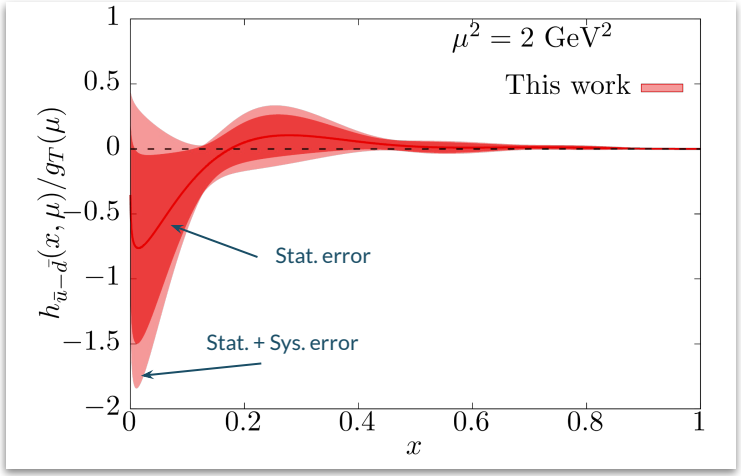
- Non-singlet antiquark distribution found to be consistent with an isospin-symmetric intrinsic sea





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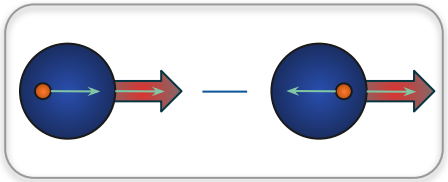
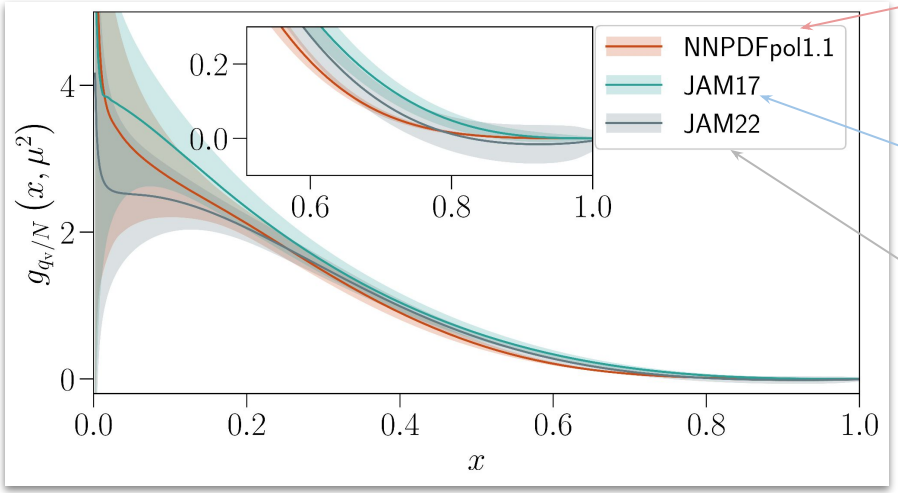
Towards the Quark Helicity PDF

Helicity asymmetry of partons within hadronic state of definite helicity





[First such global analysis]
All long. polarized DIS + jet/W-production @ STAR/PHENIX

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Simultaneous fit of PDFs/FFs in polarized (semi-)inclusive DIS

N. Sato et al., PRD 93 (2016) 7, 074005

Includes jet production @ STAR/PHENIX; PDF positivity relaxed

C. Cocuzza et al., arXiv:2202.03372 [hep-ph]

Matrix element defining helicity:

$$M^{\alpha}(p,z) = \langle N(p,S)|\,\overline{\psi}(z)\,\gamma^{\alpha}\gamma^{5}\Phi^{(f)}_{\hat{z}}(\{z,0\})\,\psi(0)\,|N(p,S)\rangle$$

$$= -2m_N S^{\alpha}\mathcal{M}(\nu,z^2) - 2im_N p^{\alpha}(z\cdot S)\,\mathcal{N}(\nu,z^2) + 2m_N^3 z^{\alpha}(z\cdot S)\,\mathcal{R}(\nu,z^2)$$

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$$-2m_N S^{+}\left[\mathcal{M}(\nu,0) + i\nu\mathcal{N}(\nu,0)\right]|_{\mu^2} \equiv -2m_N S^{+}\mathcal{Y}(\nu,0)$$

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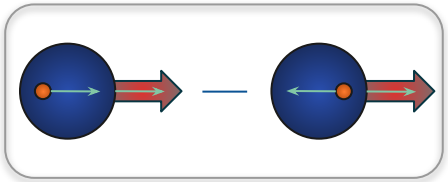

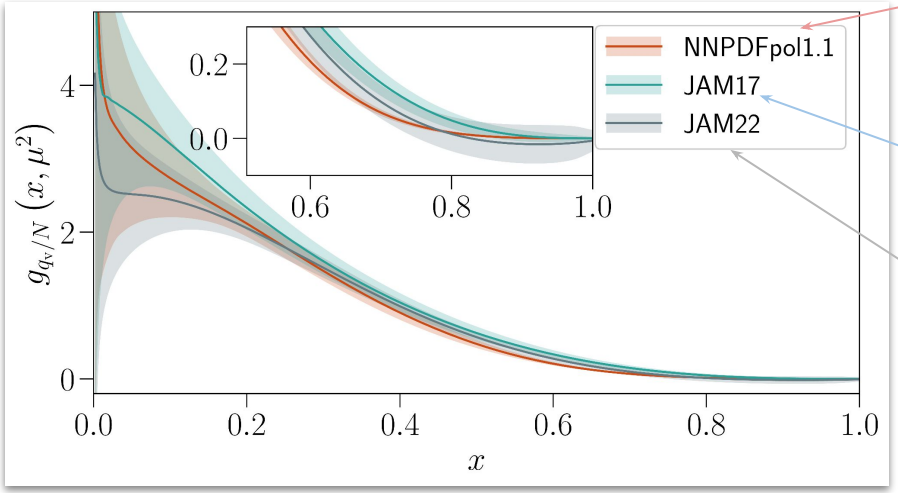

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1
- Ignore presence of contamination
- Scale-dependence modified, but fitting corrections should capture effect

2
- Demand separation of leading twist amplitude from contamination
- Overconstrained system - SVD

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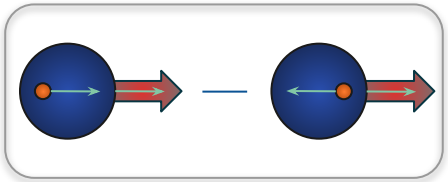

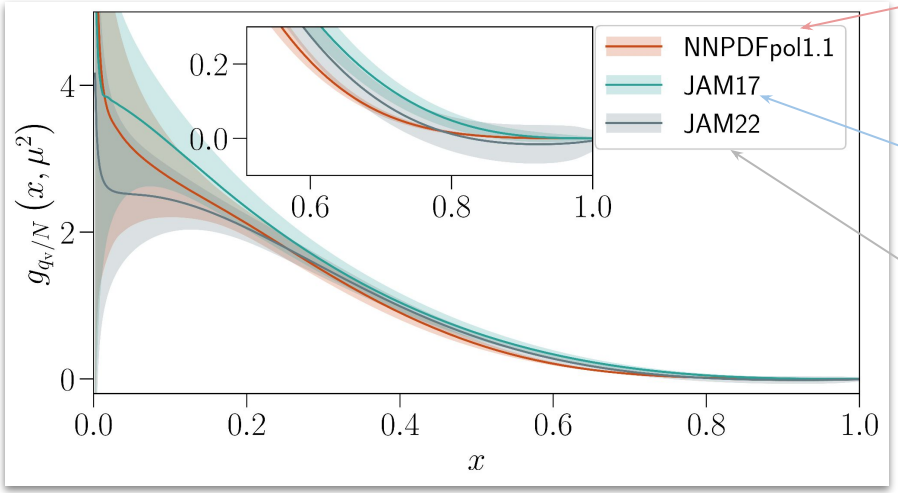

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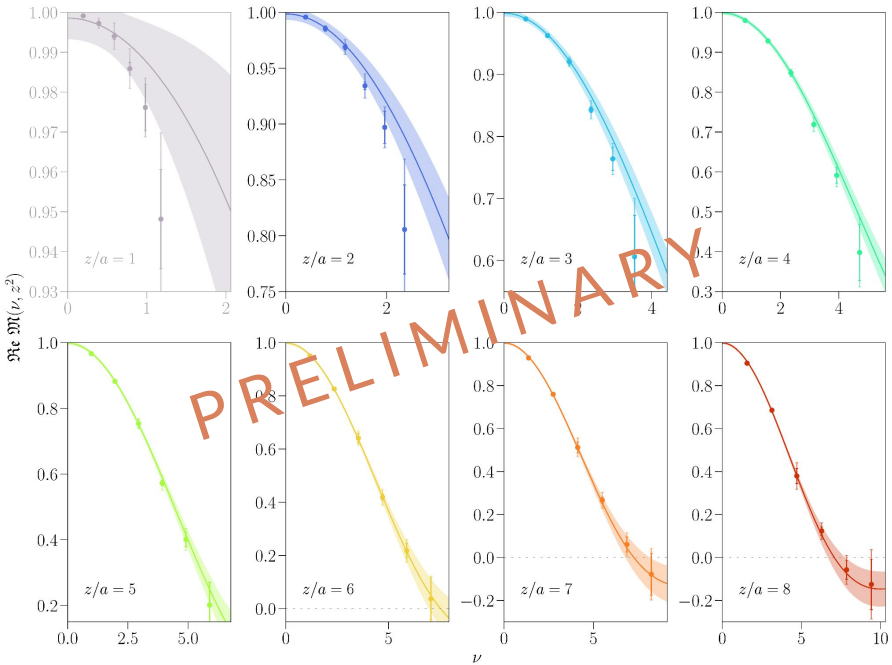
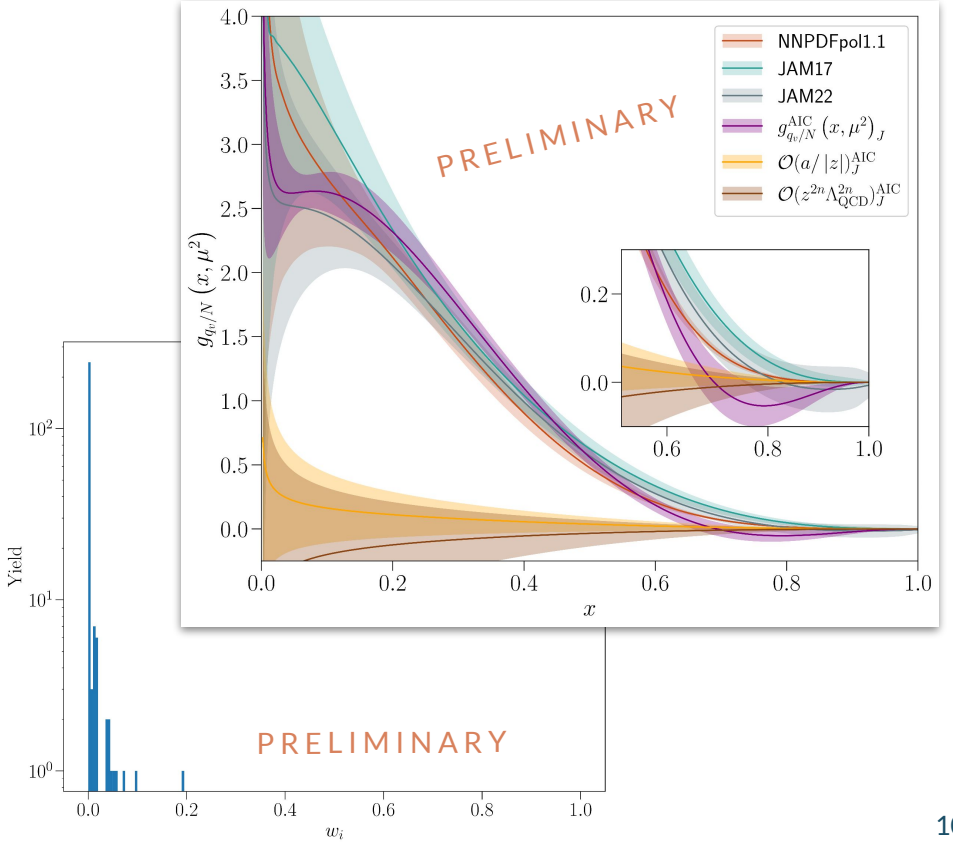
Exploit reduced rotational symmetry

Select Fit for Isovector Helicity PDFs



$$p_{\text{latt}} \in [1, 6] \qquad z/a \in [2, 8]$$
$$[N_J, N_{a/|z|}, N_{\text{ht}}, \cdots] = [3, 1, 0, \cdots]$$



- discretization effect prominent in AIC average; net higher-twist effects consistent with zero

Phenomenological Insight

Transversity is only chiral-odd twist-2 collinear PDF

➢ chiral-odd process needed [eg. transverse SSAs in SIDIS]

(Often used) Theory constraint: Soffer bound J. Soffer, Phys. Rev. Lett. 74, 1292 (1995)

$$\left|h_{q/h}\left(x,\mu^2\right)\right| \leq \frac{1}{2}\left[f_{q/h}\left(x,\mu^2\right)+g_{q/h}\left(x,\mu^2\right)\right]$$

[E.g] Z.-B. Kang et al., Phys. Rev. D 93, 014009 (2016); M. Radici et al., JHEP 05, 123 (2015)
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Do PDFs computed from LQCD support or challenge this bound?

"Soffer-PDF"

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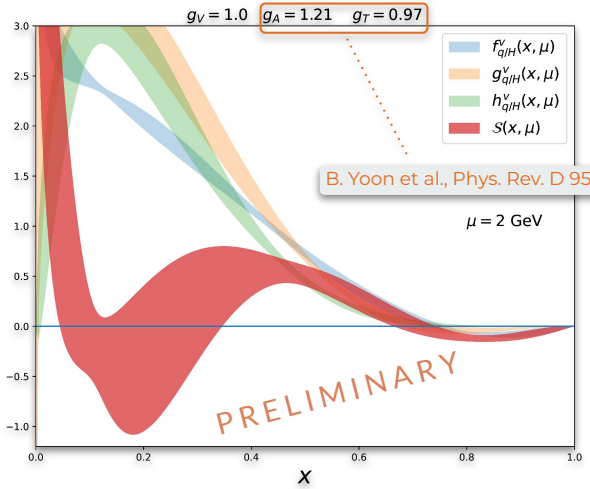

B. Yoon et al., Phys. Rev. D 95 (2017) 7, 074508

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Must rescale with renormalized charges for correct normalization

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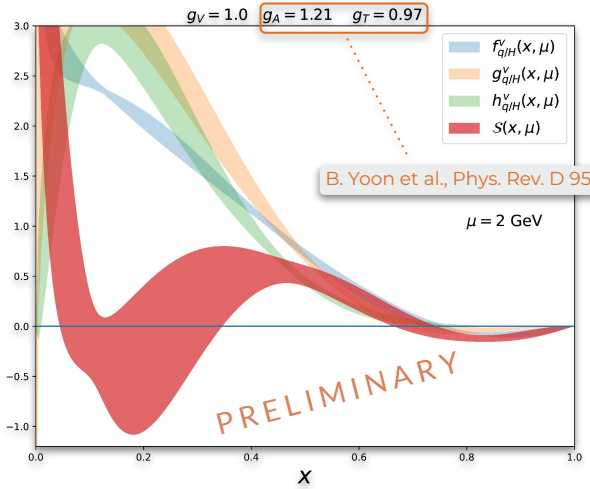

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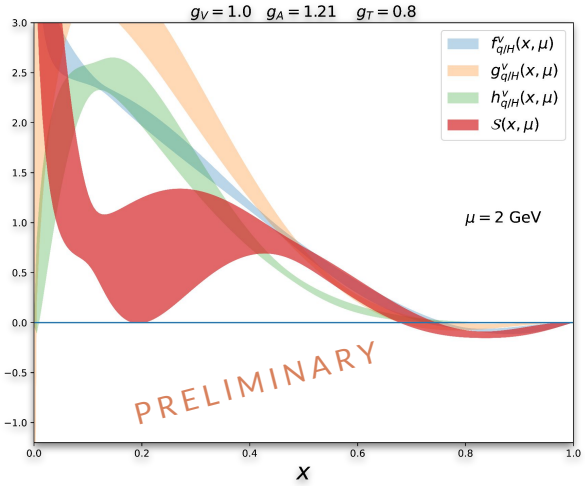
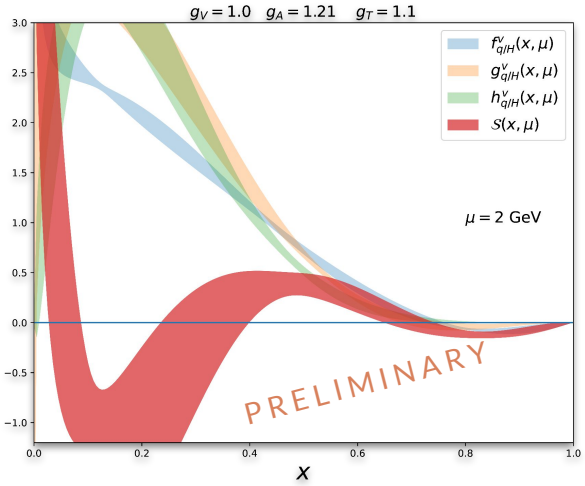




➢ shape of "Soffer-PDF" driven by g_T^{u-d}

➢ caveat - model bias (no AIC here)

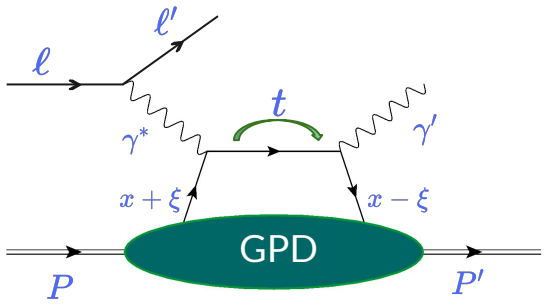
➢ (Soffer bound as a prior):
PDFs have potential to constrain, or provide precise upper bound on g_T^{u-d}

Next Steps - Off-Forward Matrix Elements

Relevant in variety of exclusive channels

- DVCS/DVMP: (e.g. E12-06-113 [HRS] & E12-11-003 [CLAS12])

Eur.Phys.J.A 52 (2016) 6, 157; Eur.Phys.J.A 52 (2016) 6, 151
Eur.Phys.J.A 52 (2016) 6, 158



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A. Radyushkin, Phys. Rev. D100, 116011 (2019)
A. Radyushkin, arXiv: Int.J.Mod.Phys.A 35 (2020) 05, 2030002

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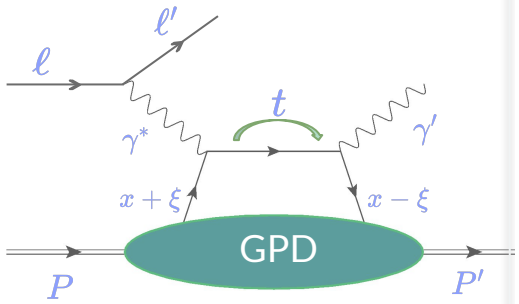

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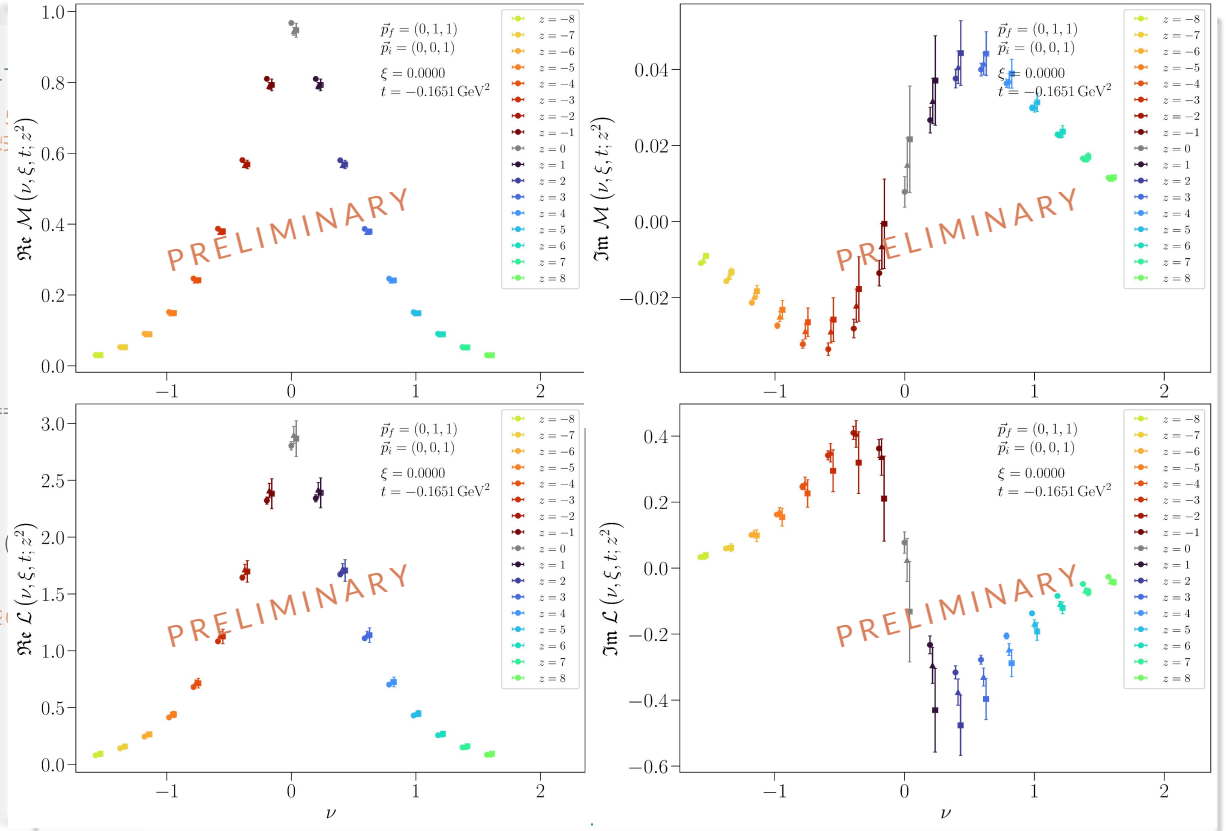

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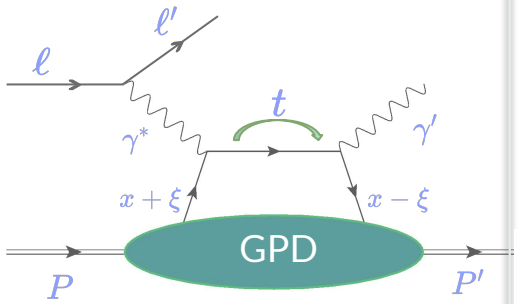
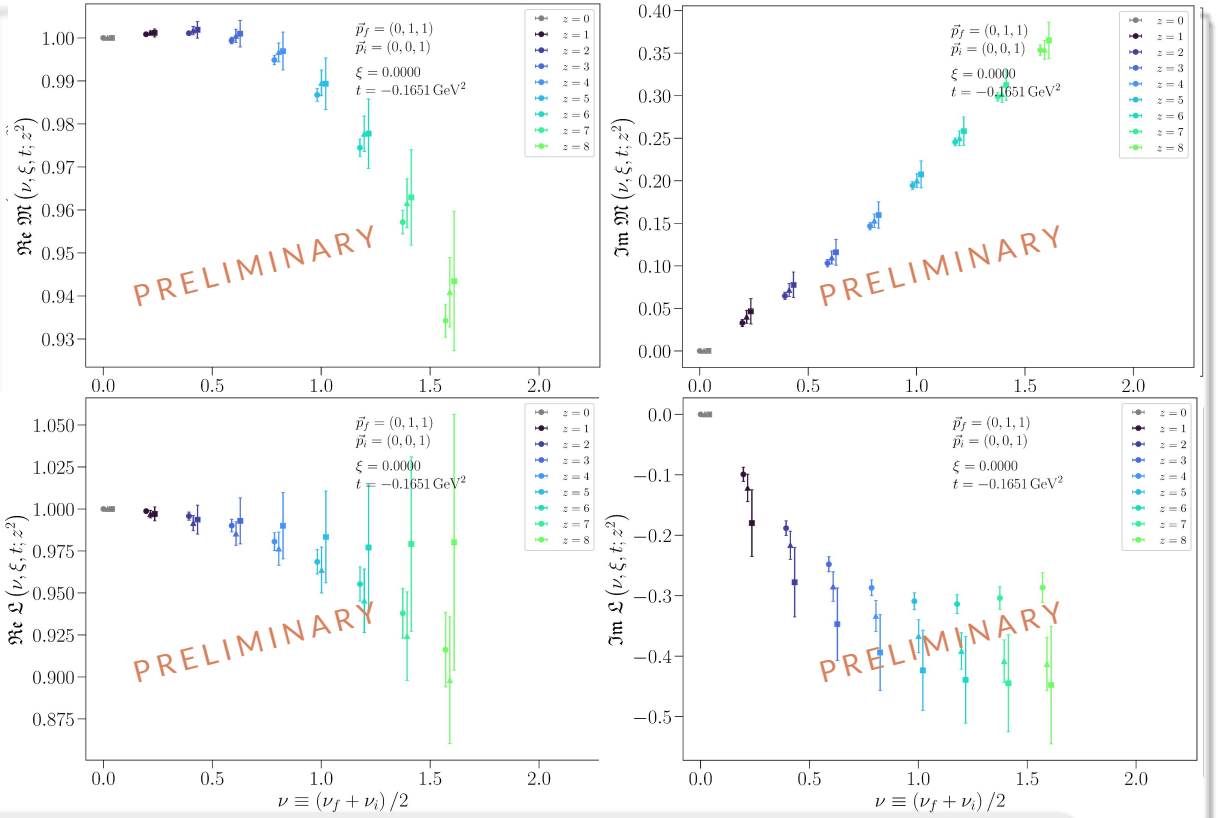


$$\mathbb{M}^{\mu}(p_f, p_i, z) \equiv \langle N(p_f)| \overline{\psi}(-z/2) \frac{\tau^3}{2} \gamma^{\mu} W(-z/2, z/2; A) \psi$$

A. Radyushkin, Phys. Rev. D100, 116011 (2019)
A. Radyushkin, arXiv: Int.J.Mod.Phys.A 35 (2020) 05, 2



Renormalize (ratio) → match onto x-dependence of GPD

$$\widetilde{\mathfrak{M}}(\nu, \xi, t, z^2) = \mathcal{K}(x\nu, \xi\nu, z^2\mu^2; \alpha_s) \otimes H(x, \xi, t, \mu^2)$$

Closing Remarks

Hadronic structure accessible from certain lattice calculable matrix elements

- ➢ short-distance factorization
- ➢ considerable progress in factorizable methods

K. Cichy & M. Constantinou, Adv.High Energy Phys. (2019), 3036904
K. Cichy, Lattice 2021, arXiv: 2110.07440 [hep-lat]; M. Constantinou, Eur. Phys. J. A 57, 77 (2021)

Isovector twist-2 quark PDFs of Nucleon

m_π[MeV] / a[fm]	$f_{q_\pm/N}(x,\mu^2)$	$g_{q_\pm/N}(x,\mu^2)$	$h_{q_\pm/N}(x,\mu^2)$
358(3) / 0.094(1)	Published	Forthcoming	Published
278(4) / 0.094(1)	Preliminary	Preliminary	Preliminary
170(5) / 0.091(2)	Ongoing	Ongoing	Ongoing

- ➢ statistical precision afforded by use of distillation and its union with momentum smearing idea
- ➢ systematic effects can be reliably addressed

Stay tuned:

- ➢ towards a continuum extrapolation
- ➢ Distillation in the off-forward regime

HadStruc Collaboration

Robert Edwards, **CE**, Nikhil Karthik,
Jianwei Qiu, David Richards, Eloy Romero, Frank Winter [1]

Balint Joó[2]

Carl Carlson, Chris Chamness, Tanjib Khan,
Christopher Monahan, Kostas Orginos, Raza Sufian[3]

Wayne Morris, Anatoly Radyushkin [4]

Joe Karpie [5]

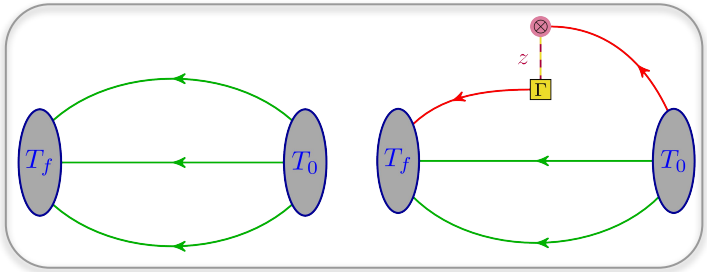
Savvas Zafeiropoulos [6]

Yan-Qing Ma [7]

Jefferson Lab [1], Oak Ridge [2], William and Mary [3], Old Dominion University [4],
Columbia University [5], Aix Marseille University [6], Peking University [7]

Obtaining the Pseudo-Distributions

Needed correlation functions:



$$C_{mn}(t) = \sum_{\vec{x},\vec{y}} \langle 0| \mathcal{O}_m(t,\vec{x})\, \mathcal{O}_n^\dagger(0,\vec{y}) |0\rangle$$
$$\equiv \mathrm{Tr}\left[\Phi_m(t) \otimes \tau(t,0)\, \tau(t,0)\, \tau(t,0) \otimes \Phi_n(0)\right]$$

Excited-state contamination + broken symmetries

➢ interpolators that best reflect properties of desired state

$$\langle 0| \hat{\mathcal{O}}(\vec{p}) |h(\vec{p})\rangle \gg \langle 0| \hat{\mathcal{O}}(\vec{p}) |h'(\vec{p})\rangle$$

Distillation: Low-rank and *non-iterative* approximation of a gauge-covariant smearing kernel (typically the Jacobi smearing kernel)

M. Peardon et al., Phys. Rev. D80, 054506 (2009)

$$\Box(\vec{x},\vec{y};t)_{ab} = \sum_{k=1}^{R_\mathcal{D}} \xi_a^{(k)}(\vec{x},t)\, \xi_b^{(k)\dagger}(\vec{y},t)$$

w/ momentum smearing algorithm:

CE et al., PRD 103 (2021) 3, 034502

Wick contractions factorize distillation space

Perambulators $\quad \tau_{\alpha\beta}^{kl}(t_f,t_0) = \xi^{(k)\dagger}(t_f)\, M_{\alpha\beta}^{-1}(t_f,t_0)\, \xi^{(l)}(t_0)$

Elementals $\quad \Phi_{\mu\nu\sigma}^{(i,j,k)}(t) = \epsilon^{abc} \left(\mathcal{D}_1 \xi^{(i)}\right)^a \left(\mathcal{D}_2 \xi^{(j)}\right)^b \left(\mathcal{D}_3 \xi^{(k)}\right)^c (t)\, S_{\mu\nu\sigma}$

$$\Xi_{\alpha\beta}^{(l,k)}(T_f,T_0;\tau,z_3) = \sum_{\vec{y}} \xi^{(l)\dagger}(T_f)\, D_{\alpha\sigma}^{-1}(T_f,\tau;\vec{y}+z_3\hat{z})\, [\Gamma]_{\sigma\rho}\, \Phi_{\hat{z}}^{(f)}(\{\vec{y}+z_3\hat{z},\vec{y}\})\, D_{\rho\beta}^{-1}(\tau,T_0;\vec{y})\, \xi^{(k)}(T_0)$$

PDF Selection

Space-like Wilson line

Amortization of inversion cost

All Dirac structures and Wilson line lengths realizable with single inversion overhead

Mapping any momentum dep. requires only contractions

Nucleon Interpolators with Distillation

Excited-state contamination

- ➢ optimize operator/state overlaps - saturate correlation functions at early temporal separations

Generic light-quark nucleon interpolator smeared with distillation

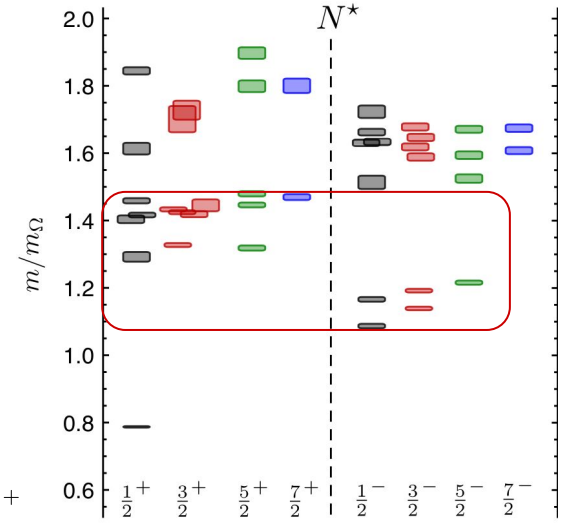
$$\mathcal{O}_i\left(t\right) = \epsilon^{abc}\left(\mathcal{D}_1\Box u\right)_a^{\alpha}\left(\mathcal{D}_2\Box d\right)_b^{\beta}\left(\mathcal{D}_3\Box u\right)_c^{\gamma}\left(t\right) S_i^{\alpha\beta\gamma}$$

Dirac structure/covariant derivatives

Discretized continuum-like interpolators of definite permutational symmetries

$$\mathcal{O}_B = \left(\mathcal{F}_{\mathcal{P}(\mathrm{F})} \otimes \mathcal{S}_{\mathcal{P}(\mathrm{S})} \otimes \mathcal{D}_{\mathcal{P}(\mathrm{D})}\right)\{q_1 q_2 q_3\} \qquad \left(N_M \otimes (\tfrac{1}{2}^+)^1_M \otimes D^{[2]}_{L=1,A}\right)^{J^P=\frac{1}{2}^+} \equiv N^2 P_A \tfrac{1}{2}^+$$

(Generally) Continuum spins reducible under octahedral group



R. Edwards, et. al., Phys. Rev. D84, 074508 (2011)

Canonical subductions

- ➢ spinors/derivatives combined into object of definite spin/parity

$$\mathcal{O}^{\{J\}}_{n_\Lambda,r} = \sum_m S^{J,m}_{n_\Lambda,r}\ \mathcal{O}^{\{J,m\}}$$

R. Edwards, et. al., Phys. Rev. D84, 074508 (2011)
J. Dudek and R. Edwards, Phys. Rev. D85, 054016 (2012)

Helicity subductions

C. Thomas, et al., Phys. Rev. D85, 014507 (2012)
C. Thomas, private communication

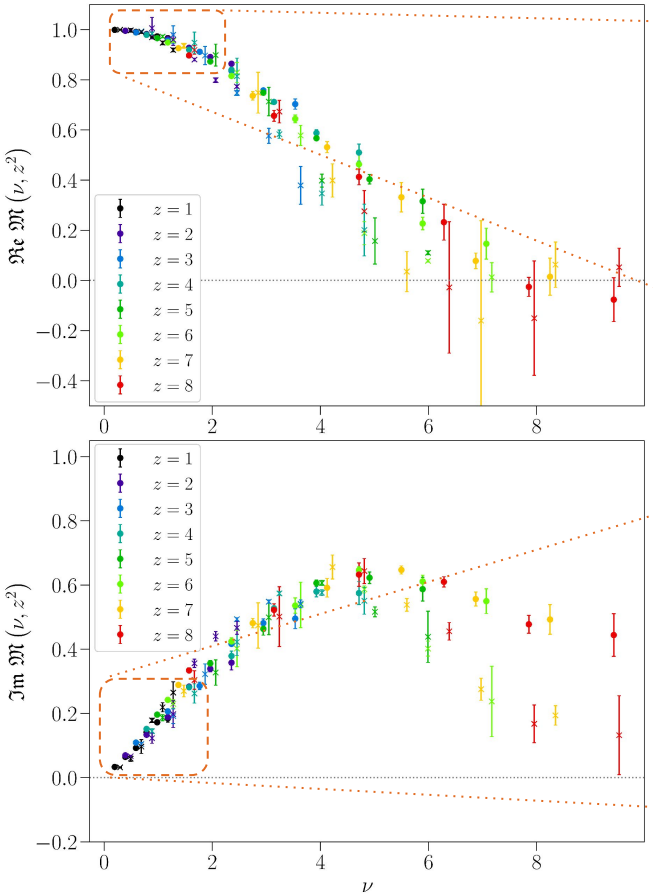
- ➢ boost breaks (double-cover) octahedral symmetry to little groups

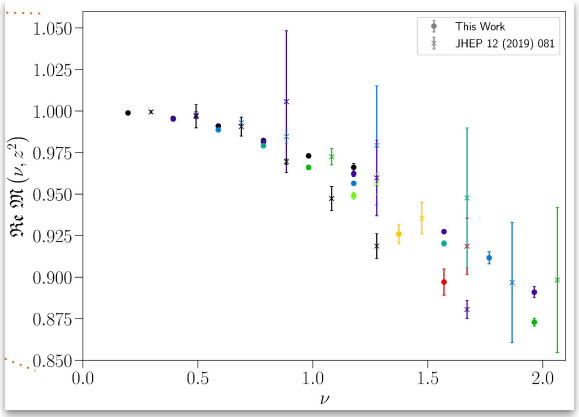
$$\left[\mathbb{O}^{J^P,\lambda}\left(\vec{p}\right)\right]^\dagger = \sum_m \mathcal{D}^{(J)}_{m,\lambda}\left(R\right)\left[O^{J^P,m}\left(\vec{p}\right)\right]^\dagger$$

- ➢ subduce into little groups

$$\left[\mathbb{O}^{J^P,|\lambda|}_{\Lambda,\mu}\left(\vec{p}\right)\right]^\dagger = \sum_{\hat{\lambda}=\pm|\lambda|} S^{\tilde{\eta},\hat{\lambda}}_{\Lambda,\mu}\left[\mathbb{O}^{J^P,\hat{\lambda}}\left(\vec{p}\right)\right]^\dagger$$

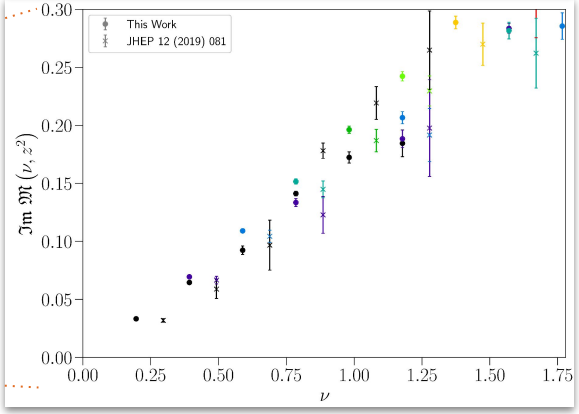
Efficacy of Distillation





B. Joó et al., JHEP 12 (2019) 081 [Gaussian smearing]

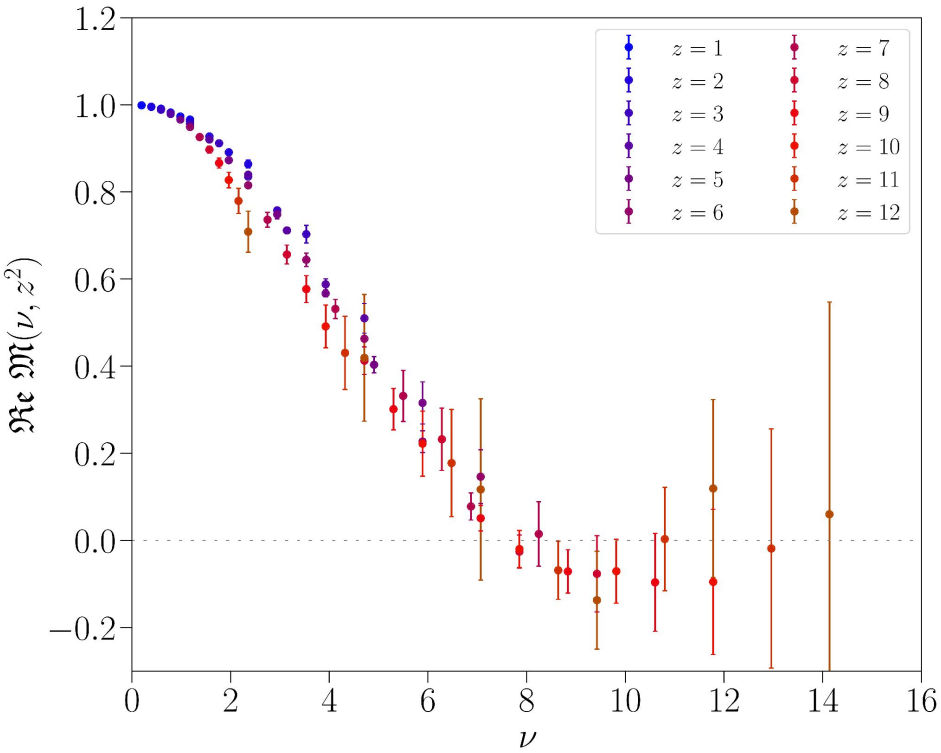
$$N_{\text{cfg}} = 417 \quad N_{\text{src}} = 8 \quad N_{\zeta} = 5$$
$$N_{\text{inv/cfg}} \simeq 8.6\text{k}$$

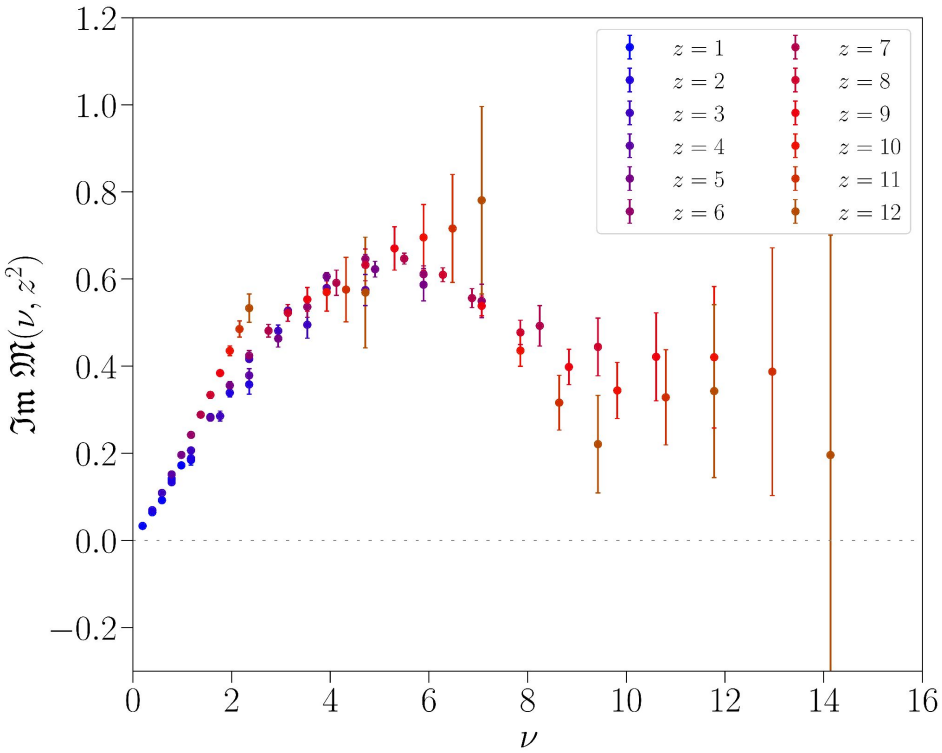


CE, R. Edwards, C. Kallidonis et al., JHEP 11 (2021) 148 [Distillation]

$$N_{\text{cfg}} = 349 \quad N_{\text{src}} = 4 \quad N_{\zeta} = 3$$
$$N_{\text{inv/cfg}} \simeq 16\text{k}$$

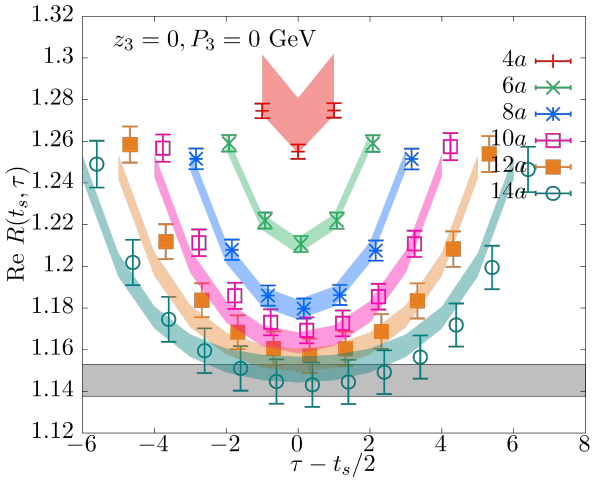
Unpolarized Reduced Pseudo-ITD

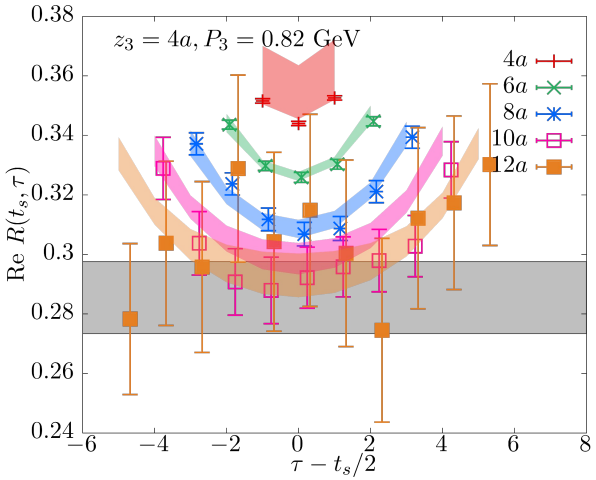


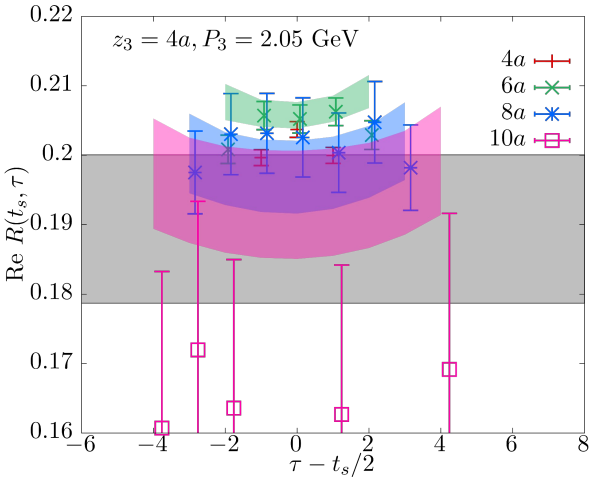


CE, R. Edwards, C. Kallidonis et al., JHEP 11 (2021) 148

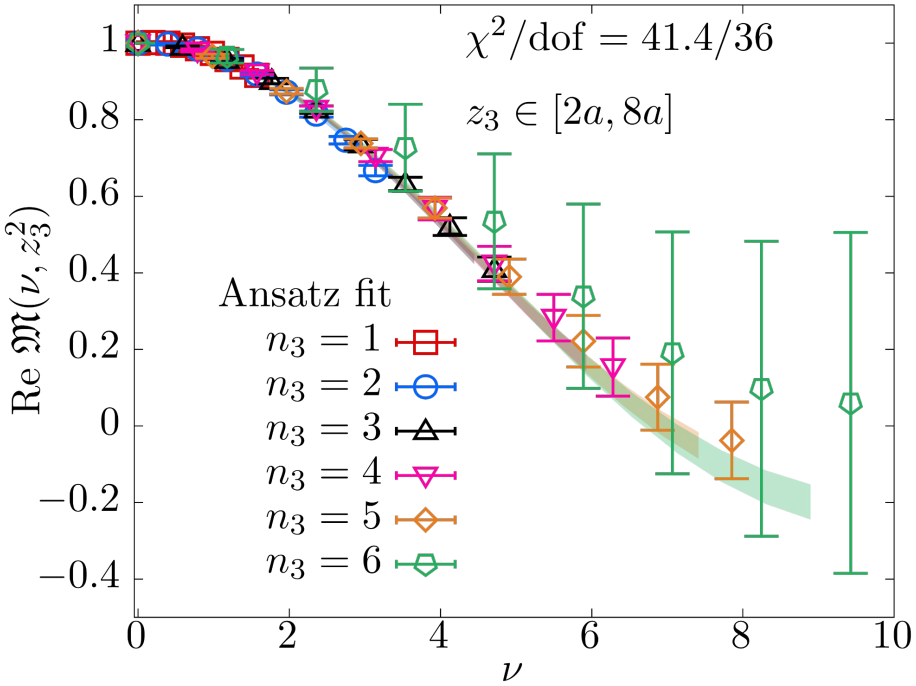
Select Transversity Matrix Element Extractions

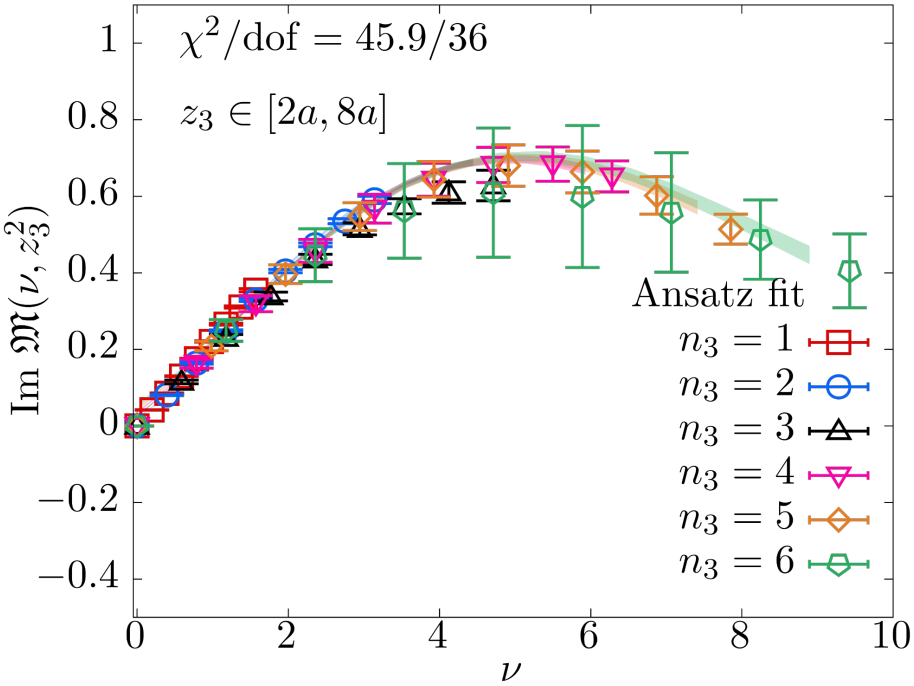






Transversity Reduced Pseudo-ITD





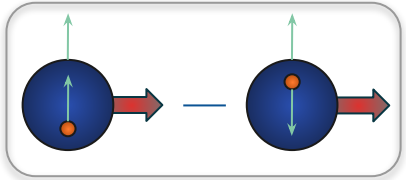
Pheno.-type parameterization

- Twist-2 OPE - Taylor series in Ioffe-time
- plus leading discretization/higher-twist

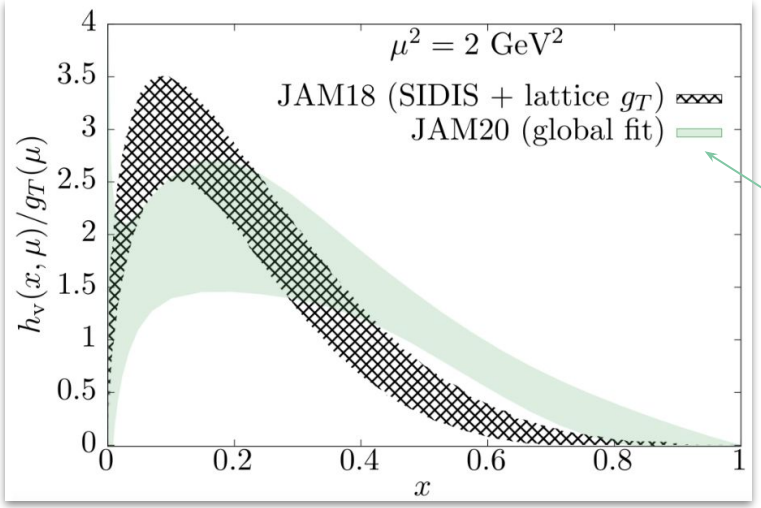
$$g_T^{-1} h_\pm(x) = N_\pm x^{\alpha_\pm} (1-x)^{\beta_\pm} \left(1 + \gamma_\pm \sqrt{x} + \delta_\pm x\right)$$

Quark Transversity from Pseudo-distributions

Distribution of transversely polarized quarks within transversely polarized hadron



➢ only chiral-odd twist-2 collinear PDF
- need to couple to chiral-odd process
[eg. transverse SSAs in SIDIS]



[1st such global analysis]
Transverse SSAs in pion production via proton/deuteron targets

H.-W. Lin, et al., PRL 120 152502 (2018)

SIDIS + transverse SSAs via SIA (e+e-) & pp-collisions

J. Cammarota et al., PRD 102 054002 (2020)

Matrix element defining transversity:

$$M^{\alpha\beta}(p,z) = \langle h(p)|\,\overline{\psi}(z)\, i\sigma^{\alpha\beta}\gamma^5 \Phi^{(f)}_{\hat{z}}(\{z,0\})\,\psi(0)\,|h(p)\rangle$$
$$= 2(p^\alpha S_\perp^\beta - p^\beta S_\perp^\alpha)\underline{\mathcal{M}(\nu,z^2)} + 2im_N^2(z^\alpha S_\perp^\beta - z^\beta S_\perp^\alpha)\mathcal{N}(\nu,z^2) + 2m_N^2(z^\alpha p^\beta - z^\beta p^\alpha)(z\cdot S_\perp)\,\mathcal{R}(\nu,z^2)$$

NLO coefficients match to transversity, but normalization left to future work

V.M. Braun, Y.Ji and A. Vladimirov et al., JHEP 10, 087 (2021)
CE, J. Karpie, N. Karthik et al., PRD 105 (2022) 3, 034507

Suitable choice of kinematics isolates amplitude sensitive to leading-twist PDF

$$\langle N(p_z,\mathbf{S}_\perp)|\,\overline{\psi}\gamma_5\gamma_4\gamma_T\Phi(\{z,0\})\,\tfrac{\tau^3}{2}\psi(0)\,|N(p_z,\mathbf{S}_\perp)\rangle = 2E\mathbf{S}_\perp\mathcal{M}(\nu,z^2)$$

$$\frac{\langle x^0\rangle}{g_T}(\mu) = \int_0^1 dx\,\frac{h(x,\mu)}{g_T(\mu)} = 1$$

Rotational symmetry - improved stats.

Matching Kernels to Quark PDFs

$$\mathfrak{M}\left(\nu, z^2\right) = \left\{\delta\left(1-u\right) - \frac{\alpha_s C_F}{2\pi}\int_0^1 du\left[\ln\left(\frac{e^{2\gamma_E+1}z^2\mu^2}{4}\right)B\left(u\right) + L\left(u\right)\right]\right\}\mathcal{Q}\left(u\nu, \mu^2\right) + \mathcal{O}\left(z^2\Lambda_{\mathrm{QCD}}^2\right)$$

Unpolarized: $B\left(u\right) = \left[\frac{1+u^2}{1-u}\right]_+$ $\quad L\left(u\right) = \left[4\frac{\ln(1-u)}{1-u} - 2\left(1-u\right)\right]_+$

Helicity: $B\left(u\right) = \left[\frac{1+u^2}{1-u}\right]_+$ $\quad L\left(u\right) = \left[4\frac{\ln(1-u)}{1-u} - 4\left(1-u\right)\right]_+$

Transversity: $B\left(u\right) = \left[\frac{2u}{1-u}\right]_+$ $\quad L\left(u\right) = 4\left[\frac{\ln(1-u)}{1-u}\right]_+$

Matching Kernels to Quark GPDs

$$\mathcal{G}_{q/h}^{[\gamma^+]} = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \langle h(p_f)|\,\overline{q}\left(-\tfrac{z}{2}\right)\gamma^+\Phi_{\hat{z}^-}^{(f)}\left(\{-\tfrac{z}{2},\tfrac{z}{2}\}\right) q\left(\tfrac{z}{2}\right)|h(p_i)\rangle\,|_{z^+=0,\mathbf{z}_T=\mathbf{0}}$$

$$\mathbb{M}^\mu(p_f,p_i,z) \equiv \langle N(p_f)|\,\overline{\psi}(-z/2)\,\frac{\tau^3}{2}\gamma^\mu W(-z/2,z/2;A)\,\psi(z/2)\,|N(p_i)\rangle$$

$$\mathbb{M}^\mu(p_f,p_i,z) = \langle\langle\gamma^\mu\rangle\rangle M\left(\nu_f,\nu_i,t;z^2\right) + \langle\langle 1\rangle\rangle z^\mu N\left(\nu_f,\nu_i,t;z^2\right) - \frac{i}{2m_N}\langle\langle\sigma^{\mu\nu}\rangle\rangle\left(p_i-p_f\right)_\nu L\left(\nu_f,\nu_i,t;z^2\right)$$

$$\widetilde{\mathfrak{M}}\left(\nu,\xi,t,z^2\right) = \widetilde{\mathcal{I}}\left(\nu,\xi,t,\mu^2\right) - \frac{\alpha_s C_F}{2\pi}\int_0^1 \mathrm{d}u\,\widetilde{\mathcal{I}}\left(u\nu,\xi,t,\mu^2\right)\left\{\ln\left[\frac{e^{2\gamma_E+1}}{4}z^2\mu^2\right]B_G\left(u,\overline{u},\xi,\nu\right) + L_G\left(u,\overline{u},\xi,\nu\right)\right\} + \mathcal{O}\left(z^2\Lambda_{\mathrm{QCD}}^2\right)$$

$$\left[\frac{2u}{1-u}\right]_+\cos\left(\overline{u}\xi\nu\right) + \frac{\sin\left(\overline{u}\xi\nu\right)}{\xi\nu} - \frac{1}{2}\delta\left(\overline{u}\right)$$

$$4\left[\frac{\ln(1-u)}{1-u}\right]_+\cos\left(\overline{u}\xi\nu\right) - 2\frac{\sin\left(\overline{u}\xi\nu\right)}{\xi\nu} + \delta\left(\overline{u}\right)$$