# Electromagnetic form factors of the proton and neutron from $N_\mathrm{f} = 2 + 1$ lattice QCD

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# Outline

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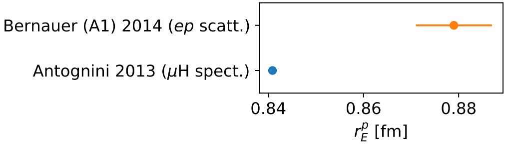
# Motivation

- Internal structure of the nucleon still an open research field in subatomic physics
- In particular, there is a discrepancy between different determinations of the electric charge radius of the proton




- Electromagnetic form factors of the proton and neutron of high interest
- For theory calculations, split up into isovector and isoscalar part
- Whereas the former only contains quark-connected contributions, in the latter also the numerically challenging quark-disconnected contributions appear
- Full calculation of the proton and neutron form factors from first principles necessitates explicit treatment of isoscalar quantities

# Outline

# Ensembles

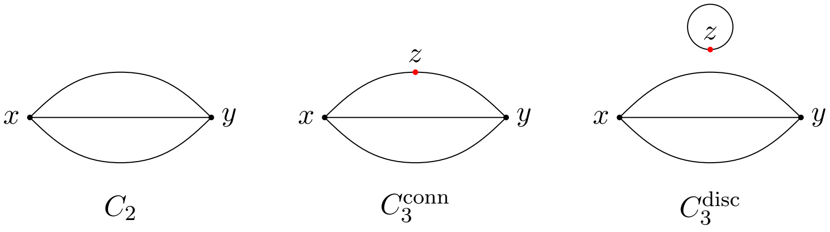
Coordinated Lattice Simulations (CLS)[1]

- Non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions, $N_\mathrm{f} = 2 + 1$
- Tree-level improved Lüscher-Weisz gauge action
- $\mathrm{tr}\, M_q =$ const.

| ID | $\beta$ | $a$ [fm] | $N_\tau$ | $N_s$ | $M_\pi$ [MeV] | $N_{\mathrm{cfg}}^{\mathrm{conn}}$ | $N_{\mathrm{cfg}}^{\mathrm{disc}}$ |
|---|---|---|---|---|---|---|---|
| C101 | 3.40 | 0.08636 | 96 | 48 | 225.0(1.2) | 2000 | 1000 |
| D450 | 3.46 | 0.07634 | 128 | 64 | 216.3(0.7) | 499 | 499 |
| E250 | 3.55 | 0.06426 | 192 | 96 | 129.9(0.9) | 400 | 400 |
| D200 | 3.55 | 0.06426 | 128 | 64 | 202.4(0.8) | 1999 | 999 |
| E300 | 3.70 | 0.04981 | 192 | 96 | 175.0(0.4) | 569 | 569 |
| J303 | 3.70 | 0.04981 | 192 | 64 | 259.6(0.8) | 1073 | 1073 |

[1]Bruno et al. 2015; Bruno, Korzec, and Schaefer 2017.

# Nucleon two- and three-point functions




- Measurement of the two- and three-point functions of the nucleon,

$$C_2(\mathbf{p}'; y_0, x_0) = \sum_{\mathbf{y}} e^{-i\mathbf{p}'\mathbf{y}} \Gamma_{\beta\alpha} \left\langle 0 \,\middle|\, N_\alpha(\mathbf{y}, y_0) \bar{N}_\beta(\mathbf{0}, x_0) \,\middle|\, 0 \right\rangle , \tag{1}$$

$$C_{3,O}^{\text{conn}}(\mathbf{p}', \mathbf{q}; y_0, z_0, x_0) = \sum_{\mathbf{y},\mathbf{z}} e^{i\mathbf{q}\mathbf{z}} e^{-i\mathbf{p}'\mathbf{y}} \Gamma_{\beta\alpha} \left\langle 0 \,\middle|\, N_\alpha(\mathbf{y}, y_0) O(\mathbf{z}, z_0) \bar{N}_\beta(\mathbf{0}, x_0) \,\middle|\, 0 \right\rangle \tag{2}$$

# Nucleon two- and three-point functions

- Construct the disconnected part of the three-point functions from the quark loops and the two-point functions,

$$C_{3,O}^{\mathrm{disc}}(\mathbf{p}', \mathbf{q}; y_0, z_0, x_0) = \left\langle L^{O,\mathrm{disc}}(\mathbf{q}; z_0) C_2(\mathbf{p}'; y_0, x_0) \right\rangle , \tag{3}$$

$$L^{O,\mathrm{disc}}(\mathbf{q}; z_0) = -\sum_{\mathbf{z}} e^{i\mathbf{q}\mathbf{z}} \,\mathrm{tr}[S(z,z)\Gamma] \tag{4}$$

- Compute the quark loops via a stochastic estimation using a frequency-splitting technique[2]
- The nucleon at the sink is at rest, *i.e.*, $\mathbf{p}' = \mathbf{0}$
- Average over the forward and backward propagating nucleon for the disconnected part

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[2] Giusti et al. 2019.

# From ratios to form factors

- Calculate the ratios[3]

$$R_O(\mathbf{q}; t_{\text{sep}}, t) = \frac{C_{3,O}(\mathbf{q}; t_{\text{sep}}, t)}{C_2(\mathbf{0}; t_{\text{sep}})} \sqrt{\frac{C_2(-\mathbf{q}; t_{\text{sep}} - t) C_2(\mathbf{0}; t) C_2(\mathbf{0}; t_{\text{sep}})}{C_2(\mathbf{0}; t_{\text{sep}} - t) C_2(-\mathbf{q}; t) C_2(-\mathbf{q}; t_{\text{sep}})}}, \quad (5)$$

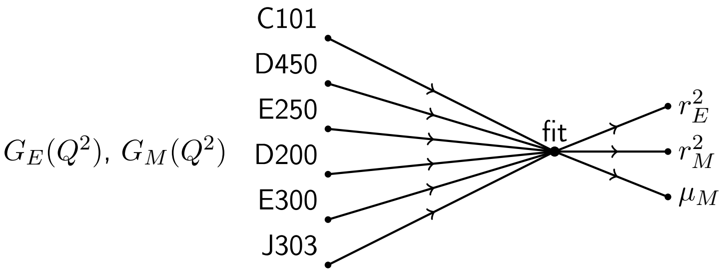
  where $t_{\text{sep}} = y_0 - x_0$ and $t = z_0 - x_0$
- Extract effective form factors $G_{E,M}^{\text{eff}}$ from suitable linear combinations of the ratios
- Apply summation method with varying starting values $t_{\text{sep}}^{\text{min}}$ for the linear fit
- Perform a weighted average over $t_{\text{sep}}^{\text{min}}$, where the weights are given by a smooth window function[4],

$$\hat{G} = \frac{\sum_i w_i G_i}{\sum_i w_i}, \qquad w_i = \tanh \frac{t_i - t_w^{\text{low}}}{\Delta t_w} - \tanh \frac{t_i - t_w^{\text{up}}}{\Delta t_w}, \quad (6)$$

  where $t_i$ is the value of $t_{\text{sep}}^{\text{min}}$ in the $i$-th fit, $t_w^{\text{low}} = 0.8$ fm, $t_w^{\text{up}} = 1$ fm and $\Delta t_w = 0.08$ fm

[3] Korzec et al. 2009; [4] Djukanovic et al. 2022 (talk by Jonna Koponen).

# Direct B$\chi$PT fits




- Combine parametrization of the $Q^2$-dependence with the chiral and continuum extrapolation
- Simultaneous fit of the pion mass and $Q^2$-dependence of the form factors to the expressions resulting from covariant chiral perturbation theory[5]

[5] Bauer, Bernauer, and Scherer 2012.

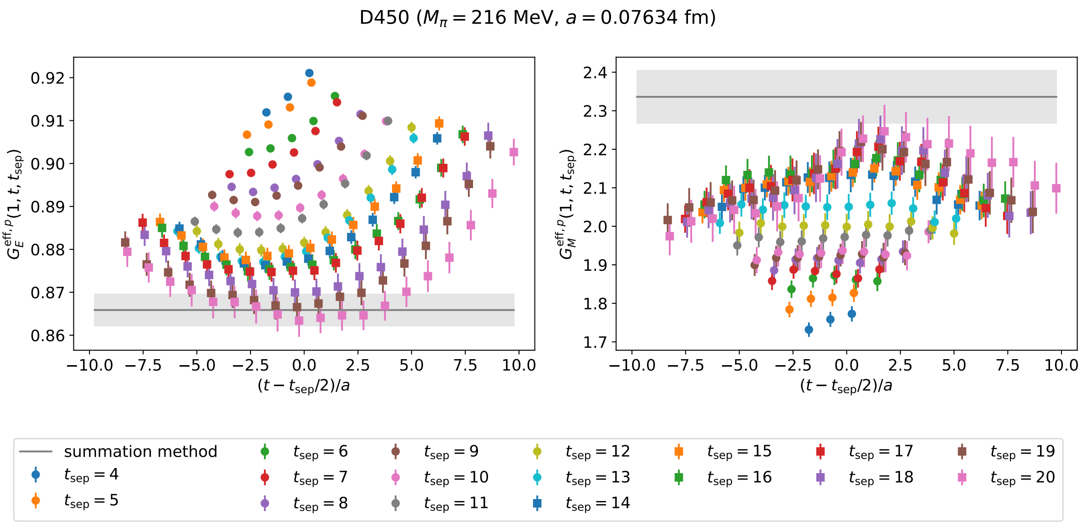
# Direct B$\chi$PT fits and model average

- Include contributions arising from the $\rho$ meson for both proton and neutron
- For the neutron, also include contributions arising from the $\omega$ and $\phi$ resonances to attain a reasonable description of the data
- Perform fits with various cuts in $M_\pi$ and $Q^2$, as well as with different models for the lattice spacing dependence, in order to estimate systematic uncertainties
- Quote naive (flat) averages over the results of all fits with a p-value above $1\,\%$, together with the average statistical uncertainty and the spread,

$$\hat{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad \sigma_{\mathrm{stat}}^2 = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^2, \qquad \sigma_{\mathrm{syst}}^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \hat{x})^2 \tag{7}$$

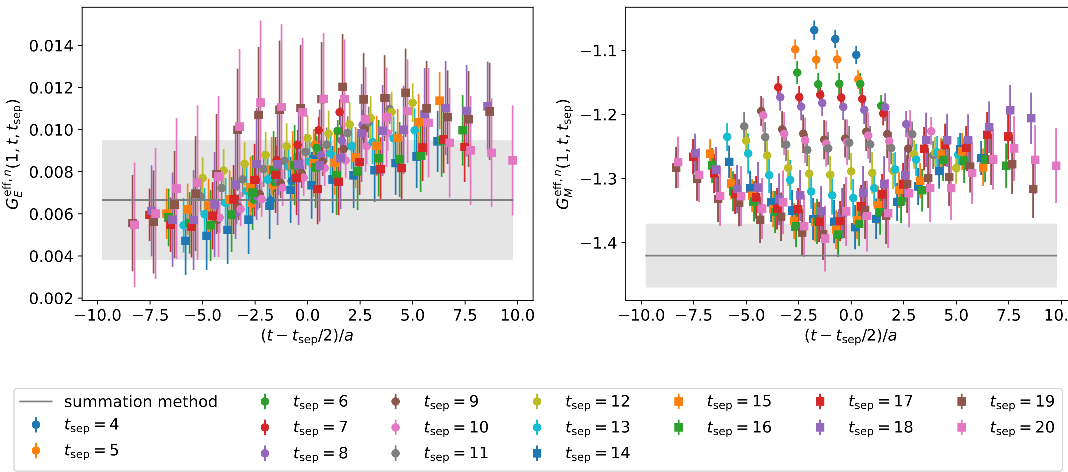
# Outline

# Effective form factors of the proton

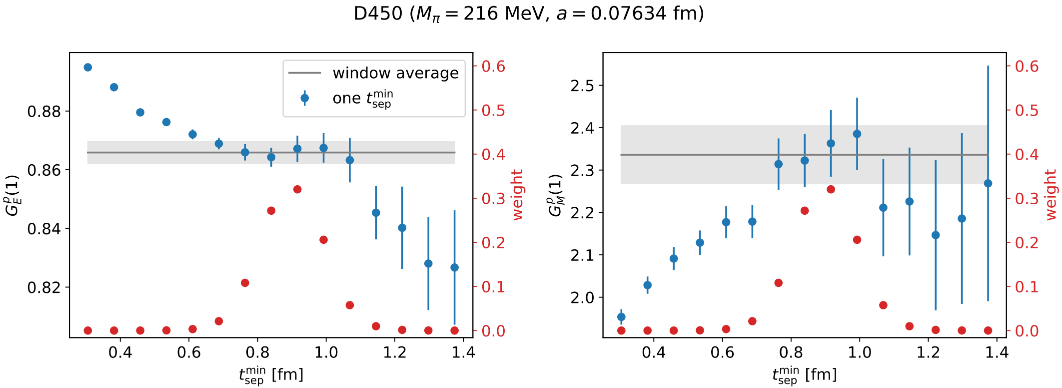


# Effective form factors of the neutron

D450 ($M_\pi = 216$ MeV, $a = 0.07634$ fm)



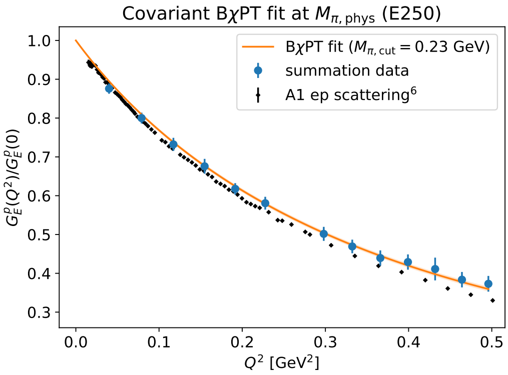
# Summation method and window average

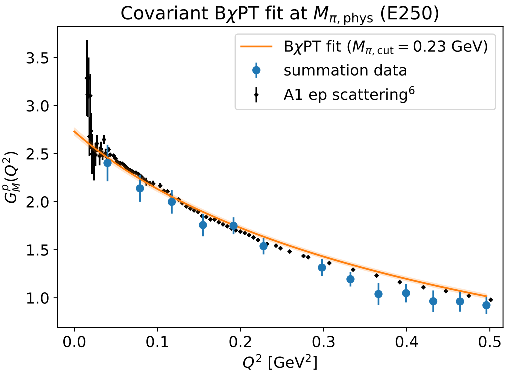



- Reliable detection of the plateau with reduced human bias (same window on all ensembles)
- Conservative error estimate

$$w_i = \tanh \frac{t_i - t_w^{\mathrm{low}}}{\Delta t_w} - \tanh \frac{t_i - t_w^{\mathrm{up}}}{\Delta t_w}$$

# $Q^2$-dependence of the proton form factors at $M_{\pi,\mathrm{phys}}$

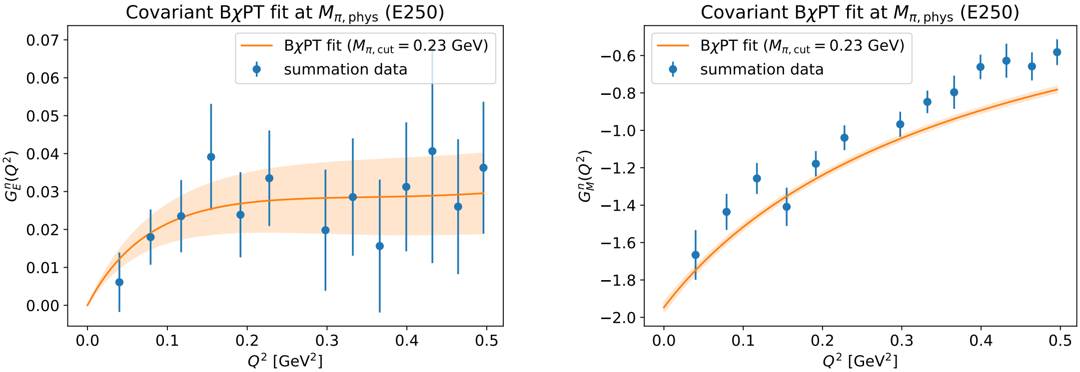



- Direct BχPT fit describes data very well
- Drastically reduced error due to the inclusion of several ensembles in one fit

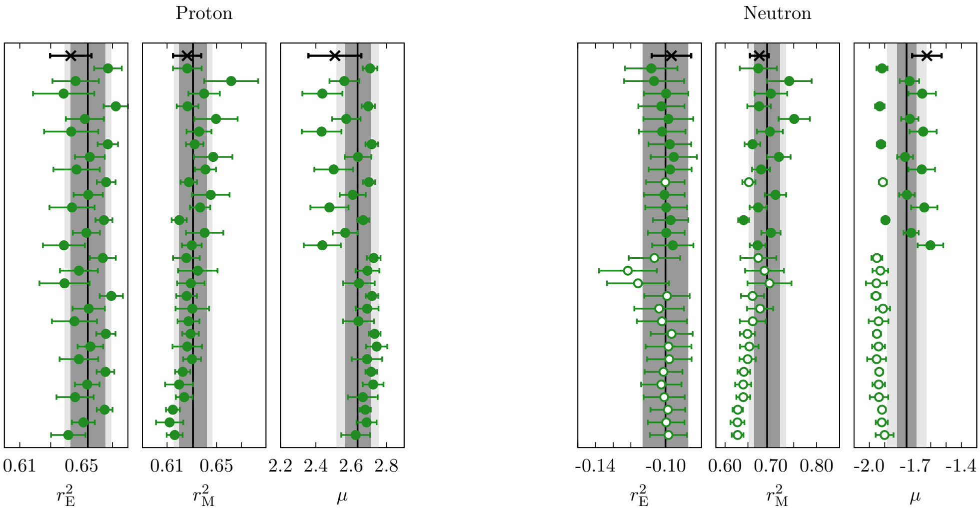
[6] Bernauer et al. 2014.

# $Q^2$-dependence of the neutron form factors at $M_{\pi,\text{phys}}$

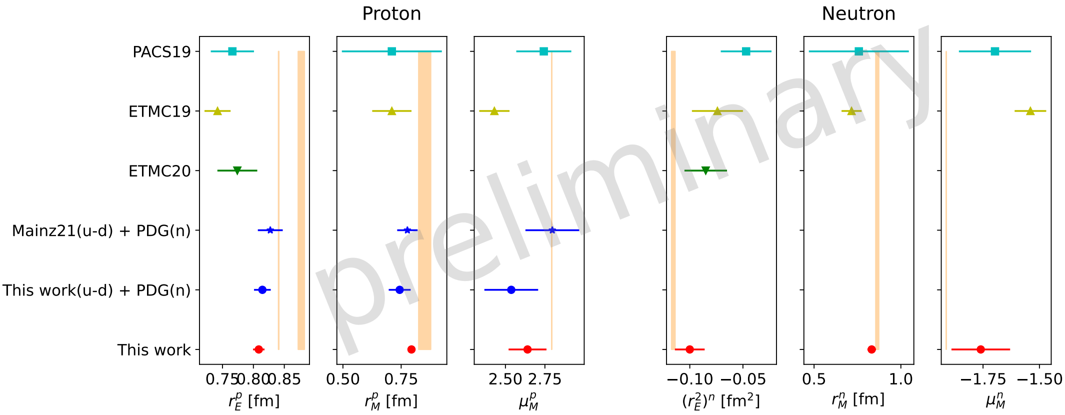



- Somewhat more deviation between fit and data than for the proton
- p-value still acceptable $(3\,\%)$

# Model average for the electromagnetic charge radii and magnetic moments



# Comparison to other studies




Vertical bands depict experimental values (PDG22 / CODATA10)

# Outline

# Conclusions

- Direct determination of the electromagnetic form factors of the proton and neutron from lattice QCD including all relevant contributions
- Chiral and continuum extrapolation via matching with the expectations from covariant chiral perturbation theory
- Preliminary results for the electromagnetic charge radii and the magnetic moments agree well with other lattice determinations and the experimental values
- Small electric charge radius of the proton favored
- Comptetitive errors
- Outlook
  - Increased statistics for the disconnected contribution on our two most chiral ensembles
  - Improved averaging of fit results and quantification of systematic uncertainties
  - Investigate some details of the analysis procedure

# Backup slides

# Nucleon two- and three-point functions

- Employ the same projection matrix $\Gamma = \frac{1}{2}(1+\gamma_0)(1+i\gamma_5\gamma_3)$ for both the two- and three-point functions
- Build the interpolating operator

$$N_\alpha(x) = \epsilon_{abc}\left(\tilde{u}_a^{\mathsf{T}}(x) C\gamma_5 \tilde{d}_b(x)\right)\tilde{u}_{c,\alpha}(x) \tag{8}$$

  for the proton using Gaussian-smeared quark fields with spatially APE-smeared gauge links
- Apply truncated-solver method with all-mode averaging and bias correction[7],

$$O = \frac{1}{N_{\mathrm{LP}}}\sum_{i=1}^{N_{\mathrm{LP}}} O^{\mathrm{LP}}(x_i) + \left[\frac{1}{N_{\mathrm{HP}}}\sum_{i=1}^{N_{\mathrm{HP}}}\left(O^{\mathrm{HP}}(x_i) - O^{\mathrm{LP}}(x_i)\right)\right] \tag{9}$$

[7] Bali, Collins, and Schäfer 2010; Blum, Izubuchi, and Shintani 2013.

# Vector current

- Use the conserved vector current,

$$V_\mu^c(n) = \frac{1}{2}\left(\bar{\psi}(n+\hat{\mu}a)(1+\gamma_\mu)U_\mu^\dagger(n)\psi(n) - \bar{\psi}(n)(1-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}a)\right), \quad (10)$$

or, more precisely, the symmetrized version

$$V_\mu^{cs}(n) = \frac{1}{2}\left(V_\mu^c(n) + V_\mu^c(n-\hat{\mu}a)\right) \quad (11)$$

- Perform $\mathcal{O}(a)$-improvement[8]
- No renormalization required

[8] Gérardin, Harris, and Meyer 2019.

# From ratios to form factors

- At zero sink momentum, the effective form factors can be computed from the ratios as

$$G_E^{\mathrm{eff}}(Q^2; t_{\mathrm{sep}}, t) = \sqrt{\frac{2E_{\mathbf{q}}}{m + E_{\mathbf{q}}}} R_{V_0}(\mathbf{q}; t_{\mathrm{sep}}, t), \tag{12}$$

$$G_M^{\mathrm{eff}}(Q^2; t_{\mathrm{sep}}, t) = \sqrt{2E_{\mathbf{q}}(m + E_{\mathbf{q}})} \frac{q_2 \operatorname{Re} R_{V_1}(\mathbf{q}; t_{\mathrm{sep}}, t) - q_1 \operatorname{Re} R_{V_2}(\mathbf{q}; t_{\mathrm{sep}}, t)}{q_1^2 + q_2^2}, \tag{13}$$

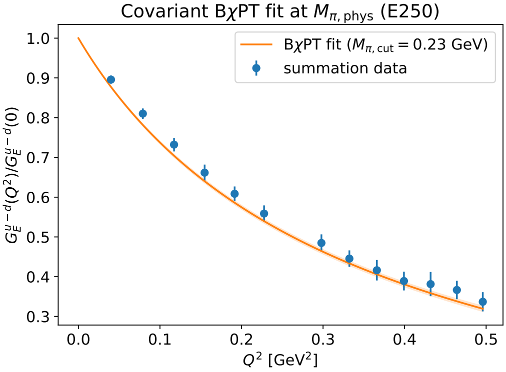
- Sum the effective form factors over the operator insertion time,

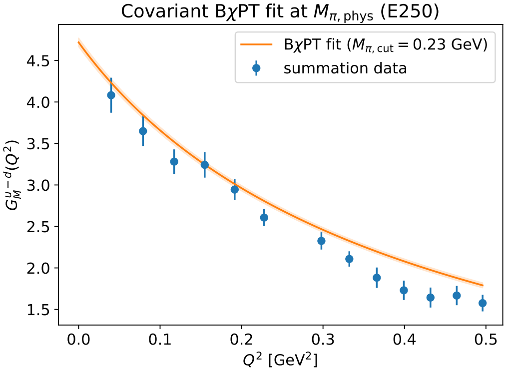
$$S_{E,M}(Q^2; t_{\mathrm{sep}}) = \sum_{t=t_{\mathrm{skip}}}^{t_{\mathrm{sep}}-t_{\mathrm{skip}}} G_{E,M}^{\mathrm{eff}}(Q^2; t, t_{\mathrm{sep}}), \quad t_{\mathrm{skip}} = 2a \tag{14}$$

- In the asymptotic limit, the slope of this as a function of $t_{\mathrm{sep}}$ is given by the ground state form factor,

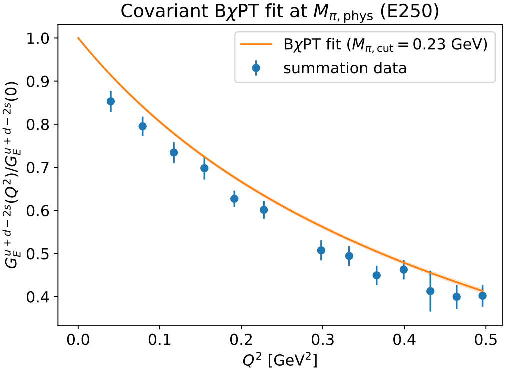
$$S_{E,M}(Q^2; t_{\mathrm{sep}}) \xrightarrow{t_{\mathrm{sep}}\to\infty} C_{E,M}(Q^2) + \frac{1}{a}(t_{\mathrm{sep}} + a - 2t_{\mathrm{skip}}) G_{E,M}(Q^2) \tag{15}$$

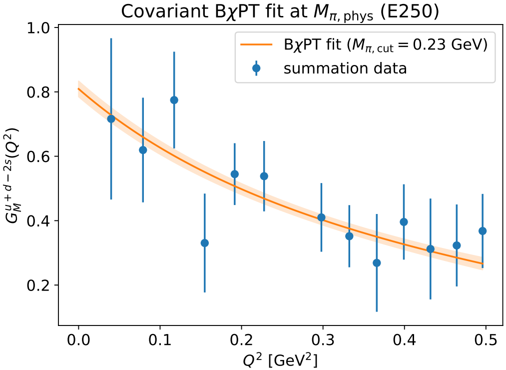
# $Q^2$-dependence of the isovector form factors at $M_{\pi,\text{phys}}$



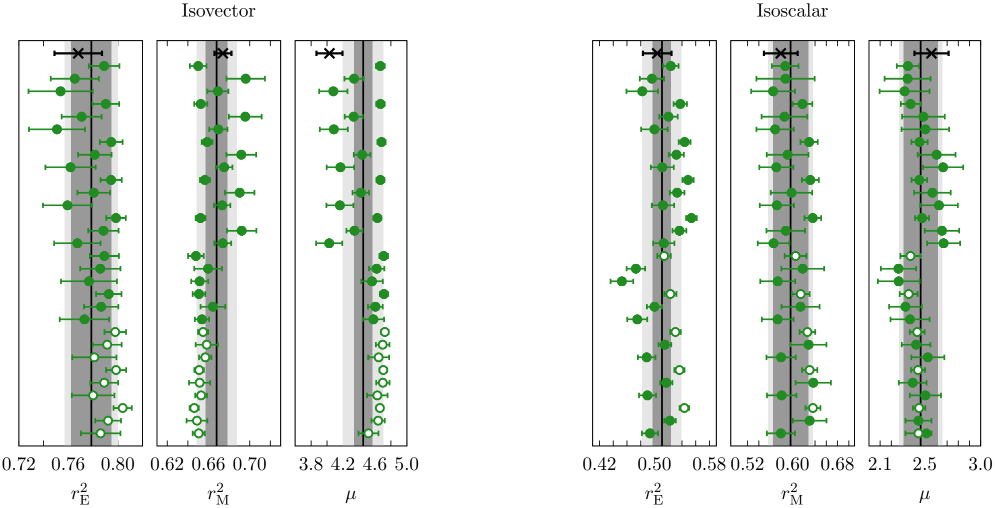



# $Q^2$-dependence of the isoscalar form factors at $M_{\pi,\mathrm{phys}}$



# Model average for the electromagnetic charge radii and magnetic moments



# AIC averages

- Perform a weighted average over the results of all fits, using weights derived from the Akaike Information Criterion[9],

$$w_i = \exp\left(-\frac{1}{2}\mathrm{AIC}_i\right) \Big/ \sum_j \exp\left(-\frac{1}{2}\mathrm{AIC}_j\right), \qquad \mathrm{AIC}_i = \chi^2_{\mathrm{min},i} + 2n_{\mathrm{fit}} + 2n_{\mathrm{cut}}, \quad (16)$$

  where $n_{\mathrm{fit}}$ is the number of fit parameters and $n_{\mathrm{cut}}$ the number of cut data points
- Determine the final cumulative distribution function (CDF) from the weighted sum of Gaussian distributions
- Quote median of this CDF together with the central $68\,\%$ quantile
- Selects 2–4 fits (depending on the channel) which carry $> 99\,\%$ of the weight
- Strongly prefers fits with low $n_{\mathrm{cut}}$, *i.e.*, the least stringent cut in $Q^2$

[9] Akaike 1974, 1998; Jay and Neil 2021.

# AIC averages

- These are sometimes located at the edge of the distribution of all fit results
- Charge radii and magnetic moment defined in terms of the low-$Q^2$ behavior of the form factors $\Rightarrow$ stricter cut in $Q^2$ favorable?!