Implementing the finite-volume scattering and decay formalism across all three-pion isospin channels

Maxwell T. Hansen

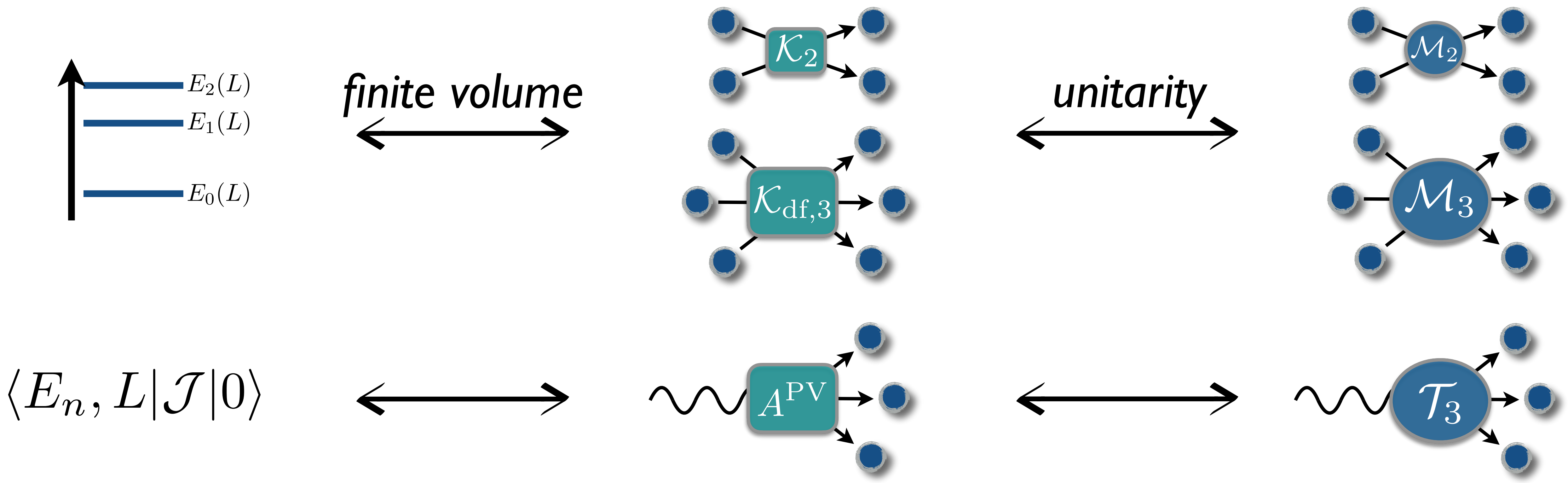
August 8th, 2022

based on work and discussion with…
Athari Alotaibi, Raul Briceño, Fabian Joswig, Felix Ziegler
(+ previous publications as cited)

UK Research and Innovation

THE UNIVERSITY of EDINBURGH

From the finite volume to amplitudes



- This talk focuses on the first step… For integral equations see: Jackura et al., PRD (2020)
  MTH, Briceño, Edwards, Thomas, Wilson, PRL 2020

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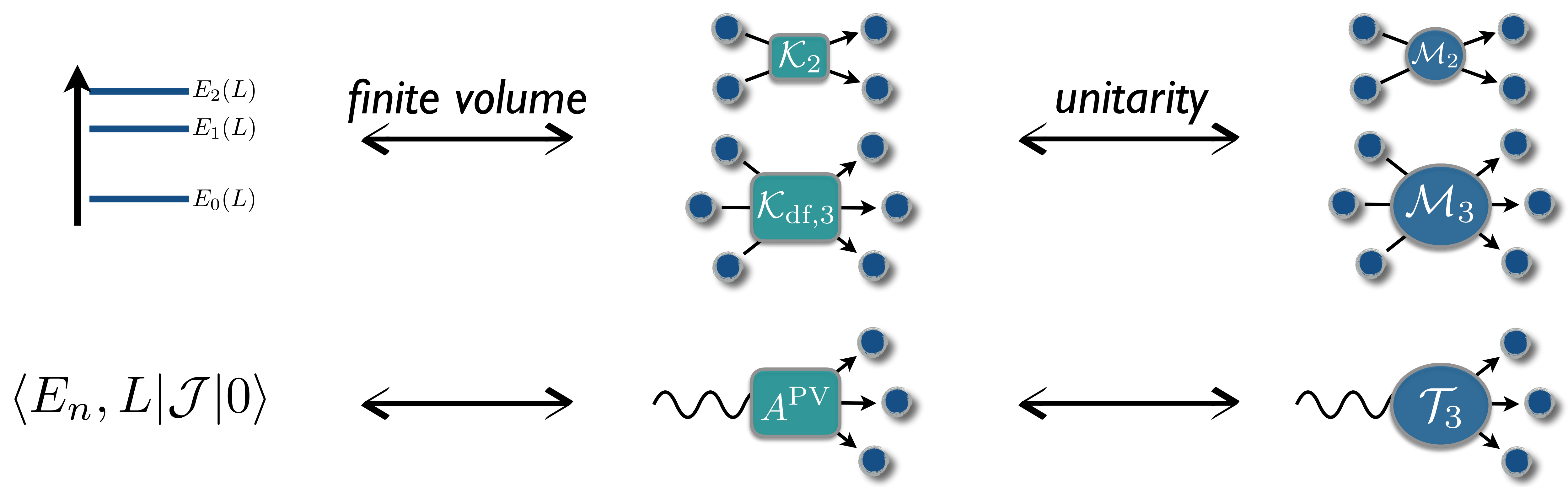

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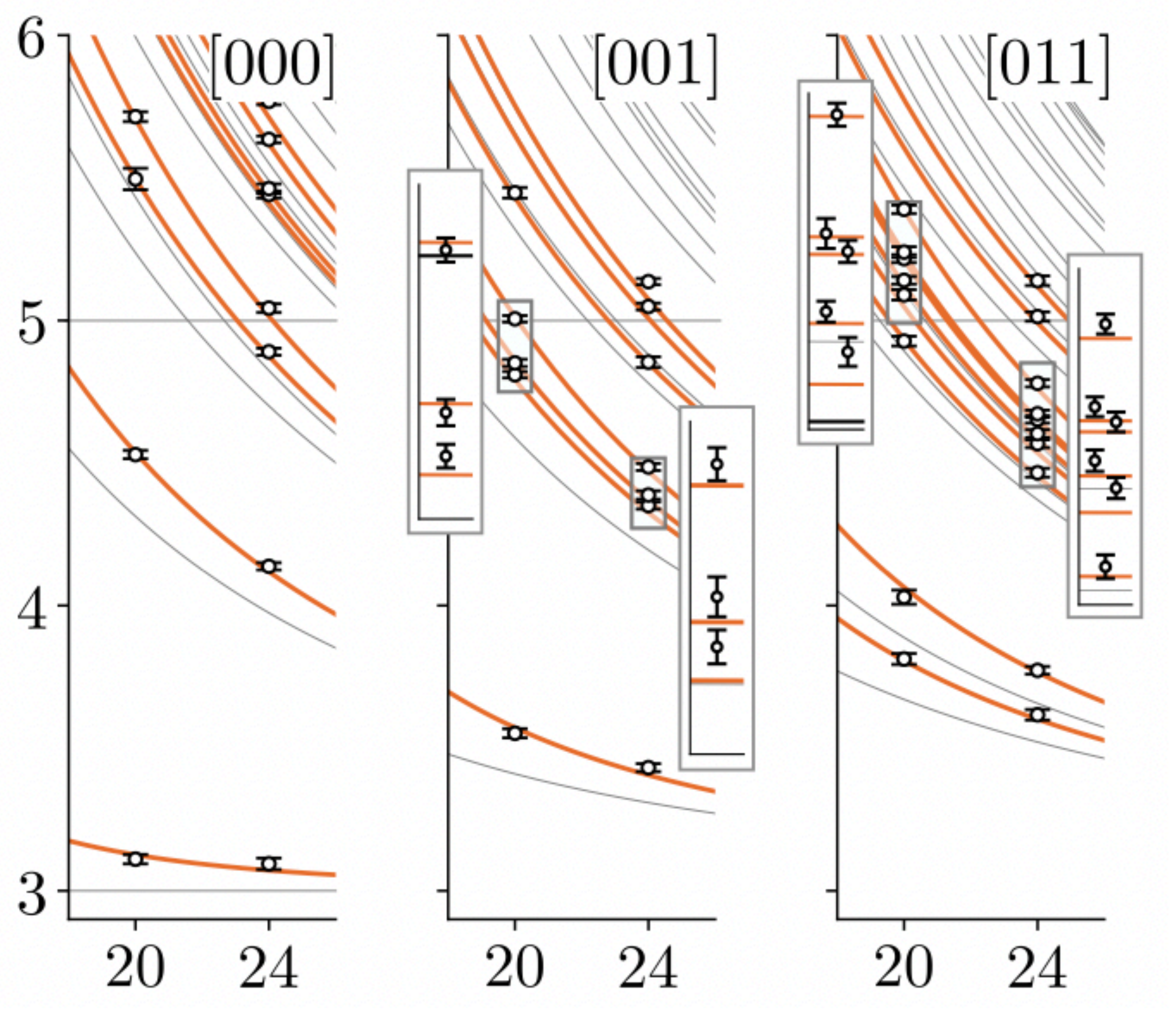
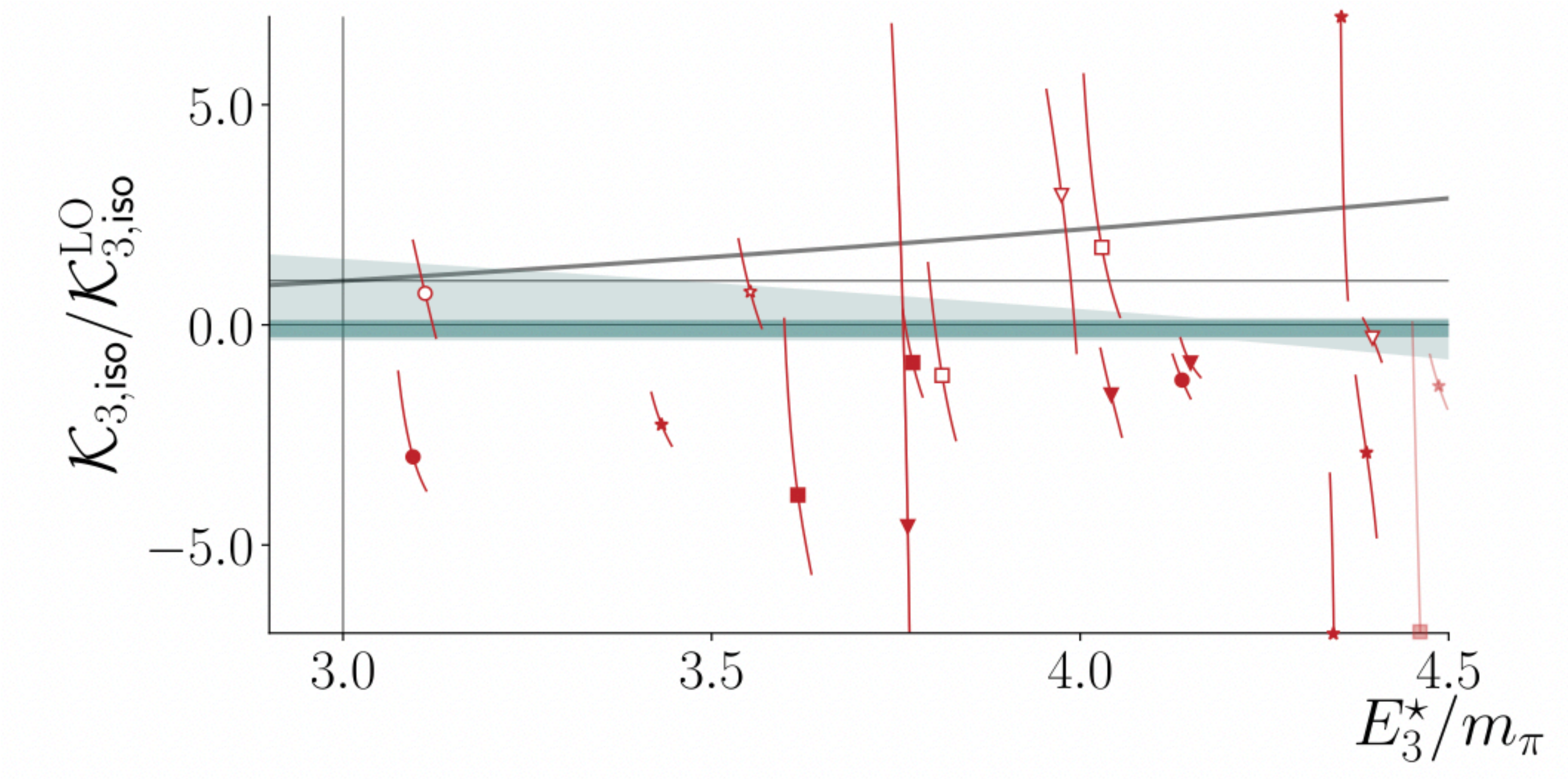
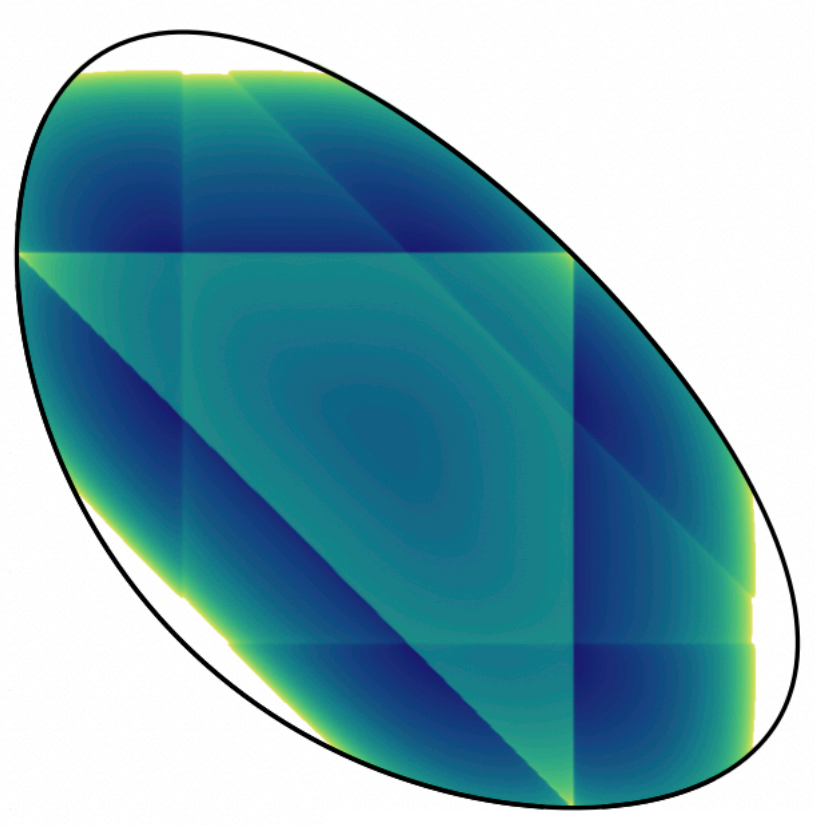






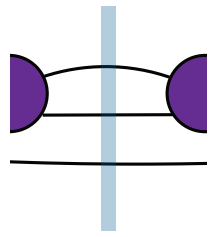
- [ ] Motivated by a Python package under development (available on GitHub… *some time this week*)

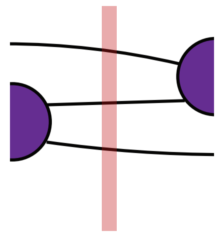
RFT quantization condition

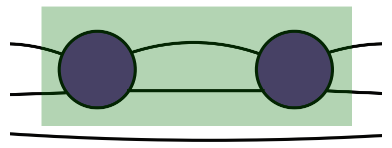
$$\det\left[\mathbf{K}_{\mathsf{df},3}^{-1}(E_{\mathsf{cm}}) + \mathbf{F}_3(E,\boldsymbol{P},L)\right] = 0$$

$\mathbf{F}_3(E,\boldsymbol{P},L) =$ Matrix of functions depending on kinematics + two-particle dynamics

$$\mathbf{F}_3 = \frac{\mathbf{F}}{3} + \mathbf{F}\mathbf{K}_2\left[1 - (\mathbf{F} + \mathbf{G})\mathbf{K}_2\right]^{-1}\mathbf{F}$$







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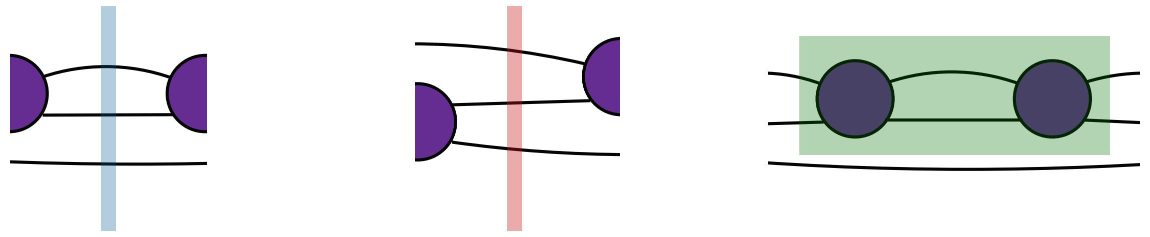


Matrices on tensor-product space: $(\text{spectator flavor space}) \otimes (\text{spectator } \boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3 \text{ space}) \otimes (\text{two-particle } \ell m)$

Holds only for three-particle energies

Neglects e^{-mL}

Requires sub-threshold continuation of $\mathbf{K}_2$

Scheme-dependent $\mathbf{K}_{\mathrm{df},3}$ related to physical amplitude via known on-shell integral equations

MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

Fitting the energies

$$\det\left[\mathbf{K}_{\text{df},3}^{-1}(E_{\text{cm}}) + \mathbf{F}_3(E, \boldsymbol{P}, L)\right]$$

- [ ] Build the quantization condition function

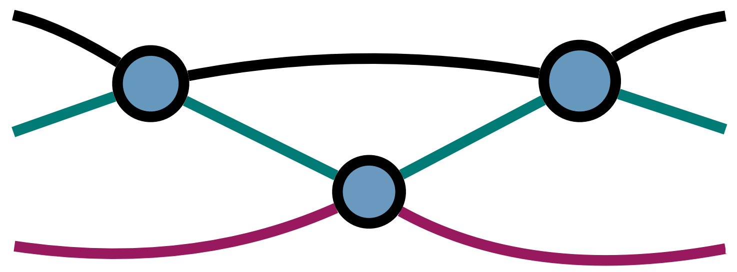
flavor-channel space N_f , e.g. $(\pi\pi\pi)_{I=0}$ or $\pi K, \pi\pi K$ …

→ spectator-channel space $N_\mathrm{s} \geq N_\mathrm{f}$, e.g. $((\pi\pi)_{I=1}\pi)_1$ or $(\pi\pi)K, (\pi K)\pi$

ℓ truncations and $(2 \to 2)_\ell$ + parametrizations $(a, r, \cdots)$ N_f lists

finite-volume set-up (geometry, total $\boldsymbol{P}$ → symmetry group, irreps)

three-body interaction scheme (definition of $K_{\mathrm{df},3}$) + parametrization $(\alpha, \beta, \cdots)$



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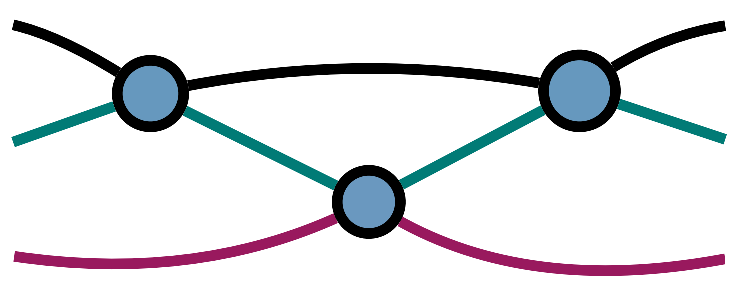

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Reduces to known function of E, L, $(a, r, \cdots)_{2\to2}$, $(\alpha, \beta, \cdots)_{3\to3}$

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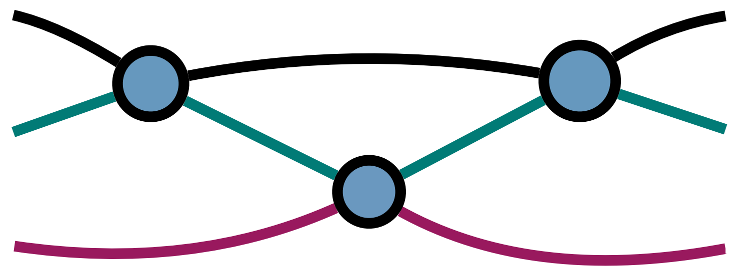


Reduces to known function of $E,\ L,\ (a, r, \cdots)_{2\to2},\ (\alpha, \beta, \cdots)_{3\to3}$

- ☐ Root-find to predict energies: $E_n(\boldsymbol{n}_P, L\,|\,a, r, \cdots|\,\alpha, \beta, \cdots)$

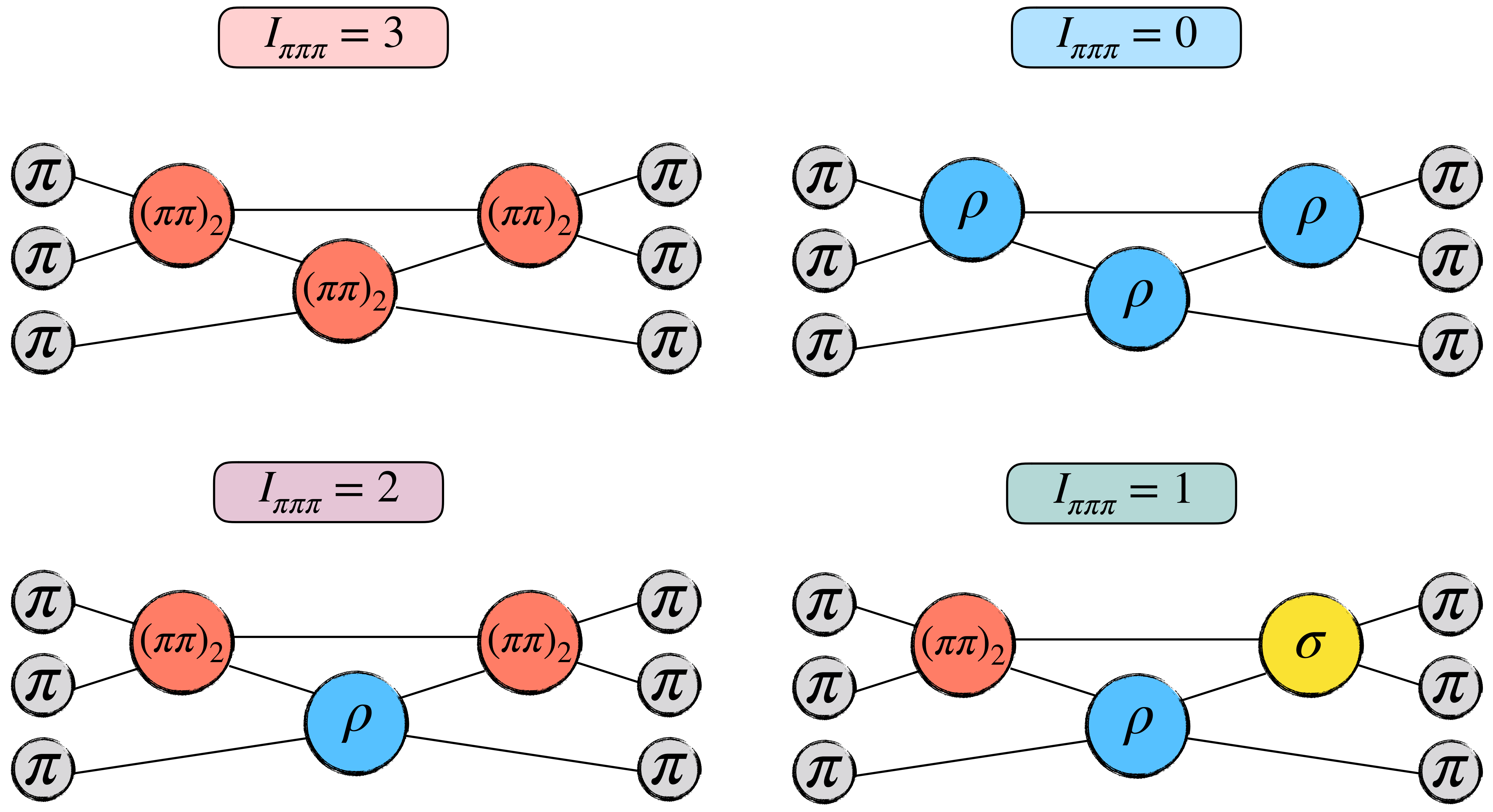
- ☐ Minimize χ^2 with lattice-determined energies → determination of parameters

Three pions with isospin

- Four possible iso-spin channels for three pions

$$1 \otimes 1 \otimes 1 = (0 \oplus 1 \oplus 2) \otimes 1 =$$
$$1 \oplus (0 \oplus 1 \oplus 2) \oplus (1 \oplus 2 \oplus 3)$$

$I_{\pi\pi} = 0$ $\quad$ $I_{\pi\pi} = 1$ $\quad$ $I_{\pi\pi} = 2$



- Four quantization conditions

$I_{\pi\pi\pi} = 0$

$$\begin{array}{c} (\rho) \\ \left(\square \right) \end{array} \; (\rho)$$

$I_{\pi\pi\pi} = 1$

$$\begin{array}{ccc} (\sigma) & (\rho) & (\pi\pi)_2 \end{array}$$
$$\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \begin{array}{c} (\sigma) \\ (\rho) \\ (\pi\pi)_2 \end{array}$$

$I_{\pi\pi\pi} = 2$

$$\begin{array}{cc} (\rho) & (\pi\pi)_2 \end{array}$$
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$I_{\pi\pi\pi} = 3$

$$\begin{array}{c} (\pi\pi)_2 \\ \left(\square \right) \end{array} \; (\pi\pi)_2$$

MTH, Romero-López, Sharpe, JHEP (2020) see also J. Baeza-Ballesteros' talk (previous session)

Constructing the index space: Example

$$(\text{spectator flavor space}) \otimes (\text{spectator } \boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3 \text{ space}) \otimes (\text{two}-\text{particle } \ell m)$$

- ❑ Flavor content → size of the first space e.g. 2 spectator channels for $I_{\pi\pi\pi} = 2$ pions

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See also Sharpe (last session) + related publications

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- Flavor content → size of the first space   e.g. 2 spectator channels for $I_{\pi\pi\pi} = 2$ pions
- Three-particle energy (E), momentum ($\boldsymbol{P}$) and volume (L) + scheme → size of the second space

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$$\left(E - \sqrt{m^2 + \boldsymbol{k}^2}\right)^2 - (\boldsymbol{P} - \boldsymbol{k})^2 \geq 0$$

or more precisely…

$$\left(E - \sqrt{m^2 + (2\pi/L)^2\boldsymbol{n}_k^2}\right)^2 - (2\pi/L)^2(\boldsymbol{n}_P - \boldsymbol{n}_k)^2 \geq 0$$

e.g. 27 spectator momenta for $E = 5m_\pi, \quad m_\pi L = 5, \quad \boldsymbol{n}_P^2 = 0$

sorted into shells $[000]_1 + [001]_6 + [011]_{12} + [111]_8$
for which $\boldsymbol{n}_P$ is crucial

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… minimal choice here is $(\pi\pi)_2$ *S-wave* and (ρ) *P-wave*

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108 x 108 matrices

and often larger for slightly different details

So, not huge matrices, but a given fit requires many evaluations

See also Sharpe (last session) + related publications

Finite-volume group theory

- Focusing on a cubic box → total momentum determines symmetry group

$\boldsymbol{n}_P$	group	dim	$n_{\text{1d irreps}}$	$n_{\text{2d irreps}}$	$n_{\text{3d irreps}}$	irrep names(dim)
[000]	O_h	48	4	2	4	$A_1^+(1), A_2^+(1), E^+(2), T_1^+(3), T_2^+(3), (+ \to -)$
[001]	C_4	8	4	1		$A_1(1), A_2(1), B_1(1), B_2(1), E_2(2)$
[011]	C_2	4	4			$A_1(1), A_2(1), B_1(1), B_2(1)$
[111]	C_3	6	2	1		$A_1(1), A_2(1), E_2(2)$

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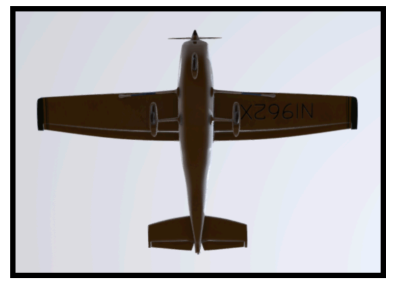
- For any object one can identify a linear combination of rotations that transforms as given irrep, row (Γ, μ)

$\Big($  $\Big)_{A_1^+} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+\cdots$

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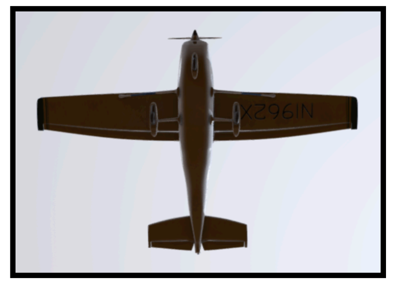
$\Big($  $\Big)_{A_1^+} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+\cdots$

Coefficients are known for all irrreps

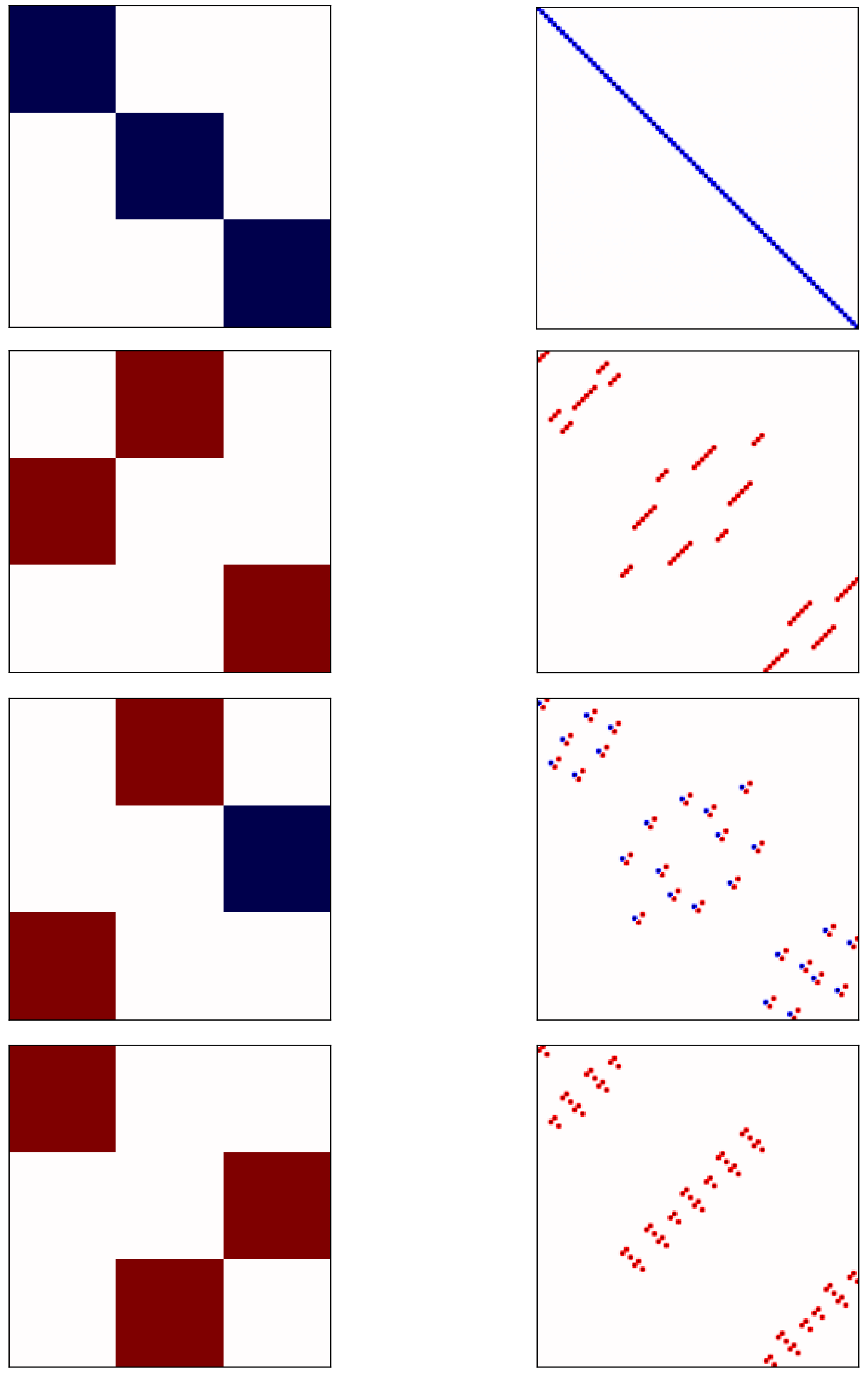
Credit C. Thomas and A. Rago for my understanding of this

Subducing the index space

- Returning to the $I_{\pi\pi\pi} = 2$ space

$$\text{(spectator flavor space)} \otimes \text{(spectator } \boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3 \text{ space)} \otimes \text{(two}-\text{particle } \ell m)$$

108 x 108 matrices



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108 x 108 matrices

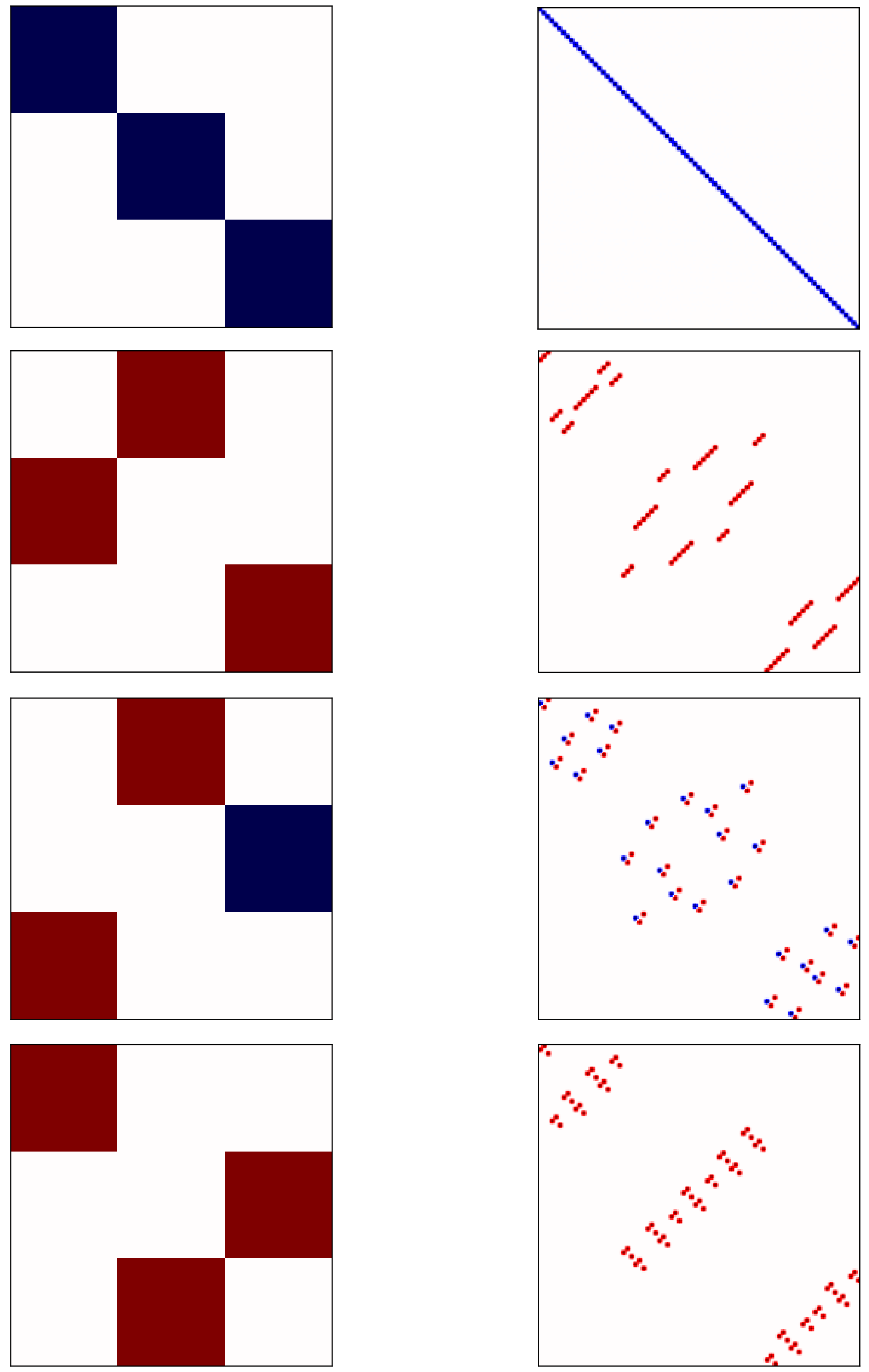
- Extract the induced representation of rotations & sum over irrep coefficients

"am-pie-ell"
Amplitudes via **Py**thon
from finite volume (**L**)

```
kellm space has size 108

    A1PLUS   covers 7x1 = 7 slots
    A2PLUS   covers 1x1 = 1 slots
    EPLUS    covers 6x2 = 12 slots
    T1PLUS   covers 4x3 = 12 slots
    T2PLUS   covers 7x3 = 21 slots
    A2MINUS  covers 3x1 = 3 slots
    EMINUS   covers 2x2 = 4 slots
    T1MINUS  covers 11x3 = 33 slots
    T2MINUS  covers 5x3 = 15 slots

Total is 108
Total matches size of kellm space
```

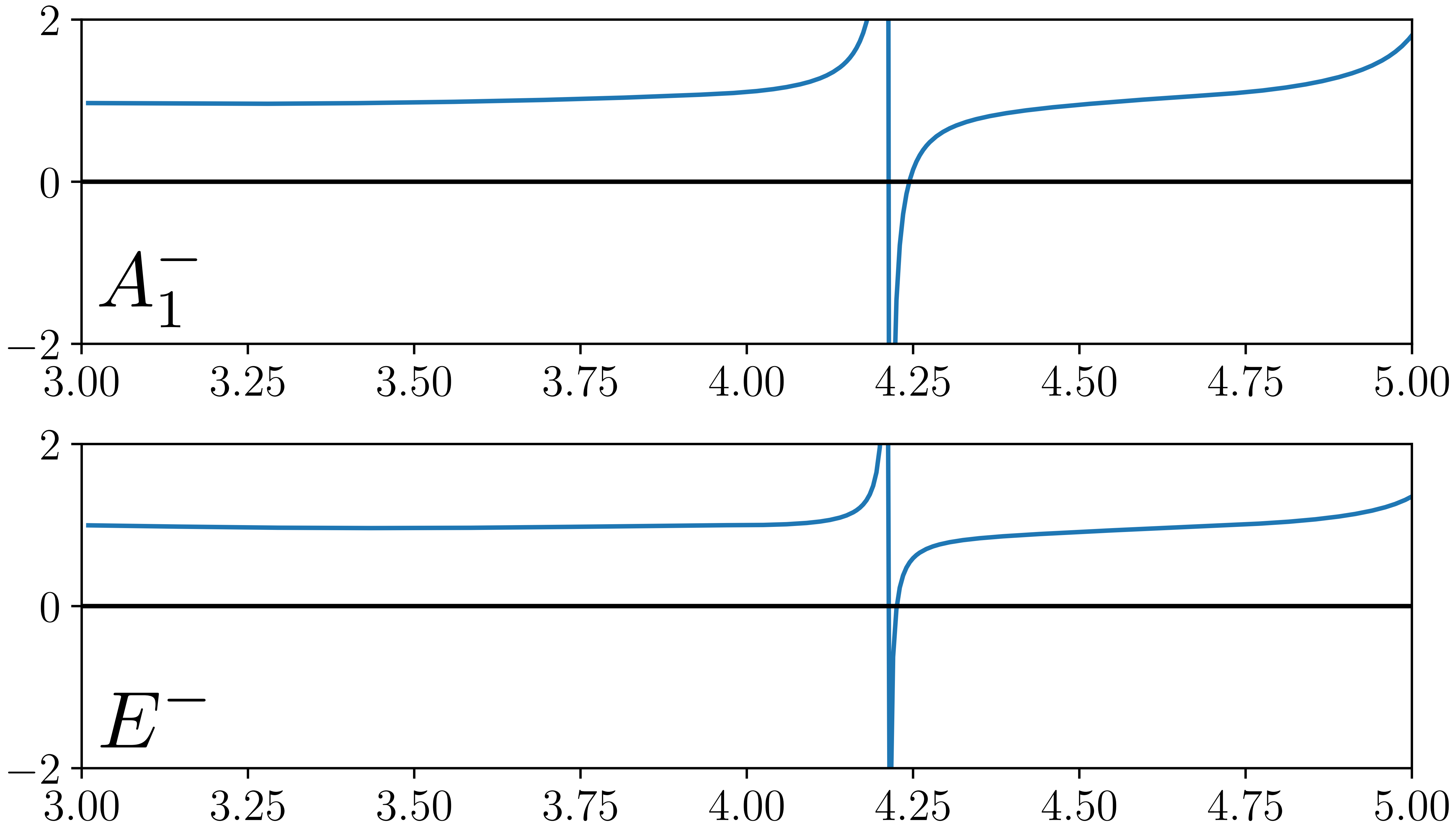


The *numerical* quantization condition

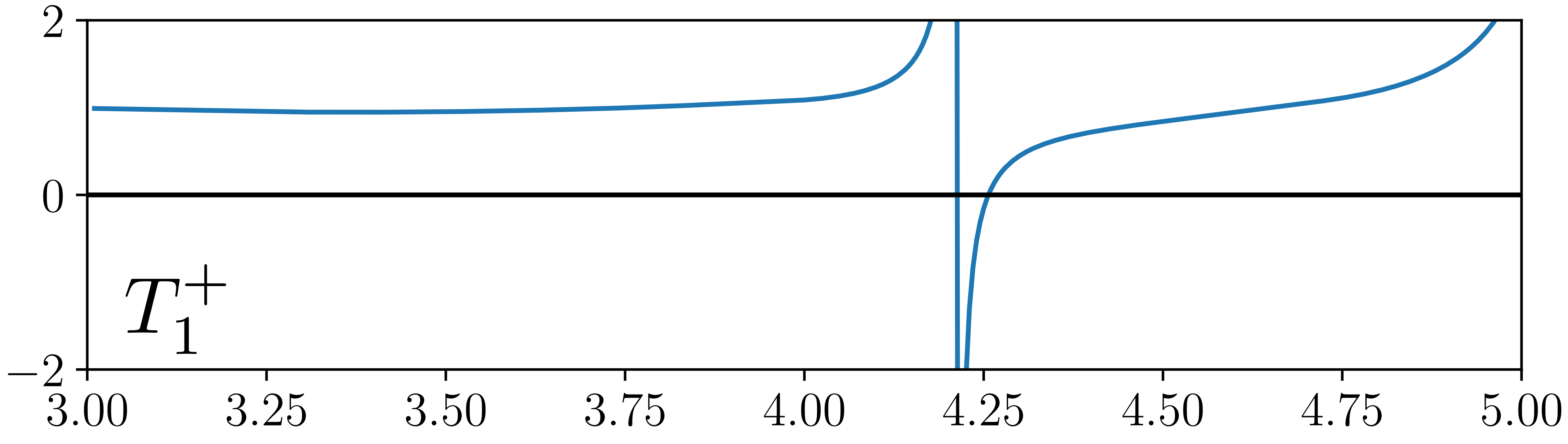
- Returning to the $I_{\pi\pi\pi} = 2$ $(\rho\pi)$ space

$$\det\left[\mathbf{K}_{\mathrm{df},3}^{-1}(E_{\mathrm{cm}}) + \mathbf{F}_3(E, \boldsymbol{P}, L)\right]$$

Here for $\mathbf{K}_{\mathrm{df},3} = 0$ and single parameters in $\mathbf{K}_2$ (weakly repulsive)



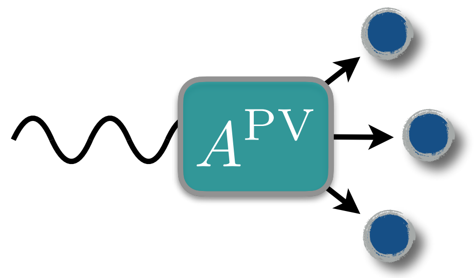
108x108 matrices for the high-energy side of the plots (does dot imply 108 solutions!)

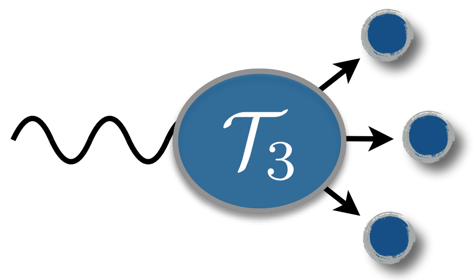


zero-crossings give solutions

Decay formalism

- Generalisation of the Lellouch-Lüscher formalism is available for three-pion states

$\langle E_n, L|\mathcal{J}|0\rangle \longleftrightarrow$ 

$\longleftrightarrow$ 

MTH, Romero-López, Sharpe (2021)

See also previous session + F. Joswig (Thursday)

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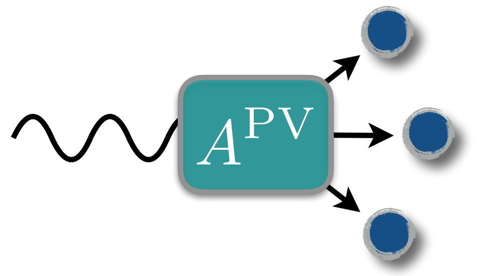
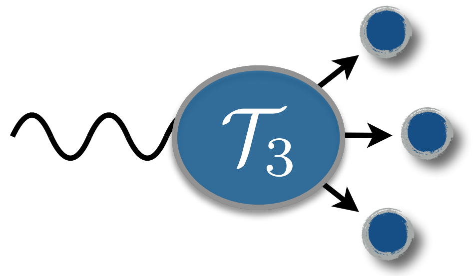
$\langle E_n, L|\mathcal{J}|0\rangle$ $\longleftrightarrow$ 

$\longleftrightarrow$ 

$$\lim_{E\to E_n(L)} (E - E_n(L)) \left[\frac{1}{\mathbf{K}_{\mathsf{df},3}(E) + \mathbf{F}_3^{-1}(E,L)} \right]_{\Gamma,\mu} = \mathcal{E}_n^{\mathsf{pv}} \mathcal{E}_n^{\mathsf{pv}^{\mathsf{T}}}$$

$$\mathcal{E}_n^{\mathsf{pv}} \cdot \langle 3\pi, \mathsf{pv}|\mathcal{J}|0\rangle = \langle n, L|\mathcal{J}|0\rangle$$

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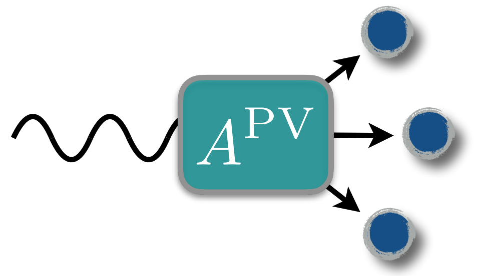
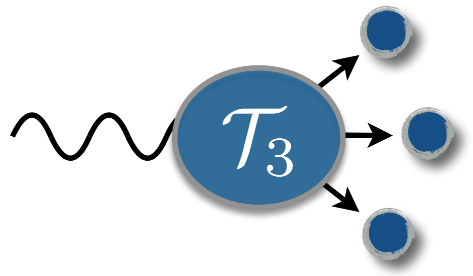
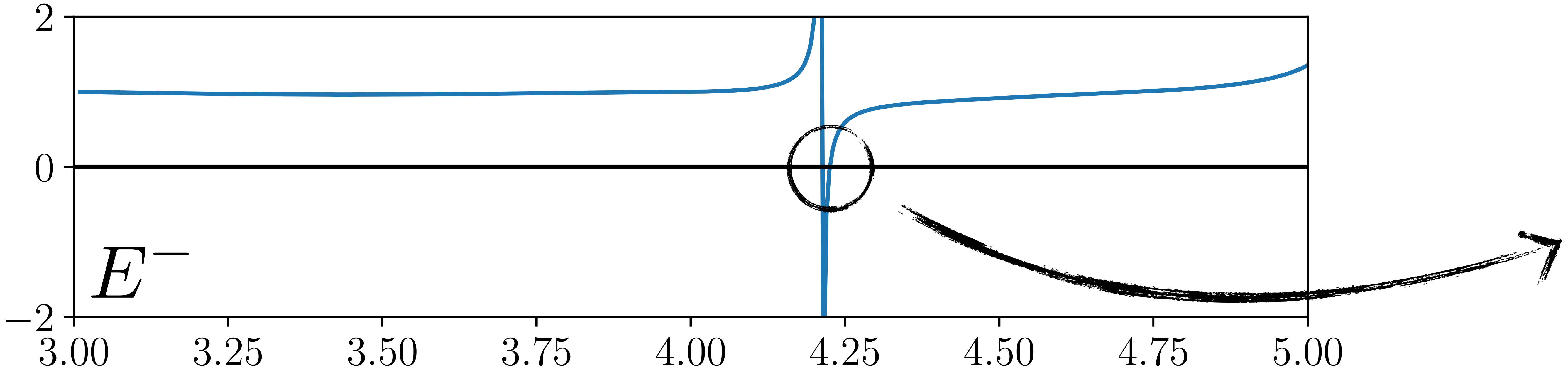
$\langle E_n, L|\mathcal{J}|0\rangle \longleftrightarrow$  $\longleftrightarrow$ 

$$\lim_{E\to E_n(L)} (E - E_n(L)) \left[\frac{1}{\mathbf{K}_{\mathsf{df},3}(E) + \mathbf{F}_3^{-1}(E,L)} \right]_{\Gamma,\mu} = \mathcal{E}_n^{\mathsf{pv}} \mathcal{E}_n^{\mathsf{pv}^{\mathsf{T}}}$$

$$\mathcal{E}_n^{\mathsf{pv}} \cdot \langle 3\pi, \mathsf{pv}|\mathcal{J}|0\rangle = \langle n, L|\mathcal{J}|0\rangle$$



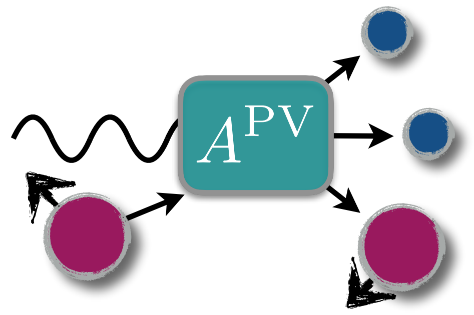
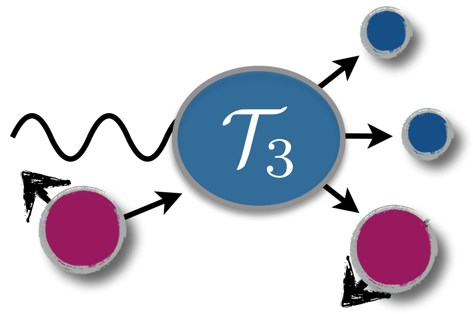
$|\mathcal{E}_n^{\mathsf{pv}}| =$ 

MTH, Romero-López, Sharpe (2021)

See also previous session + F. Joswig (Thursday)

Summary and outlook

- RFT formalism already describes many interesting systems: $(\pi\pi\pi)_{I_{\pi\pi\pi}}, \ (\pi\pi K), \ (\pi K K)$
- Formalism for general three-particle states is "around the corner"

$\langle E_n, L | \mathcal{J} | N \rangle \longleftrightarrow$  $\longleftrightarrow$ 

- Analysis is getting expensive!
- Strategies to speed things up:

  prepare quantization condition space as much as possible!

  group theory → prepare projectors (various tricks when projection is planned)

- "am-pie-ell" Python package to appear on Github for easy implementation and comparison

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Thanks for listening!