The momentum sum rule via the Feynman Hellmann Theorem

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Motivation

- Long-standing question re: nucleon momentum:

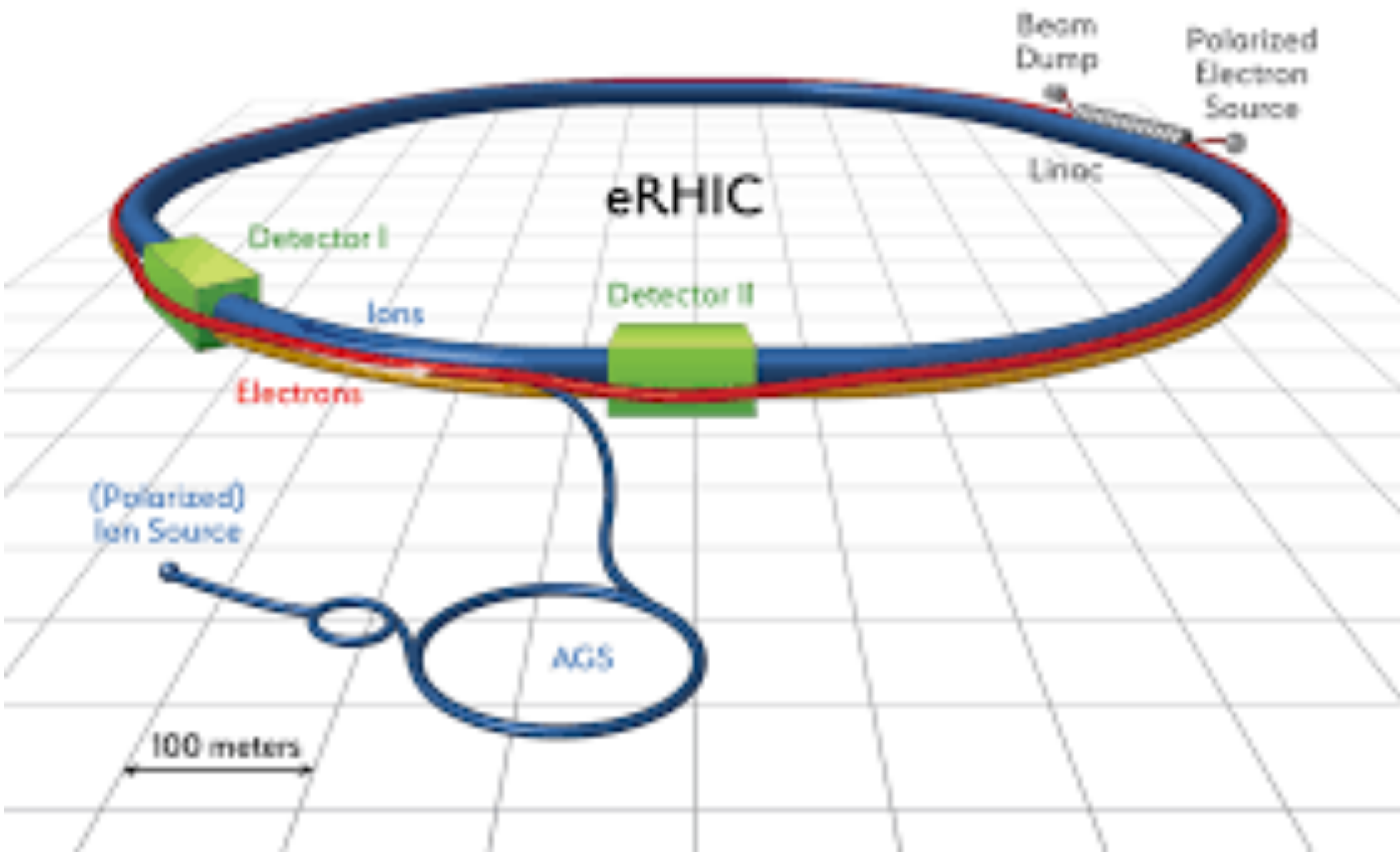
  How is the nucleon's momentum distributed amongst its constituents?

- Addressed experimentally @ JLab (now), EIC (future)
- Must satisfy the momentum rule

$$\sum_q \langle x \rangle_q + \langle x \rangle_g = 1$$

  where

  $\langle x \rangle_f =$ fraction of nucleon momentum carried by parton $f=q,g$

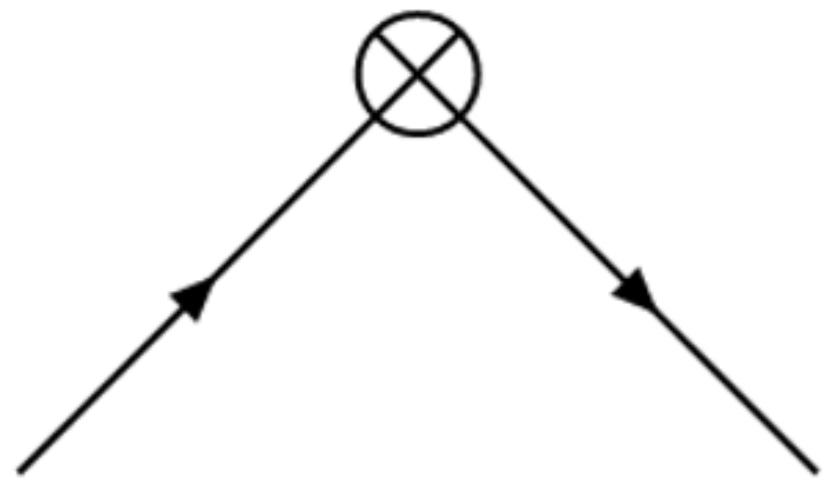


- Experimentally : $\langle x \rangle_g \sim \frac{1}{2}$
- Received much interest from Lattice QCD, but with challenges,
  - e.g. statistical noise in $\langle x \rangle_g$ due disconnected nature

Motivation

$Z_g = Z_{qg} + Z_{gg}$

Renormalisation: Mixing between $\langle x \rangle_q$ and $\langle x \rangle_g$

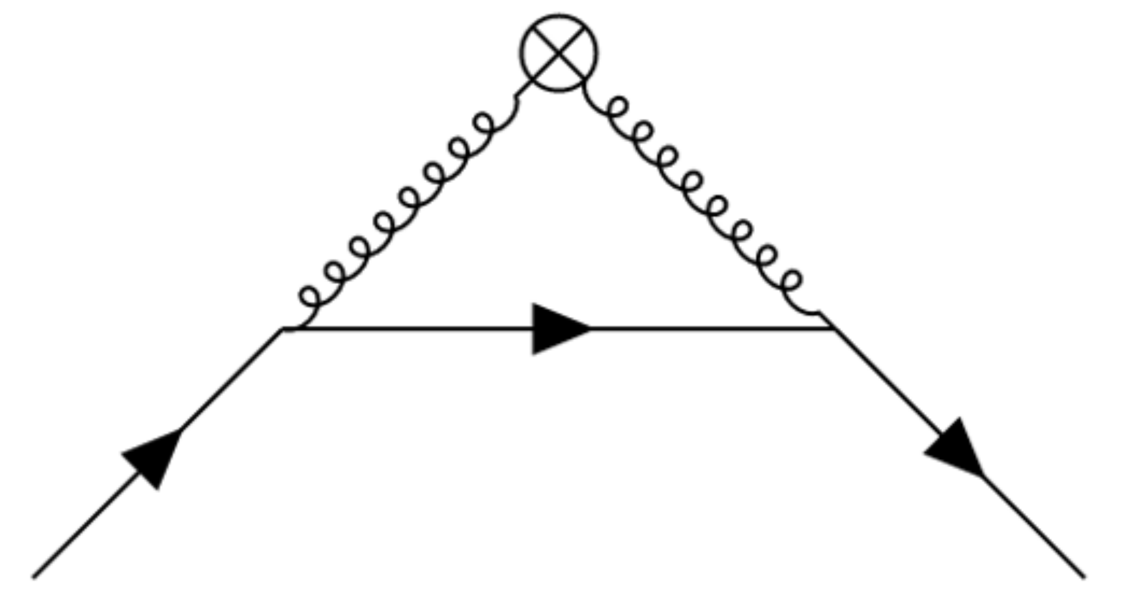


i.e. $$\sum_q \langle x \rangle_q^R + \langle x \rangle_g^R \;=\; 1 \;=\; Z_q \sum_q \langle x \rangle_q^{lat} + Z_g \langle x \rangle_g^{lat}$$

does not necessarily mean

$$\langle x \rangle_q^R = Z_q \langle x \rangle_q^{lat} \quad \text{or} \quad \langle x \rangle_g^R = Z_g \langle x \rangle_g^{lat}$$

e.g. $$\boxed{\begin{pmatrix} \langle x \rangle_g \\ \langle x \rangle_q \end{pmatrix}^R = \begin{pmatrix} Z_{gg} & Z_{gq} \\ Z_{qg} & Z_{qq} \end{pmatrix} \begin{pmatrix} \langle x \rangle_g \\ \langle x \rangle_q \end{pmatrix}^{lat}}$$



$$\Rightarrow \quad Z_g = Z_{gg} + Z_{qg} \qquad\qquad Z_q = Z_{gq} + Z_{qq}$$

Recent progress in NP determination of Z_{gg}

Mixing due to Z_{qg}, Z_{qg} often ignored or computed perturbatively

Determining $\langle x \rangle_{q,g}$

Require matrix elements

$$\langle N(\vec{p})\,|\,\mathcal{O}_f^{(b)}\,|\,N(\vec{p})\rangle = 2(m_N^2 + \frac{4}{3}\vec{p}^{\,2})\langle x \rangle_f$$

$$\mathcal{O}^{(b)} = \mathcal{O}_{44} - \frac{1}{3}\mathcal{O}_{ii}$$

which can be computed at $\vec{p} = 0$ (for $\mathcal{O}^{(b)}$)

$$\mathcal{O}_{\mu\nu}^{(g)} = -\,\mathrm{Tr}_c F_{\mu\alpha}F_{\nu\alpha}\ , \quad \mathcal{O}_g^{(b)} = \frac{2}{3}\mathrm{Tr}_c(-\vec{E}^{\,2} + \vec{B}^{\,2})$$

$$\mathcal{O}_{\mu\nu}^{(q)} = \bar{q}\gamma_\mu \overleftrightarrow{D}_\nu q\ , \quad \mathcal{O}_q^{(b)} = \bar{q}\gamma_4 \overleftrightarrow{D}_4 q - \frac{1}{3}\bar{q}\gamma_i \overleftrightarrow{D}_i q$$

Typically obtained via 3-point functions

$$\mathcal{O}(\tau) = \int d^3x\, \mathcal{O}(\tau, \vec{x})$$

This work: Feynman-Hellmann theorem *[following QCDSF(2012)]*

Compute 2-point functions in the presence of a modification to the action $S \to S(\lambda) = S + \lambda \sum_z \mathcal{O}(z)$

Matrix elements determined from energy shifts $\left.\frac{\partial E_\lambda}{\partial \lambda}\right|_{\lambda=0} = \frac{1}{2E}\left\langle N \left| : \frac{\partial S_\lambda}{\partial \lambda} : \right| N \right\rangle\Big|_{\lambda=0}$

The modified action

Wilson gluonic action: $S_g = \frac{1}{3}\beta \sum_{x\,\mu<\nu} \mathrm{Re\,Tr}_c\left[1 - U^{\square}_{\mu\nu}\right] = \sum_{\tau} \frac{1}{2}\left[\mathcal{E}^{a\,2}(\tau) + \mathcal{B}^{a\,2}(\tau)\right]$

Modify with gluon operator $\mathcal{O}_g^{(b)}$:

$$S_g(\lambda_g) = \frac{1}{3}\beta(1+\lambda_g)\sum_{x,i} \mathrm{Re\,Tr}_c(1 - U^{\square}_{i4}(x)) + \frac{1}{3}\beta(1-\lambda_g)\sum_{x,i<j} \mathrm{Re\,Tr}_c(1 - U^{\square}_{ij}(x))$$

anisotropic action

Similary modify Wilson/Clover action with $\mathcal{O}_q^{(b)}$:

$$S_q^W(\lambda) = \sum_x \bar{q}(x)q(x) - \kappa\Big[\sum_x \bar{q}(x)\big(1-(1+\lambda_q)\gamma_4\big)\, U_4(x)\, q(x+\hat{4}) \;+\; \sum_x \bar{q}(x+\hat{4})\big(1+(1+\lambda_q)\gamma_4\big)\, U_4^\dagger(x)\, q(x) \;+$$

$$\sum_{x,i} \bar{q}(x)\big(1-(1-\frac{1}{3}\lambda_q)\gamma_i\big)\, U_i(x)\, q(x+\hat{i}) \;+\; \sum_{x,i} \bar{q}(x+\hat{i})\big(1+(1-\frac{1}{3}\lambda_q)\gamma_i\big)\, U_i^\dagger(x)\, q(x)\Big]$$

modified hopping term

Simulation details

Quenched QCD ➡ *no disconnected quarks and* $Z_{qg} = 0$

Volume: $24^3 \times 48$

Wilson glue, $\beta = 6.0 \implies a = 0.1\text{fm}$

5 values of λ_g

N_s	N_t	β	λ_g	β_{input}	ξ_{input}	N_{cfg}
24	48	6.0	-0.0666	5.9867	0.9354	1000
24	48	6.0	-0.0333	5.9967	0.9672	1000
24	48	6.0	0	6.0	1	1000
24	48	6.0	$+0.0333$	5.9967	1.0340	1000
24	48	6.0	$+0.0666$	5.9867	1.0689	1000

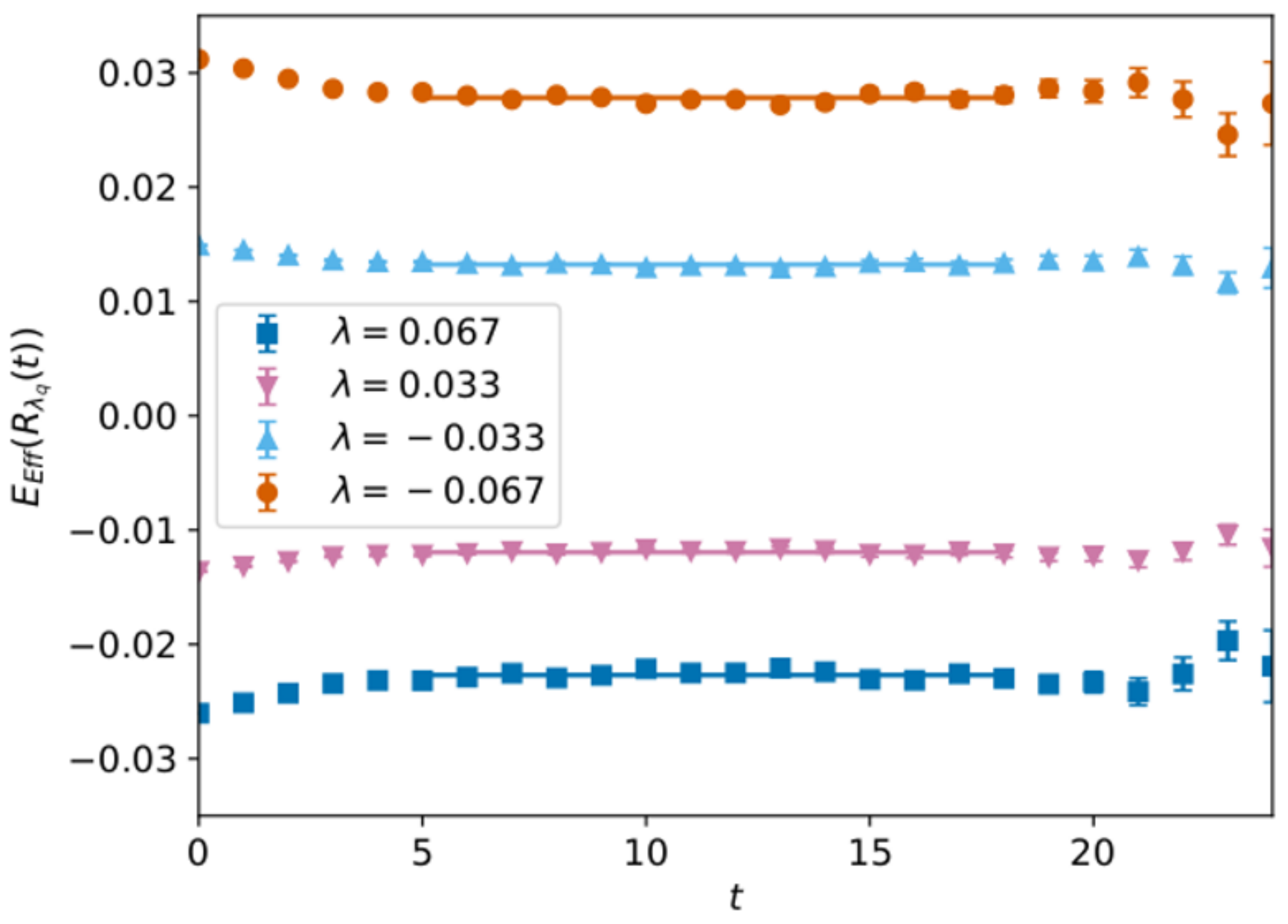
NP-clover action for valence quarks

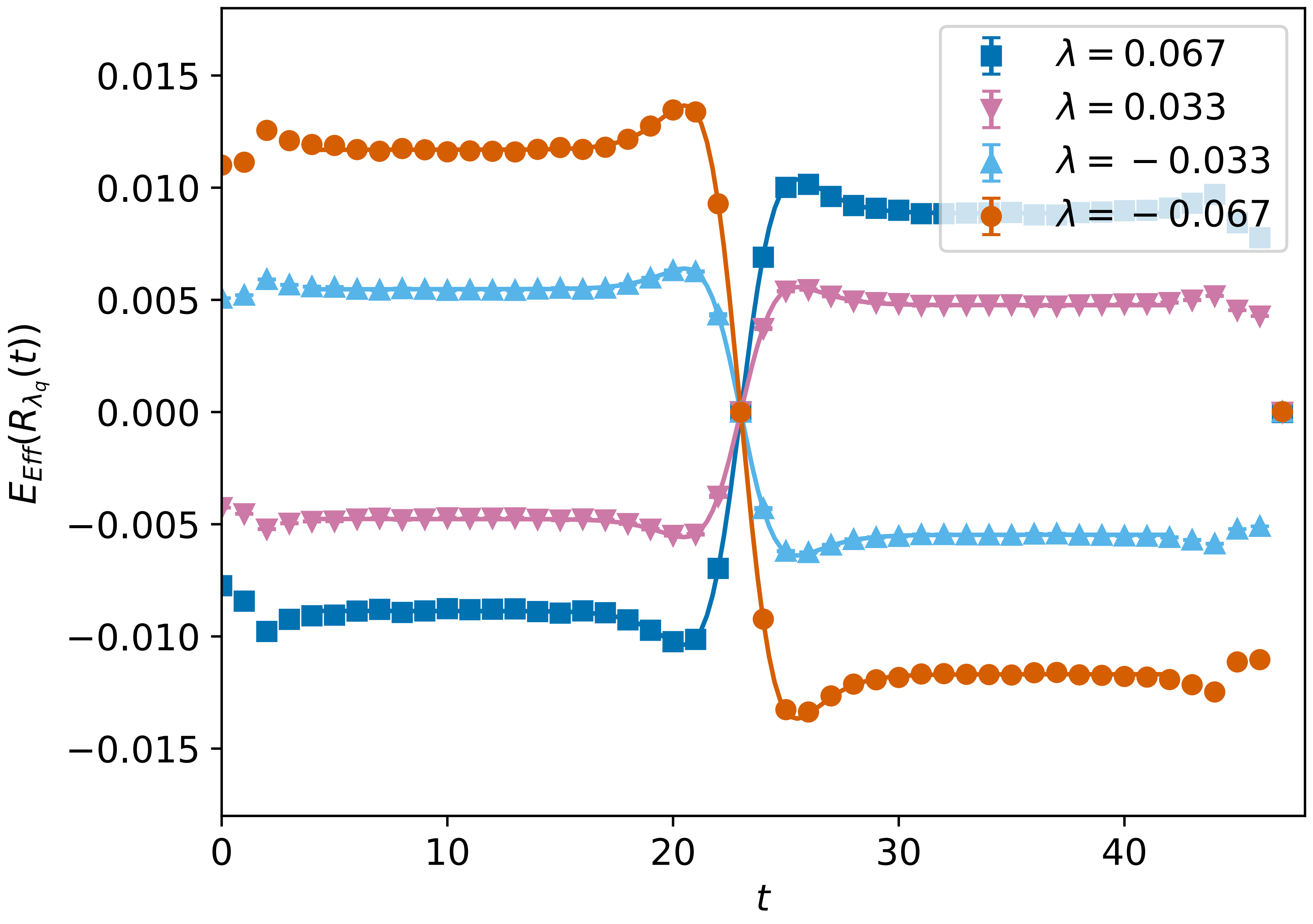
$\kappa = 0.1320, 0.1333, 0.1342$ ➡ $m_\pi \approx 1080, 820, 600\,\text{MeV}$

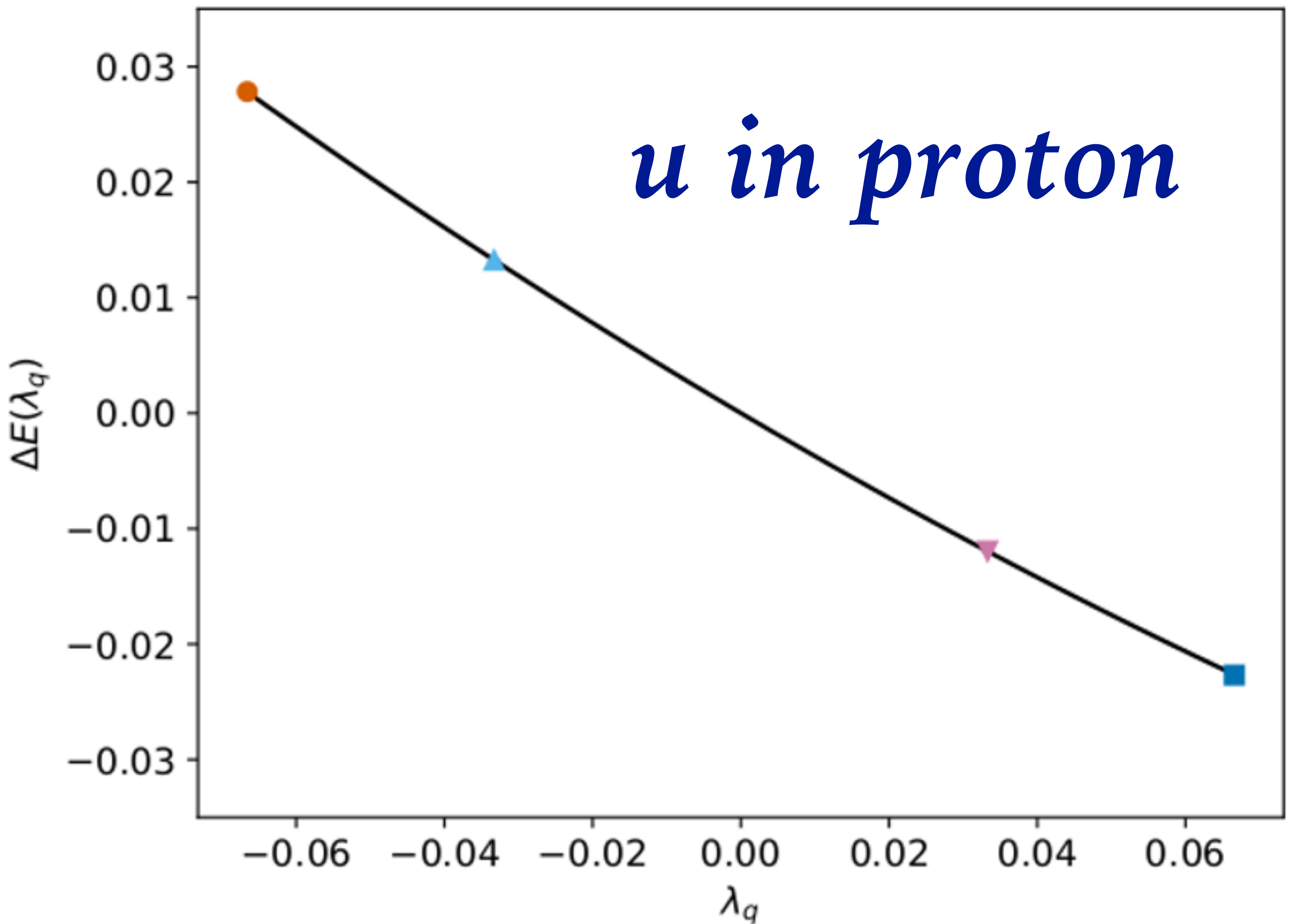
5 values of $\lambda_q = -0.0666,\ -0.0333, 0,\ +0.0333,\ +0.0666$

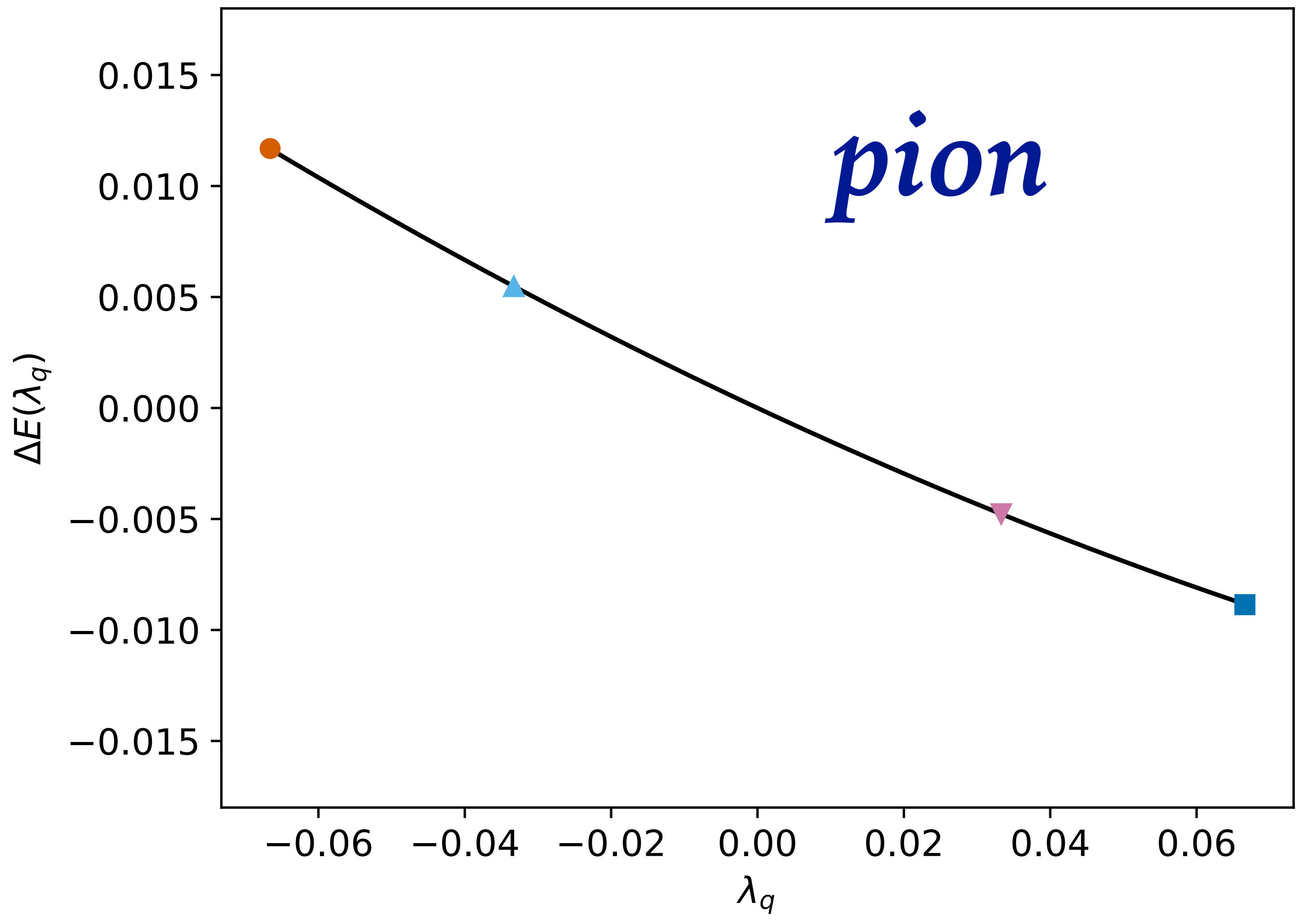
Energy shifts: Quark operator

$m_\pi \approx 1065\,\text{MeV}$



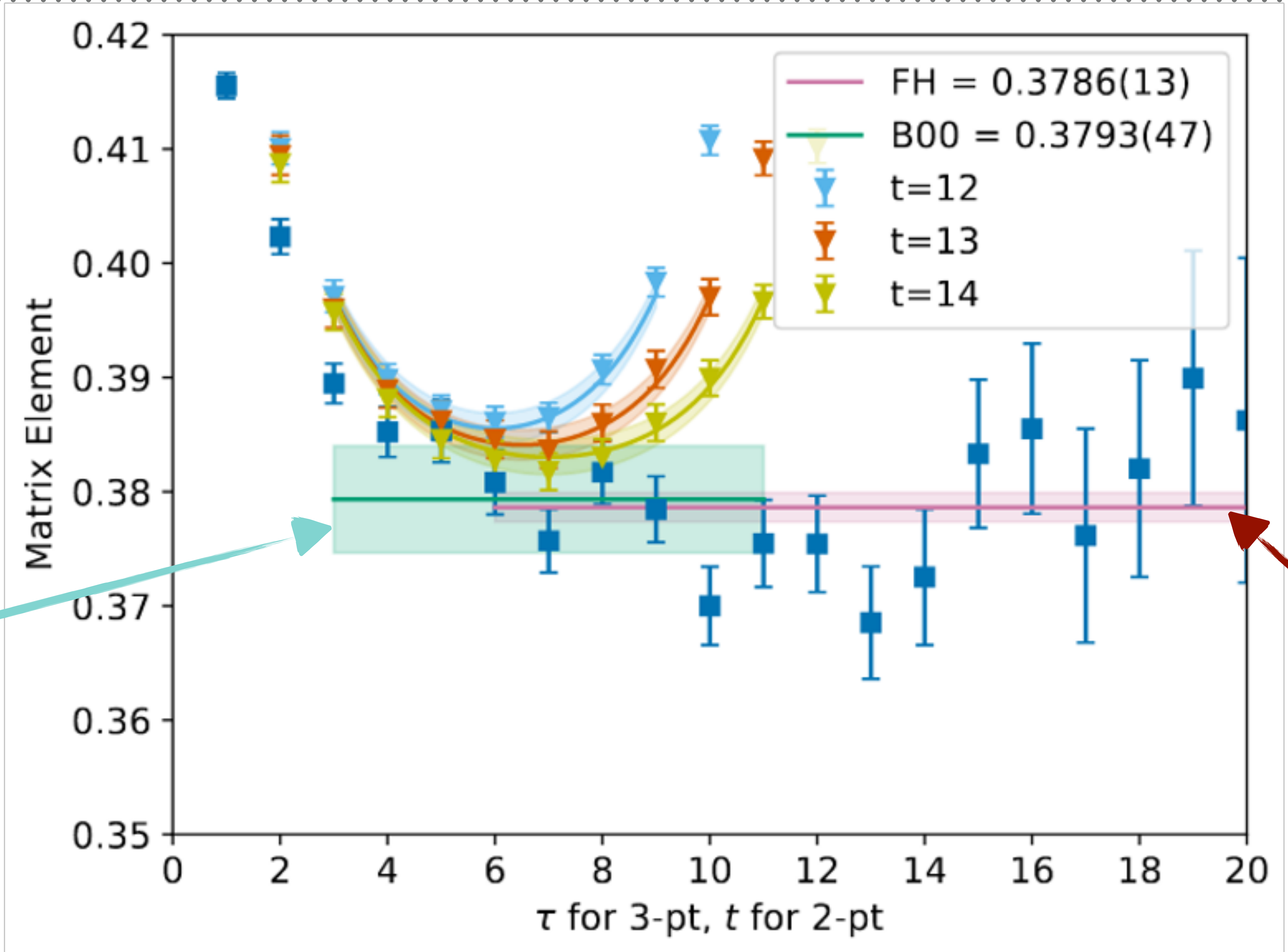






Quark operator - comparison to 3-point functions

$m_\pi \approx 1065\,\mathrm{MeV}$



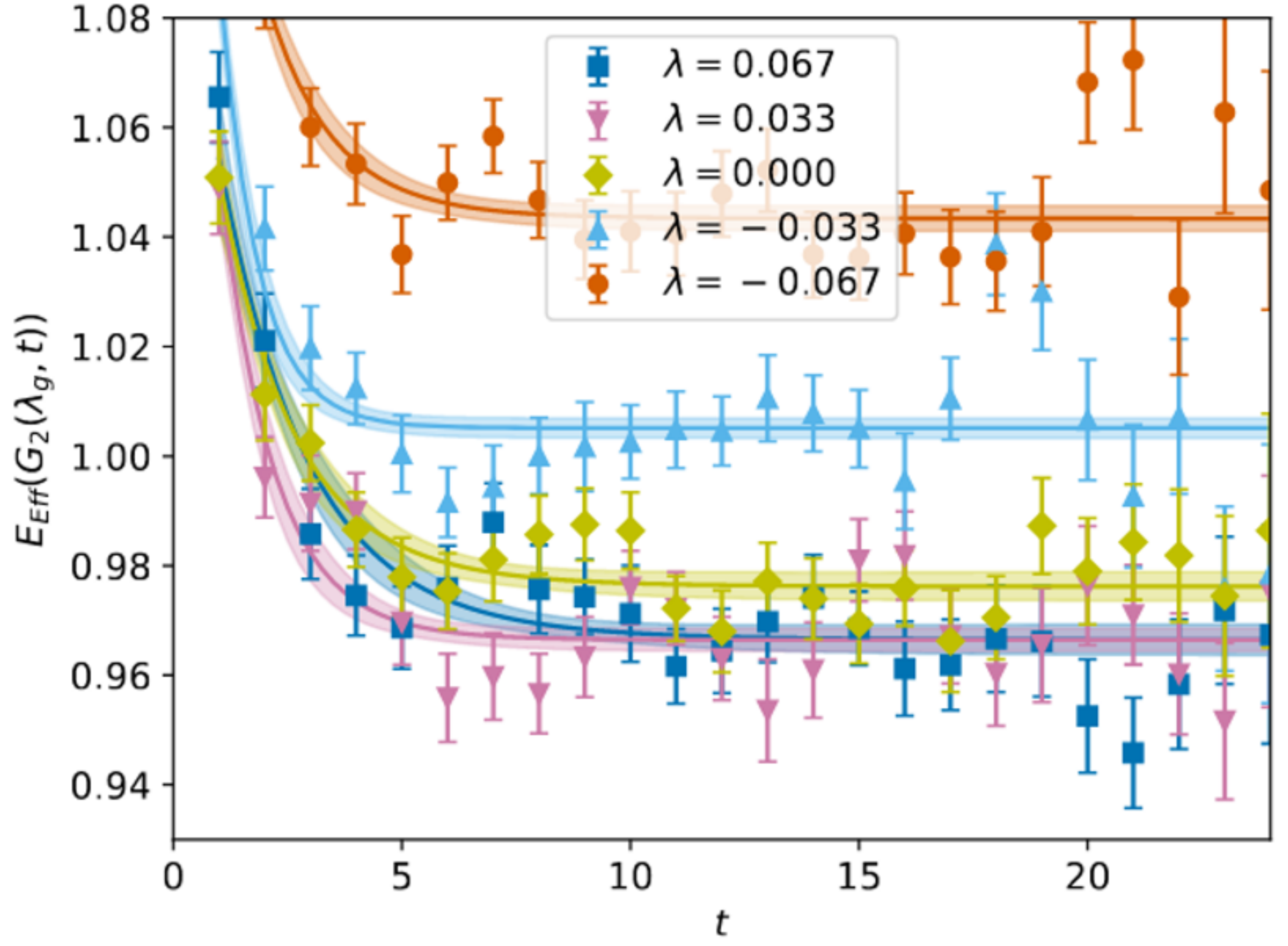
2-state fit

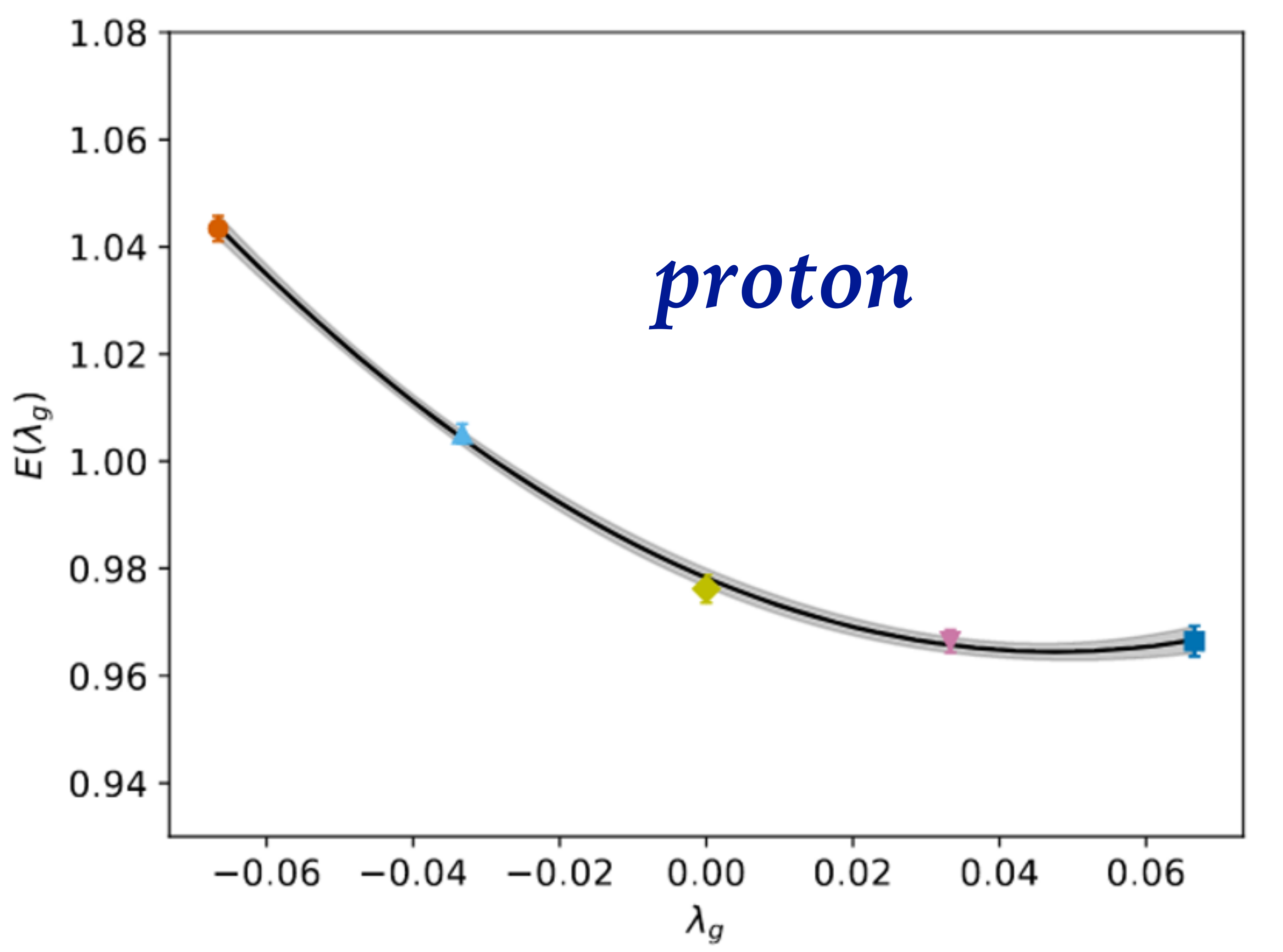
Feynman-Hellmann

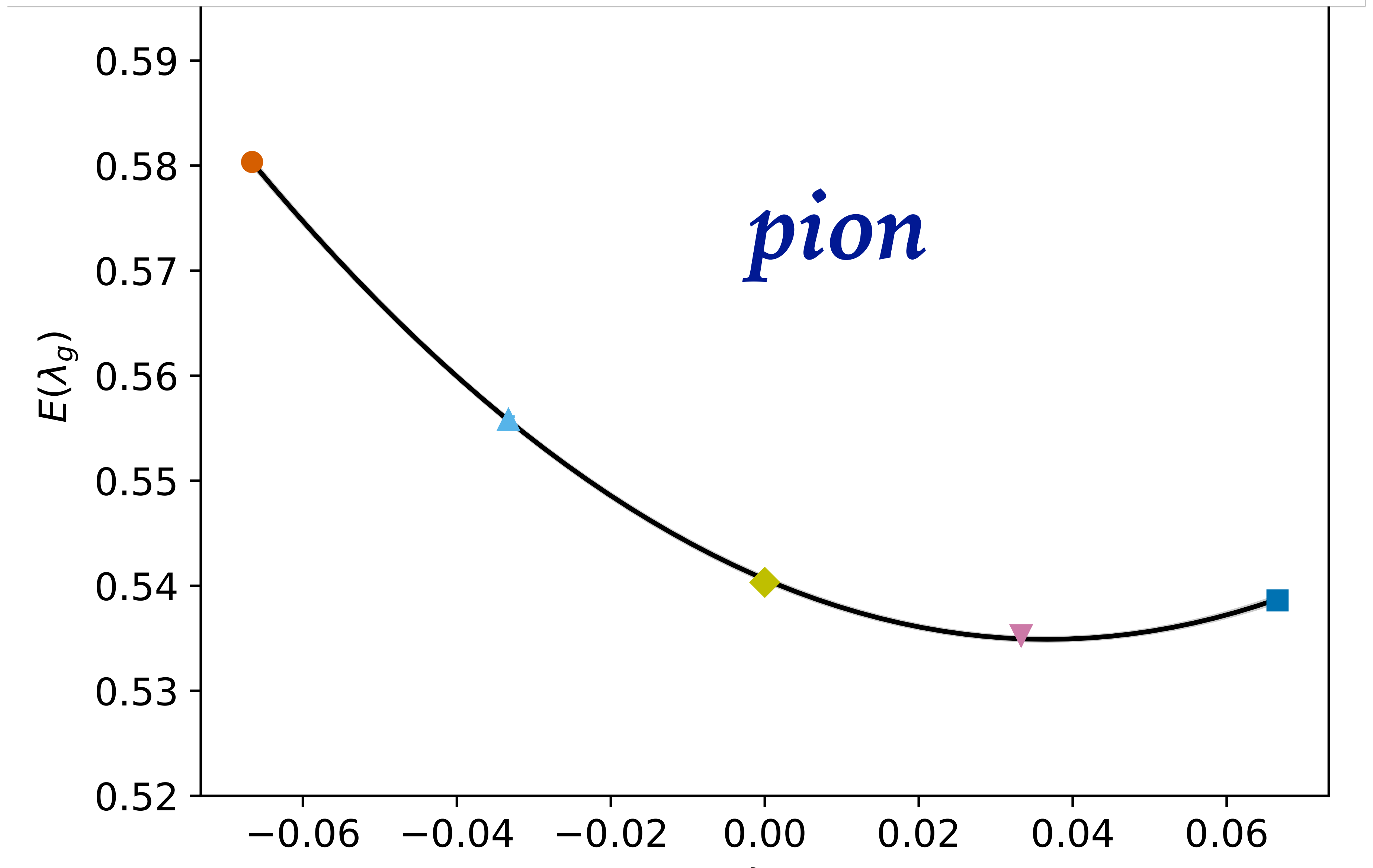
Excellent agreement between Feynman-Hellmann and standard 3-point function methods

Energies: Gluon operator

$m_\pi \approx 1065\,\mathrm{MeV}$



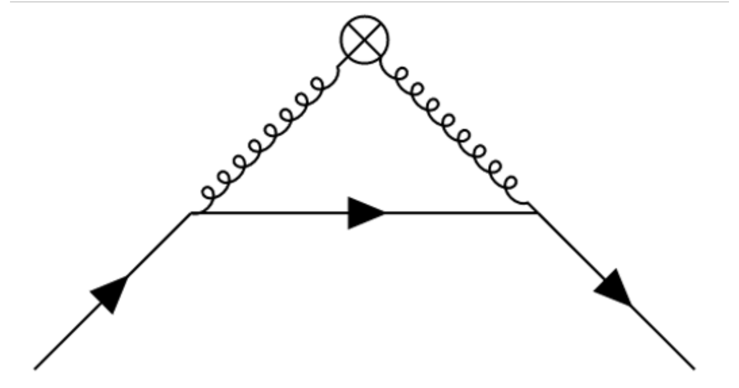




Data points are entirely uncorrelated, from separate ensembles.

Good agreement with quadratic fit, no significant cubic term.

Renormalisation



Recall quark-glue mixing under renormalisation $\begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_q \end{pmatrix}^R = \begin{pmatrix} Z_{gg} & Z_{gq} \\ Z_{qg} & Z_{qq} \end{pmatrix} \begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_q \end{pmatrix}^{lat}$

But $Z_{qg} = 0$ in quenched QCD

$$\begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_u \\ \langle x\rangle_d \end{pmatrix}^R = \begin{pmatrix} Z_{gg} & Z_{gq} & Z_{gq} \\ 0 & Z_{qq} & 0 \\ 0 & 0 & Z_{qq} \end{pmatrix} \begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_u^{con} \\ \langle x\rangle_d^{con} \end{pmatrix}^{lat}$$

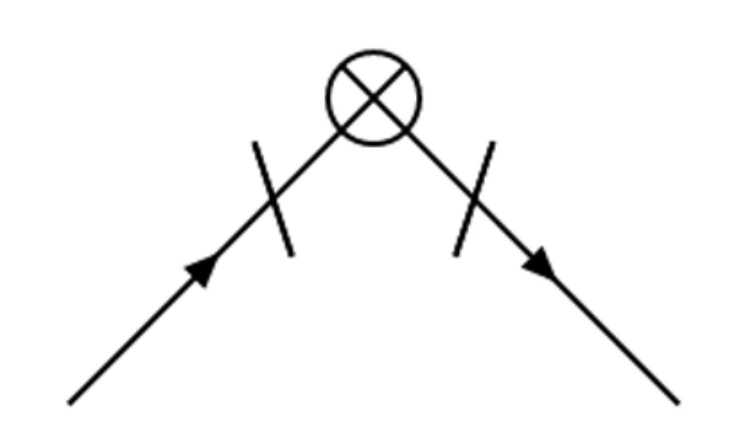
for $n_f = 0$ with $m_u = m_d$

and

$$\left(\langle x\rangle_g + \langle x\rangle_u + \langle x\rangle_d\right)^R = Z_g \langle x\rangle_g^{lat} + Z_q \left(\langle x\rangle_u + \langle x\rangle_u\right)^{lat} = 1$$

with Z_g, Z_q depending only on coupling g and

$$Z_g = Z_{gg} \qquad \text{and} \qquad Z_q = Z_{gq}^{\overline{\text{MS}}} + Z_{qq}^{\overline{\text{MS}}}$$

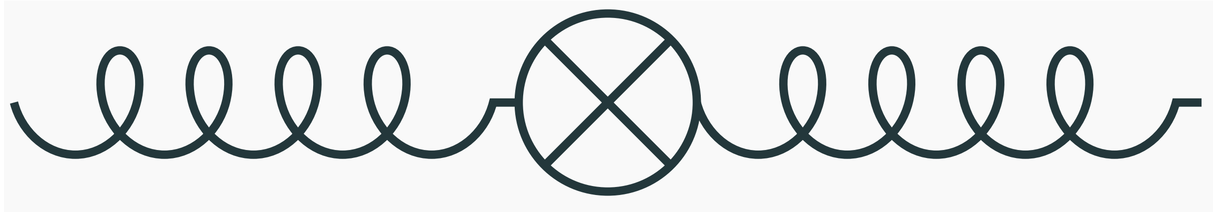
We will employ RI'-MOM, e.g. $\frac{1}{12}\mathrm{Tr}\left(\Gamma^R [\Gamma^{\text{Tree}}]^{-1}\right) = 1, \quad \Gamma^R = Z_{\mathcal{O}} Z_\psi^{-1} \Gamma^{lat}$ and $\Gamma^{lat} =$ 

Renormalisation - FH

Similar to: QCDSF(2015)

Extract 3-point functions from perturbed quark/gluon propagators

Generate propagators on same modified gauge fields as above

Gluon: $\left.\frac{\partial D_{\lambda_g}(p)}{\partial \lambda_g}\right|_{\lambda_g=0} = -\langle A(p)O(0)A(-p)\rangle^{lat} =$ 

with $\langle A(p)O_g(0)A(-p)\rangle^R = Z_A Z_{\mathcal{O}_g} \langle A(p)O_g(0)A(-p)\rangle^{lat}$ $\quad D(p)^R = Z_A D(p)^{lat}$

To avoid mixing with non-physical operators in the EMT [Collins&Scalise(1994),Shanahan&Detmold(2019)]

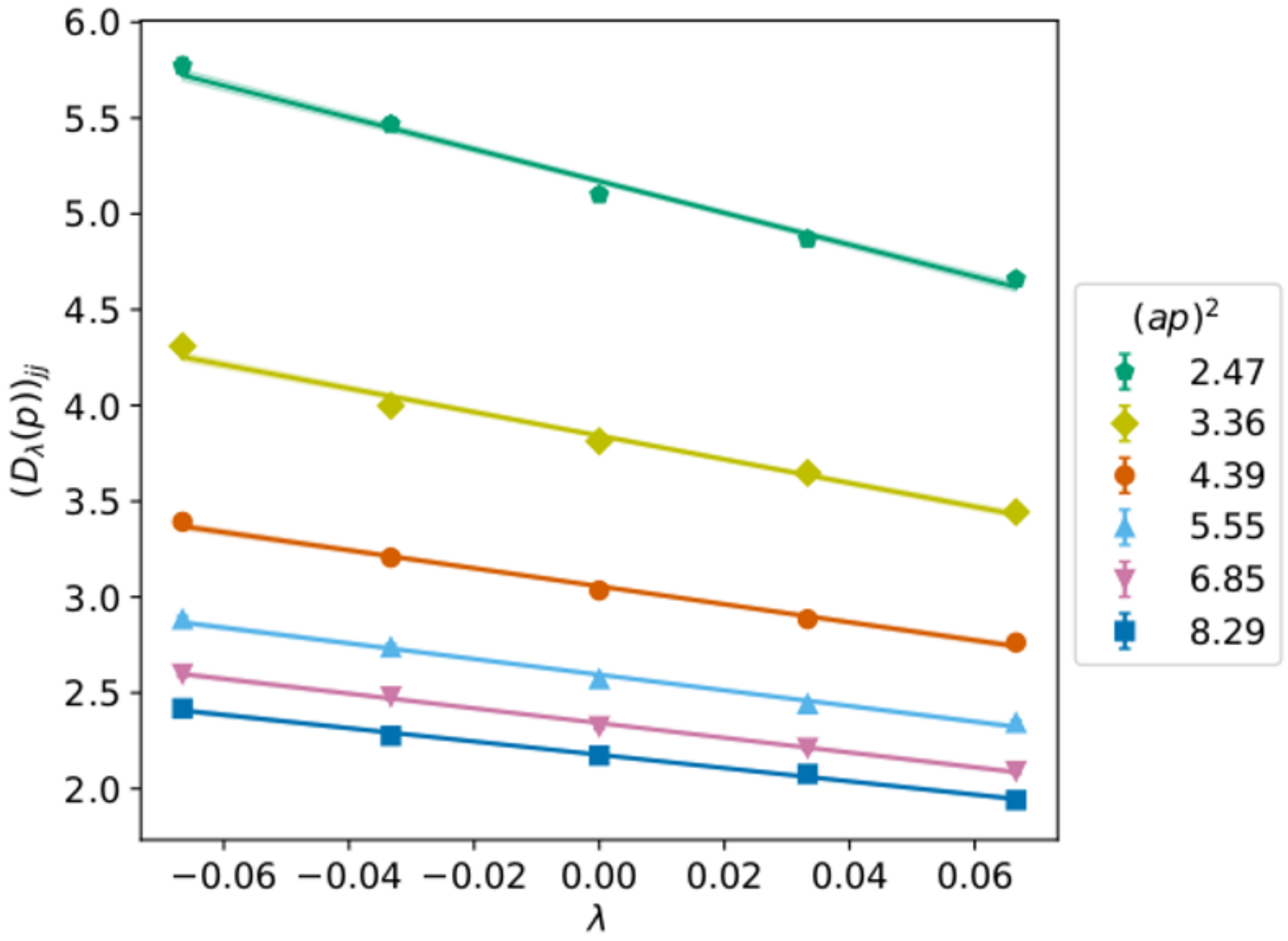
→ take combination $\left\langle A_\rho(p)\bar{T}^g_{44}A_\tau(-p)\right\rangle - \left\langle A_\rho(p')\bar{T}^g_{44}A_\tau(-p')\right\rangle = 2p_4^2$

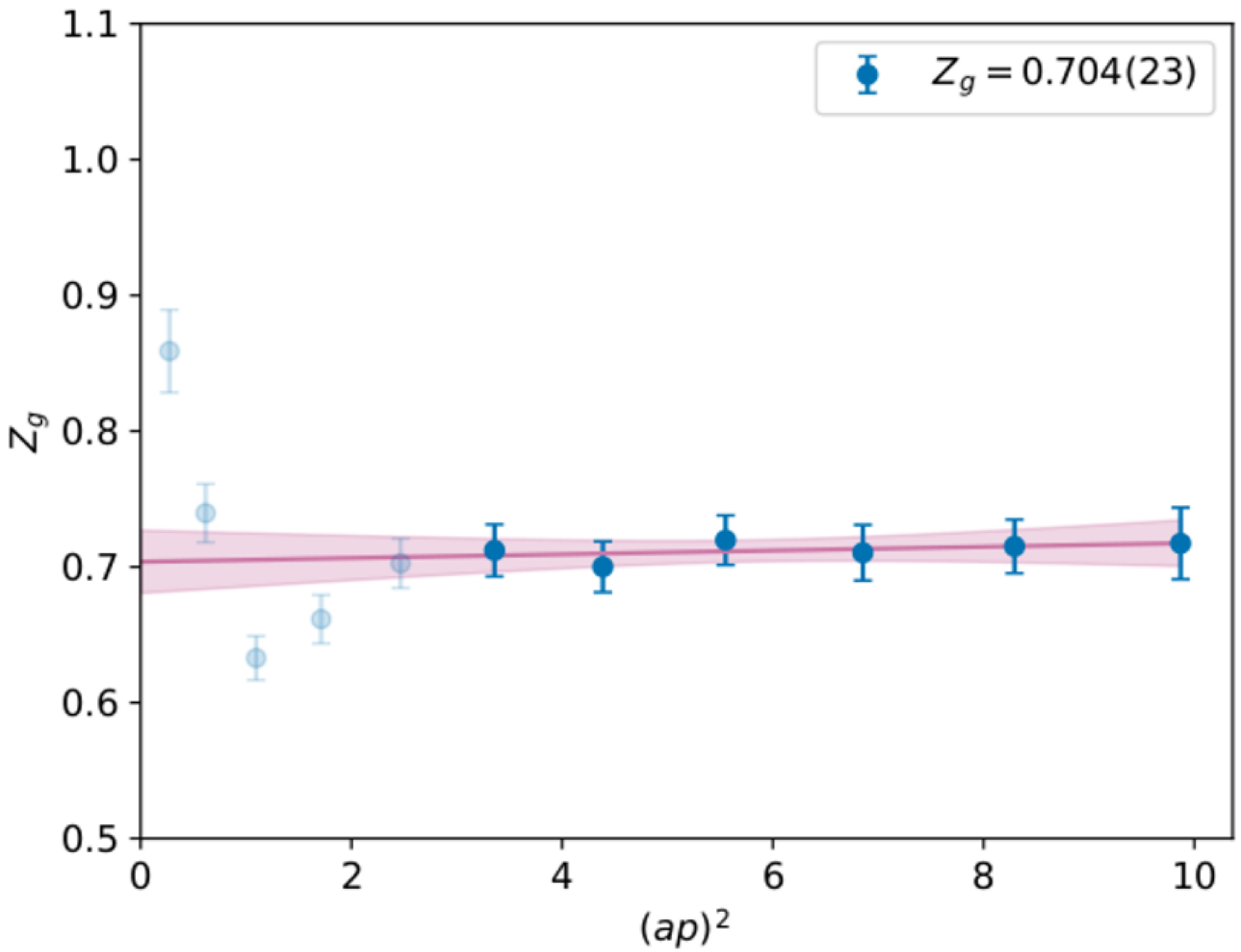
when $\rho \neq 4, p_4 \neq 0, p'_4 = 0, p_\rho = p'_\rho = 0$ and $p^2 = p'^2$

$$\bar{T}^g_{44} = \frac{3}{4}\mathcal{O}_g^{(b)}$$

→ $$Z_g(\mu) = 2p_4^2 p^2 D_0^{lat}(p) \left[\left.\frac{\partial (D^{latt}_{\lambda_g}(p))_{jj}}{\partial \lambda_g}\right|_{\lambda_g=0} - \left.\frac{\partial (D^{latt}_{\lambda_g}(p'))_{jj}}{\partial \lambda_g}\right|_{\lambda_g=0} \right]^{-1} \Bigg|_{\substack{p_j=p'_j=0 \\ p^2=p'^2=\mu^2}}$$

Renormalisation - glue





Data points at different λ_g are entirely uncorrelated, from separate ensembles.

Appear linear in λ_g, quadratic terms small, no significant cubic term.

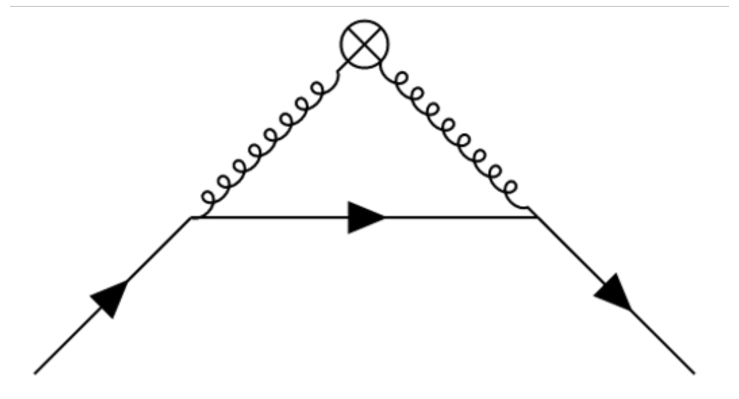
Good signal for Z_g

Renormalisation - quark

Need to account for quark-glue mixing

Z_{qq} can be obtained via usual RI'-MOM (e.g. QCDSF(2005))

To account for mixing, generate quark propagators on same modified gauge fields

$$\left.\frac{\partial S_{\lambda_g}(p)}{\partial \lambda_g}\right|_{\lambda_g=0} = -\langle \bar{q}(p)O(0)q(p)\rangle =$$ 

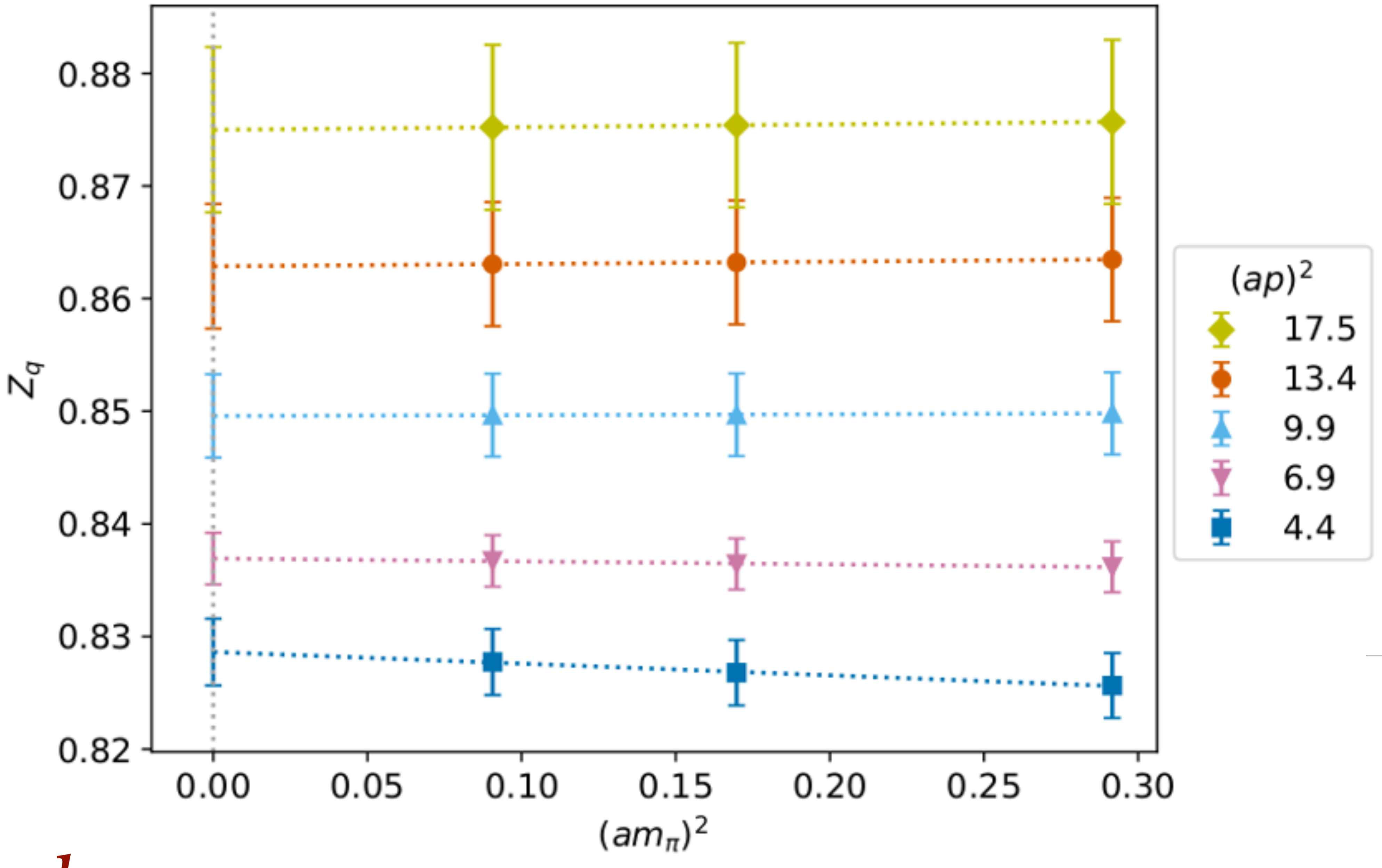
and invoke $\mathcal{O}_q^R + \mathcal{O}_g^R = Z_q \mathcal{O}_q^{lat} + Z_g \mathcal{O}_g^{lat}$ with $Z_q = Z_{qq} + Z_{gq}$ and $Z_g = Z_{qg} + Z_{gg}$ $\quad n_f = 0 : Z_{qg} = 0$

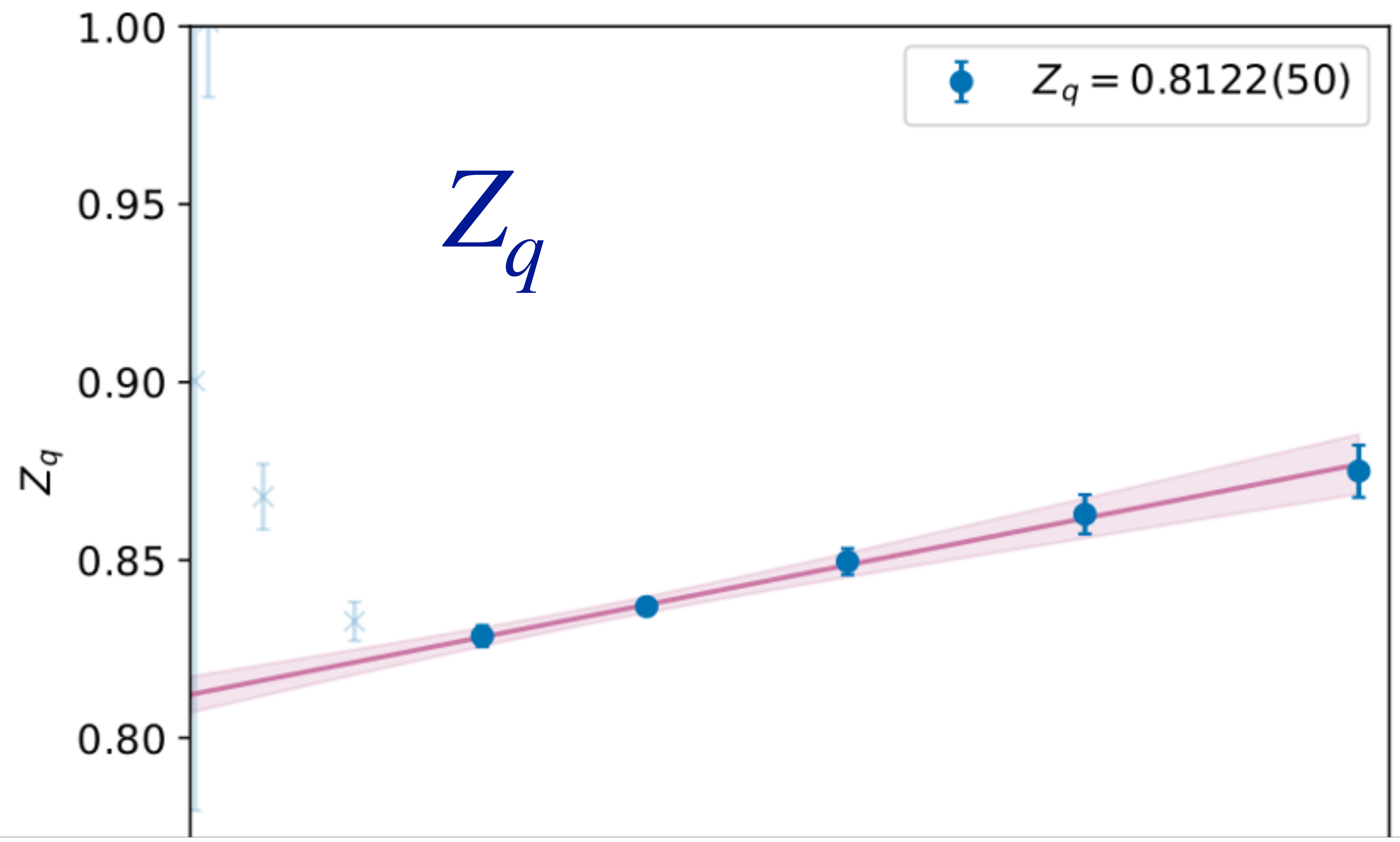
$$Z_q^{-1}(\mu) = \frac{1}{12}\mathrm{Tr}\left\{ \Gamma_{qq}^{lat}(p)\left[Z_\psi(p)\Gamma_{qq}^{\mathrm{Born}}(p) - Z_g(p)\left(\left[S_0^{lat}(p)\right]^{-1} \left.\frac{\partial S_{\lambda_g}(p)}{\partial \lambda_g}\right|_{\lambda_g=0} \left[S_0^{lat}(p)\right]^{-1} \right)\right]^{-1} \right\}\Bigg|_{p^2=\mu^2}$$

use standard quark 3-point methods

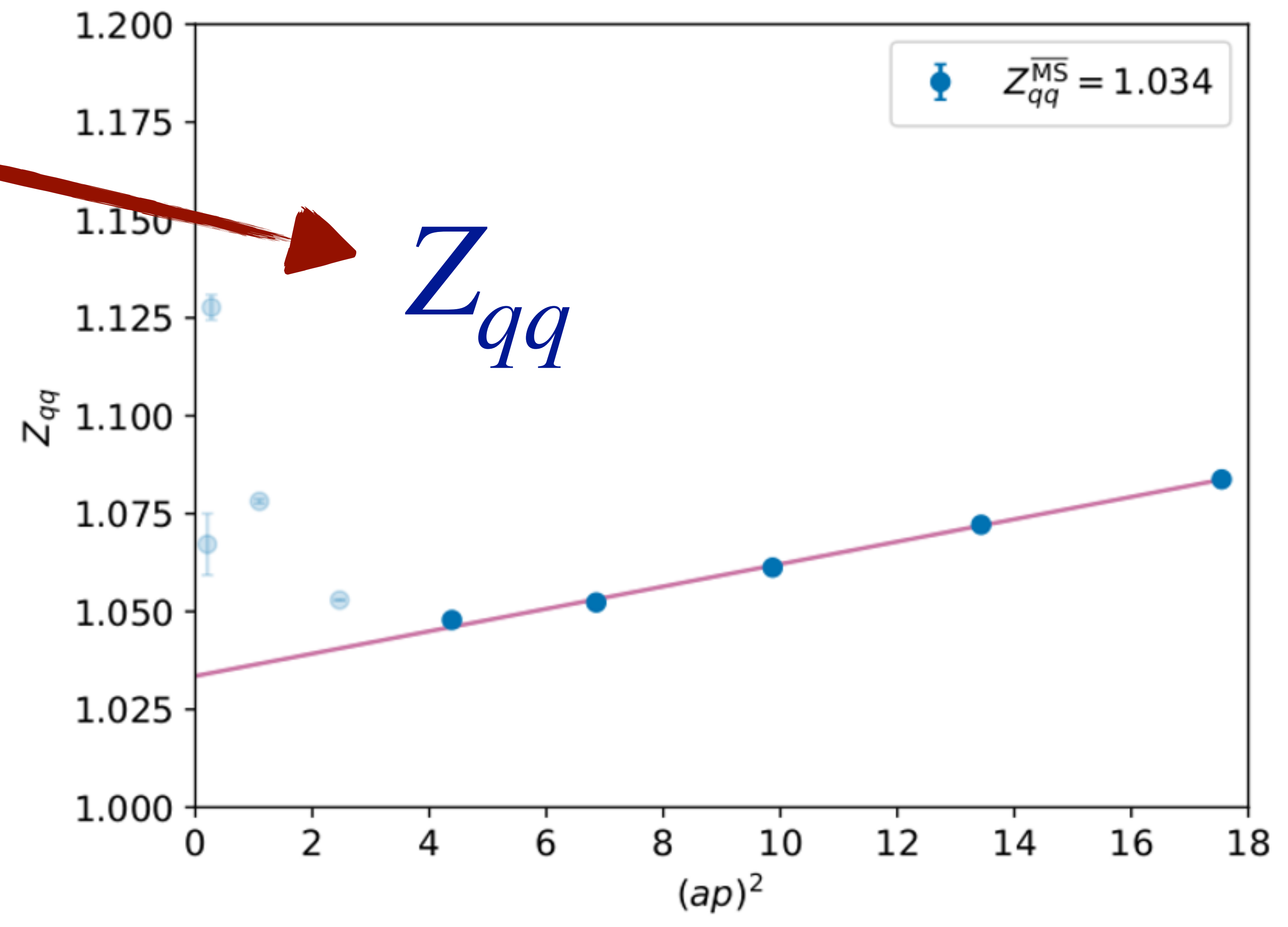
then isolate mixing term $Z_{gq} = Z_q - Z_{qq}$

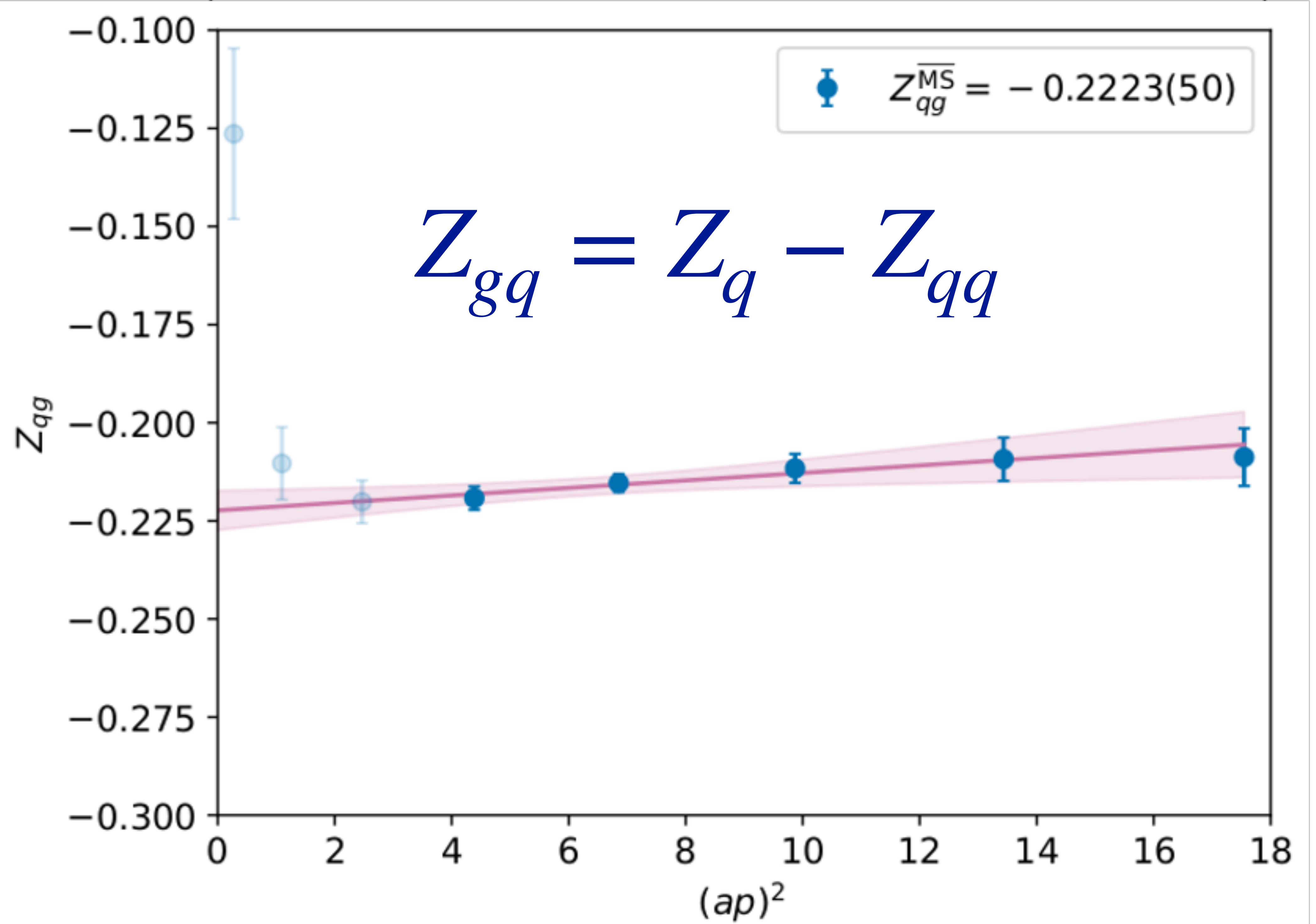
Renormalisation - quark





use standard RI'-MOM with quark 3-point methods





Momentum sum rule

$$\begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_u \\ \langle x\rangle_d \end{pmatrix}^{R} = \begin{pmatrix} Z_{gg} & Z_{gq} & Z_{gq} \\ 0 & Z_{qq} & 0 \\ 0 & 0 & Z_{qq} \end{pmatrix} \begin{pmatrix} \langle x\rangle_g \\ \langle x\rangle_u^{con} \\ \langle x\rangle_d^{con} \end{pmatrix}^{lat}$$

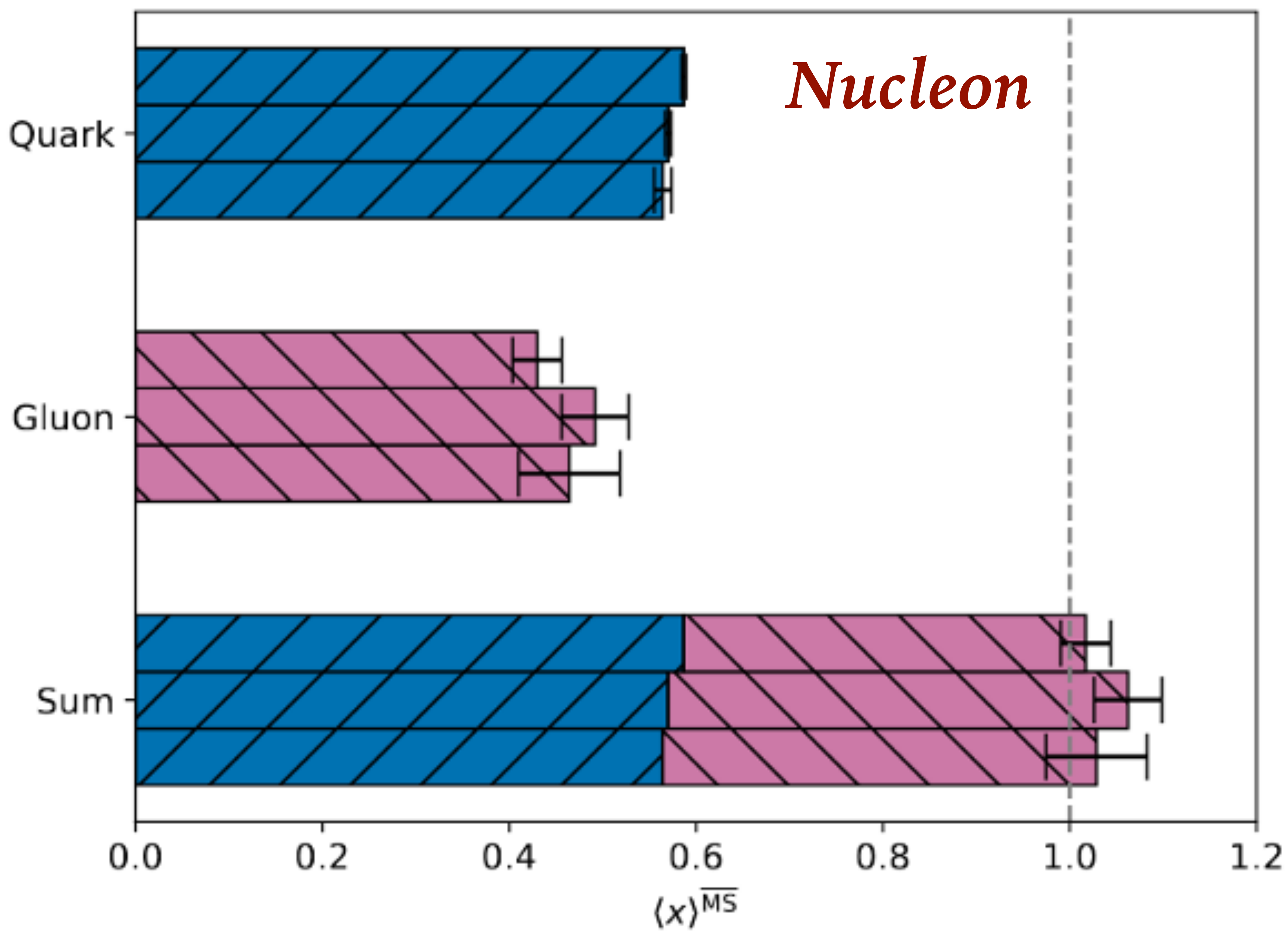
$$Z^{\overline{\mathrm{MS}}} = \begin{pmatrix} 0.704(23) & -0.2223(50) & -0.2223(50) \\ 0 & 1.034(1) & 0 \\ 0 & 0 & 1.034(1) \end{pmatrix}$$

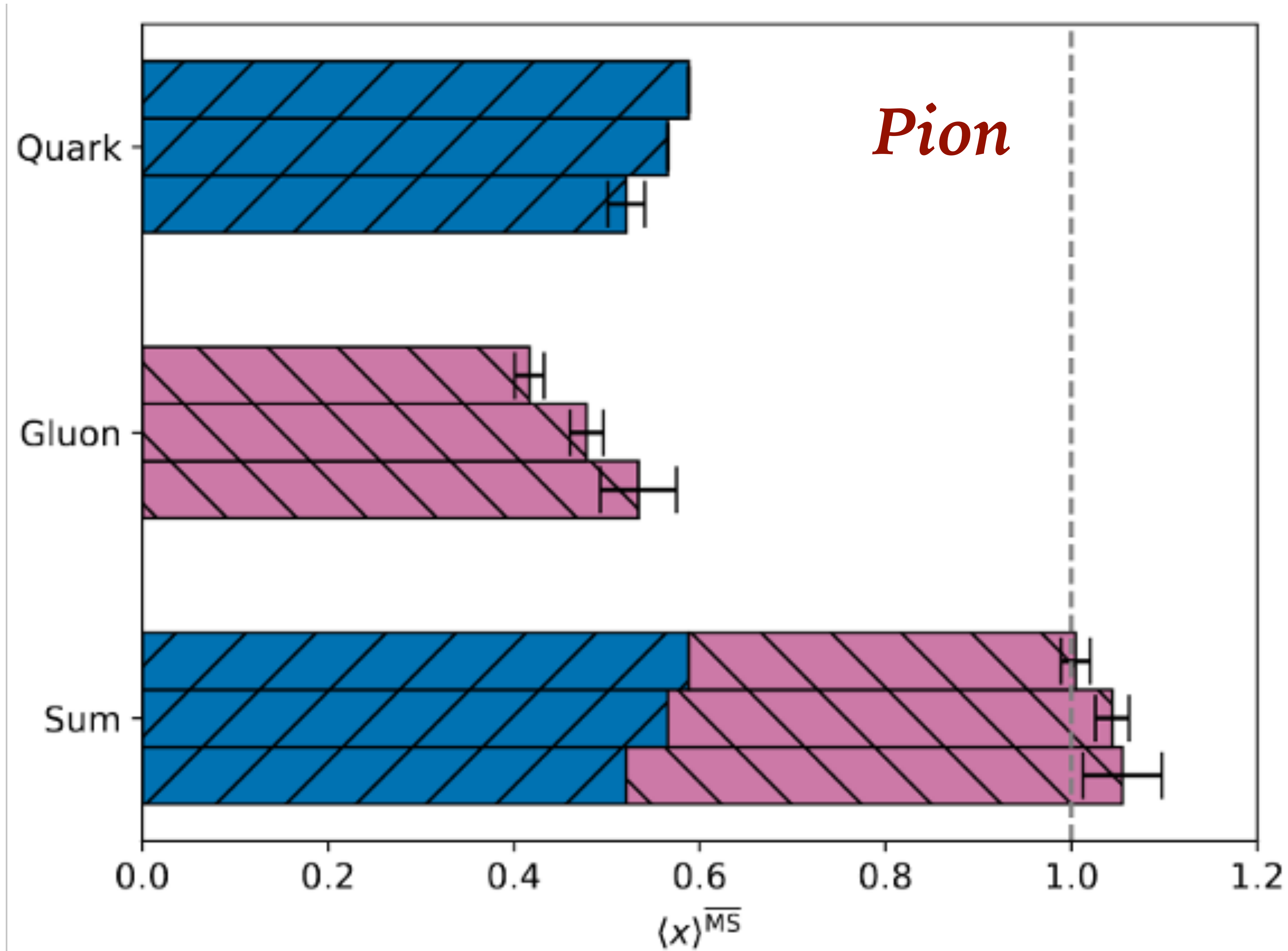
Nucleon

am_π	$\langle x\rangle_q^{\overline{\mathrm{MS}}}$	$\langle x\rangle_g^{\overline{\mathrm{MS}}}$	$\langle x\rangle_q^{\overline{\mathrm{MS}}} + \langle x\rangle_g^{\overline{\mathrm{MS}}}$
0.540	0.5869(23)	0.430(26)	1.018(27)
0.412	0.5703(31)	0.492(36)	1.063(36)
0.300	0.5645(92)	0.464(54)	1.029(54)

Pion

am_π	$\langle x\rangle_q^{\overline{\mathrm{MS}}}$	$\langle x\rangle_g^{\overline{\mathrm{MS}}}$	$\langle x\rangle_q^{\overline{\mathrm{MS}}} + \langle x\rangle_g^{\overline{\mathrm{MS}}}$
0.540	0.58803(58)	0.417(16)	1.005(16)
0.412	0.56569(80)	0.478(18)	1.045(18)
0.300	0.521(20)	0.534(41)	1.056(42)





Summary and outlook

Feynman-Hellmann method provides a viable alternative to 3-point functions for

- matrix elements
- renormalisation factors

In quenched QCD with heavy quark masses reveals for both π and N $\langle x \rangle_q \sim 0.5 - 0.6, \quad \langle x \rangle_g \sim 0.4 - 0.5$

Currently generating dynamical ensembles with:

- $n_f = 2$ NP Clover fermions with $m_\pi \sim 600\,\mathrm{MeV}$
- 3 values each of λ_q and λ_g
- Z matrix more complicated:

$$\begin{pmatrix} \langle x \rangle_g \\ \langle x \rangle_u^{con} \\ \langle x \rangle_d^{con} \\ \langle x \rangle_u^{dis} \\ \langle x \rangle_d^{dis} \end{pmatrix}^R = \begin{pmatrix} Z_{gg} & Z_{gq} & Z_{gq} & Z_{gq} & Z_{gq} \\ 0 & Z_a - Z_b & 0 & 0 & 0 \\ 0 & 0 & Z_a - Z_b & 0 & 0 \\ Z_{qg} & Z_b & Z_b & Z_a & Z_b \\ Z_{qg} & Z_b & Z_b & Z_b & Z_a \end{pmatrix} \begin{pmatrix} \langle x \rangle_g \\ \langle x \rangle_u^{con} \\ \langle x \rangle_d^{con} \\ \langle x \rangle_u^{dis} \\ \langle x \rangle_d^{dis} \end{pmatrix}^{lat}$$

$$Z_{qq}^{NS} = Z_a - Z_b$$

$$Z_{qq}^{S} = Z_{qq}^{NS} + n_f Z_b$$

BACKUP

(slide from T.Howson, Cairns, Australia, 2022)

TROUBLE WITH THE GLUE

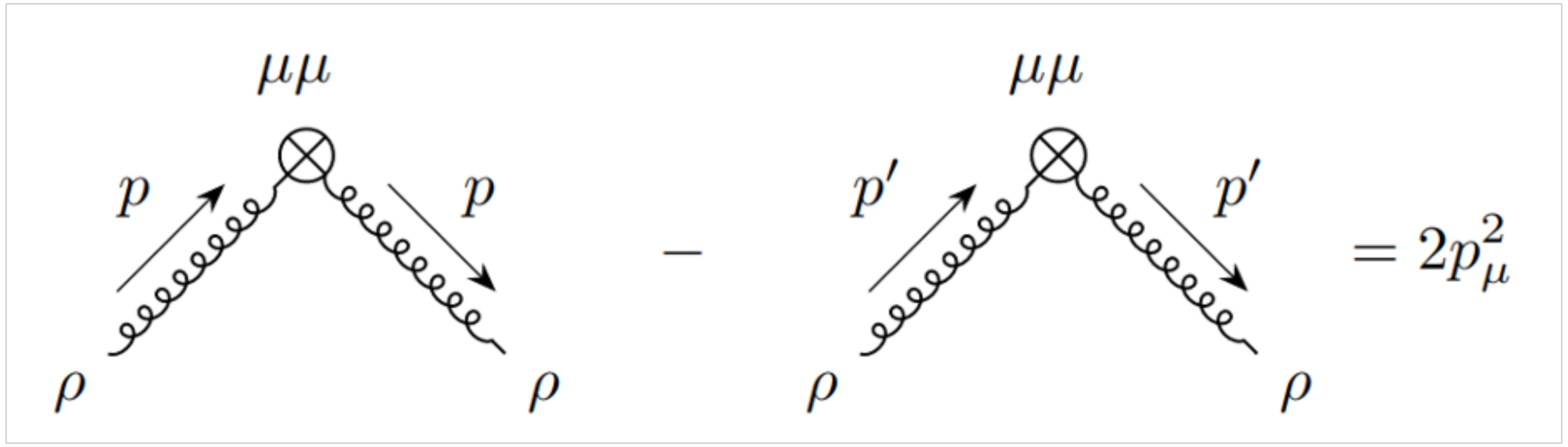
We can write out the gluon 3-point function from the EMT,

$$\langle A_{\sigma'}|\,\overline{T}^{g}_{\mu\nu}\,|A_{\sigma}\rangle = D_{\sigma'\rho}(p)\times$$

Gauge Dependent Terms, Will Mix

$$\Big(2p_\mu p_\nu \delta_{\rho\tau} - p_\mu p_\rho \delta_{\nu\tau} - p_\tau p_\nu \delta_{\rho\mu}) - p_\rho p_\nu \delta_{\mu\tau} + p^2(\delta_{\rho\nu}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\rho\nu}) + \delta_{\mu\nu}(p_\rho p_\tau - p^2\delta_{\rho\tau})\Big) D_{\tau\sigma}(p)$$

Will want to extract the gauge independent term, vanish all other terms.



$$\rho \neq \mu,$$
$$p_\mu \neq 0,$$
$$p'_\mu = 0,$$
$$p'_\rho = p_\rho,$$
$$p^2 = p'^2$$