Two- and three-particle scattering in the (1+1)-dimensional O(3) non-linear sigma model

Jorge Baeza-Ballesteros

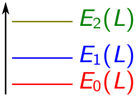
In collaboration with M. T. Hansen

IFIC, University of Valencia-CSIC

Lattice22 - 8th August 2022

Finite-volume quantization conditions

Finite-volume spectrum:



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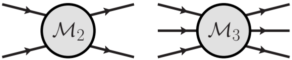

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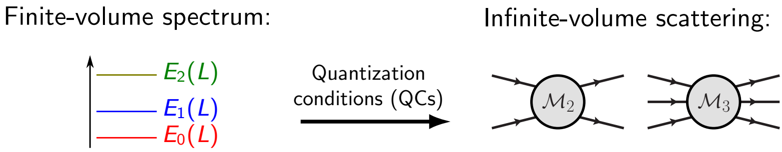

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$$\det[\mathcal{K}_2^{-1} + F(P, L)] = 0$$

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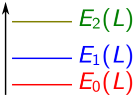

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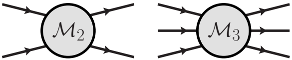

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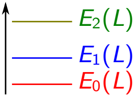

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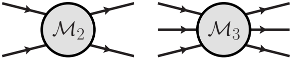

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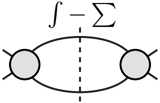

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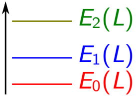

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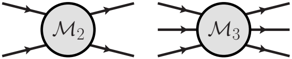

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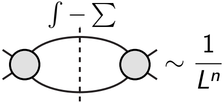

Three-particle QC: various approaches

- Relativistic field theory (RFT) [Hansen, Sharpe (2014, 2015)]
- Non-relativistic effective field theory (NREFT) [Hammer, et al. (2017)]
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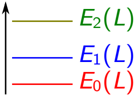

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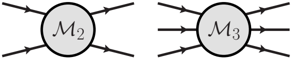

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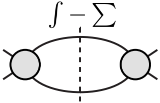

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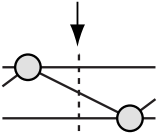

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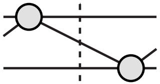

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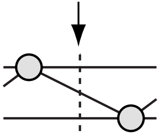

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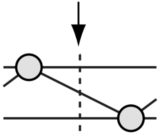

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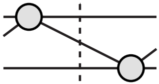

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Goal: Test RFT formalism on a solvable toy model

O(3) non-linear sigma model in (1+1) dimensions

We study the O(3) non-linear sigma model (NLSM) in 1+1 dimensions

$$S[\sigma] = \frac{\beta}{2} \int \mathrm{d}^2 x \, \partial_\mu \sigma(x) \cdot \partial_\mu \sigma(x) \qquad \begin{aligned} \sigma &= (\sigma^1, \sigma^2, \sigma^3) \\ \sigma(x) \cdot \sigma(x) &= 1 \end{aligned}$$

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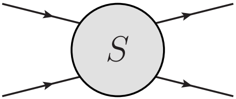
Integrable at low energies [Zamolodchikov, Zamolodchikov (1977)]:

Unitarity
+
Crossing symmetry
+
Factorization

→ Analytic S-matrix:



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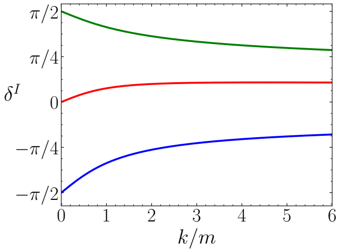

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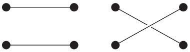
$$C_{I=2} = A_2 + A_3$$
$$C_{I=1} = A_2 - A_3$$



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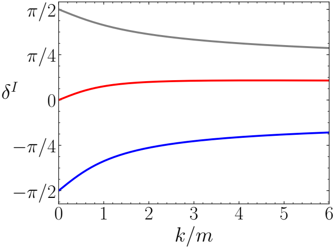

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Three particles: 7 irreps → 4 isospin channels

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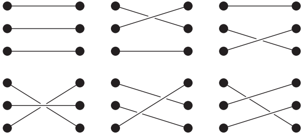

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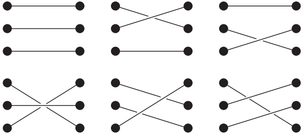

2×2 matrix for each momenta combination

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$\omega_k = E_{CM}/2$:

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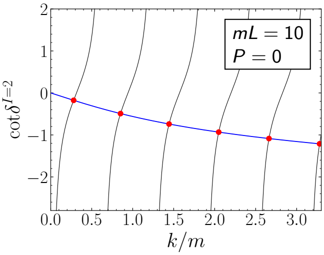

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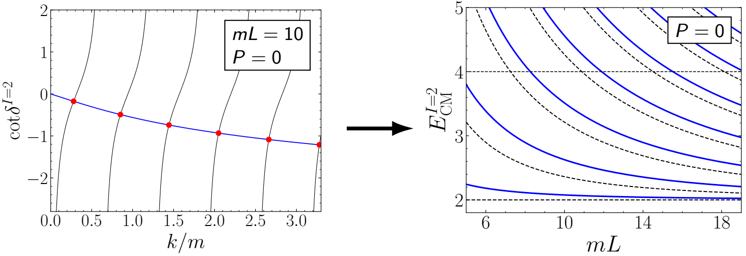

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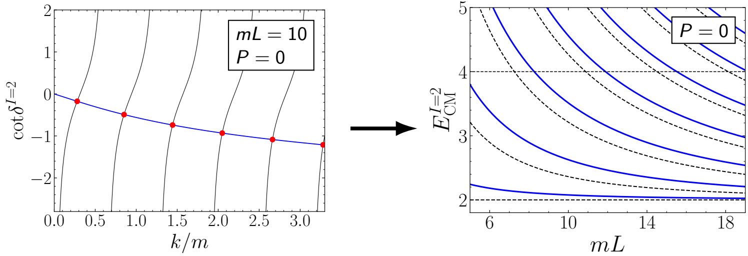

- Three-particle QC: analogous to (3+1) [Work in progress]

Lattice simulations

We use the standard lattice action

$$S[\sigma] = -\beta \sum_x \sum_\mu \sigma(x) \cdot \sigma(x + a\hat{\mu})$$

We generate the configurations and evaluate *n*-point functions using a **cluster algorithm**:

[Single-cluster: Wolff (1989), Two-cluster: Lüscher, Wolff (1990)]

- ⋆ Overcomes critical slowing down
- ⋆ Improves the signal-to-noise ratio

Single-cluster algorithm

Cluster algorithm:

1. Choose a random unit vector $\boldsymbol{r} \in \mathbb{R}^3$ and a random "seed" site
2. Grow the cluster, C, by adding neighbors

$$p_{\text{add}} = 1 - \exp[\min\{-2\beta\sigma_r(x)\sigma_r(x + a\hat{\mu}), 0\}] \qquad \sigma_r(x) = \sigma(x) \cdot r$$

3. Update the cluster

$$\sigma(x) \to \sigma(x) - 2\sigma_r(x) r$$

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Need to average over many updates!

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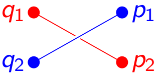
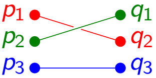
Example: Four- and six-point function



$$\propto \langle \sigma_r(\tau, q_2)\sigma_u(\tau, q_1)\sigma_u^*(0, p_2)\sigma_r^*(0, p_1)\rangle$$



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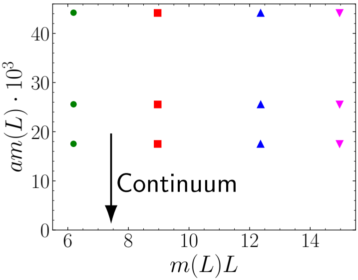
Ensembles: $[mL \sim 6, 9, 12, 15] \times [\beta = 1.63, 1.72, 1.78] = 12$ ensembles



⋆ Tuned mL and $mT \sim 20.5$

Our ensembles

We have used the "o3_cluster" code [Bulava, 2021]

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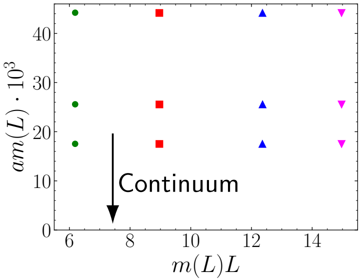

- ⋆ Tuned mL and $mT \sim 20.5$
- ⋆ 256/512/1024 replicas
- ⋆ $\sim$ 10 million thermalization updates
- ⋆ Averaged over $\sim$ 1 million measurement updates

Energy spectrum from lattice simulations

Solve GEVP and extract energy spectrum

$$C^{-1/2}(t_0)C(t)C^{-1/2}(t_0)v_n = \lambda_n(t)v_n \longrightarrow \lambda_n(t) \xrightarrow{T\gg t\gg t_0} A_n e^{-E_n t}$$

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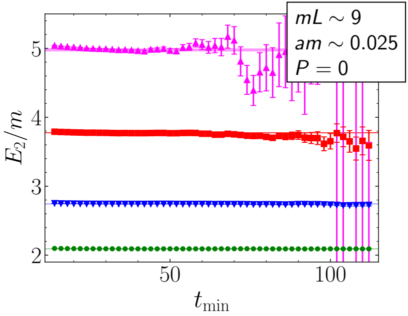
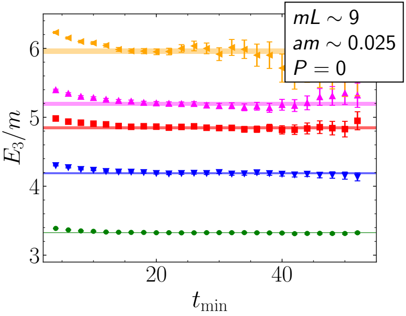
2 particles, $I = 2$ channel



3 particles, $I = 3$ channel



Continuum limit

The O(3) NLSM has large discretization effects [Balog, et al (2009, 2010)]:

$$Q(ma) = Q(0) + C(ma)^2\beta^3 \left[1 + \sum_{k=1}^{\infty} c_k \beta^{-k}\right] + \mathcal{O}(a^4)$$

with c_1 and c_2 known

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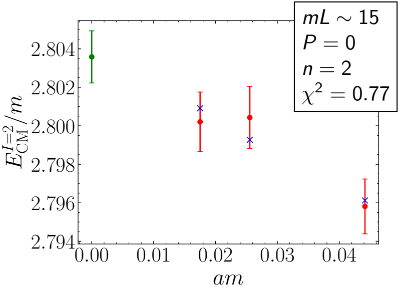
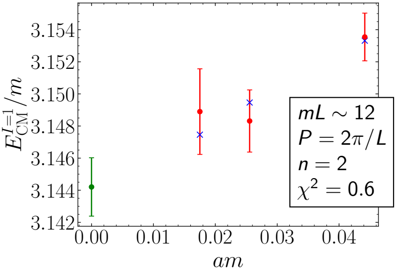


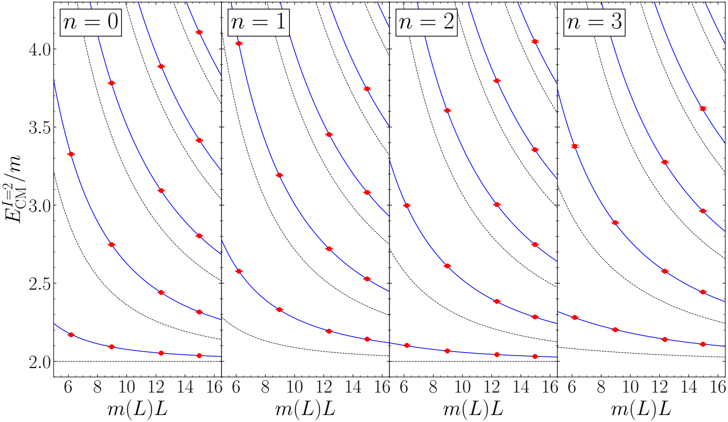
2 particles, $I = 1$ channel



Two-particle scattering

We compare 2-particle energies against analytical predictions

$P = \frac{2\pi}{L} n$ $\quad I = 2$ channel



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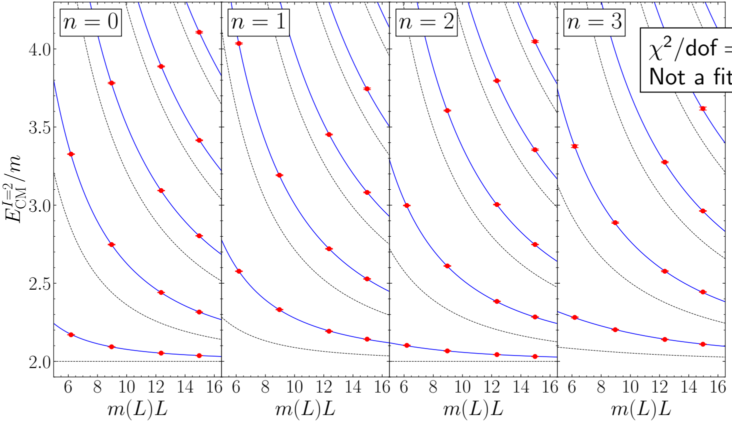

$\chi^2/\text{dof} = 1.34$
Not a fit!

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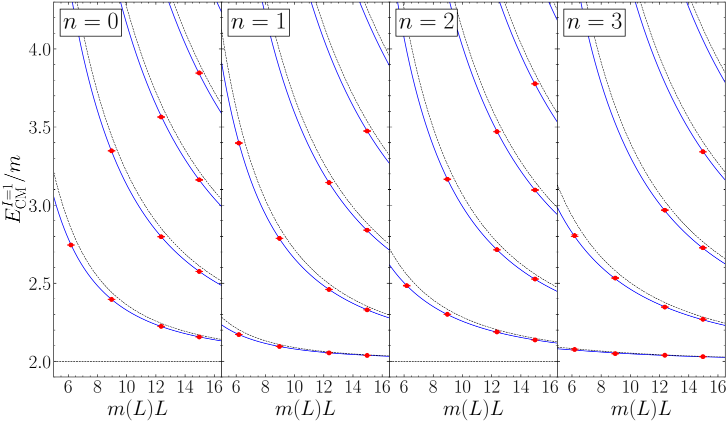

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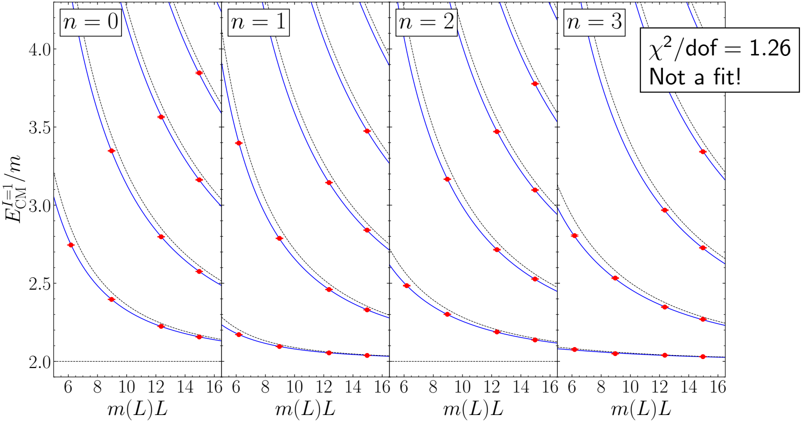
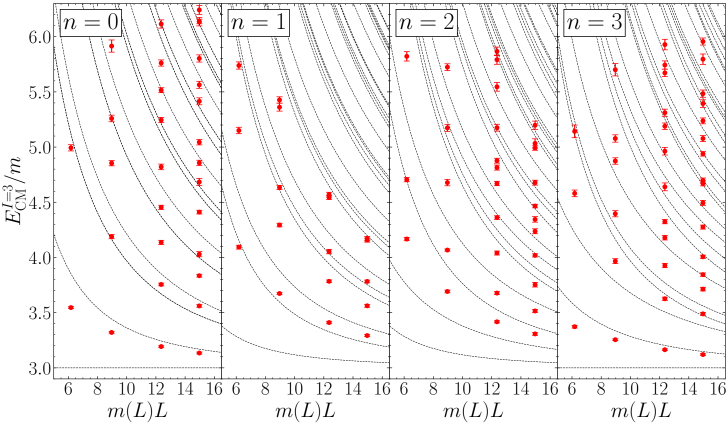


Three-particle scattering

We have determined 3-particle finite-volume energies at maximal isospin

$P = \frac{2\pi}{L} n$

$I = 3$ channel (Preliminary)



Summary and outlook

Goal: Test 3-particle RFT formalism on the (1+1)-dimensional O(3) non-linear sigma model

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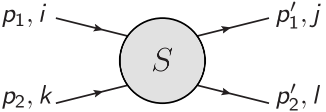
Thank you for your attention!

Exact two-particle S-matrix

Integrable at low energies [Zamolodchikov, Zamolodchikov (1977)]:

Unitarity + Crossing symmetry + Factorization



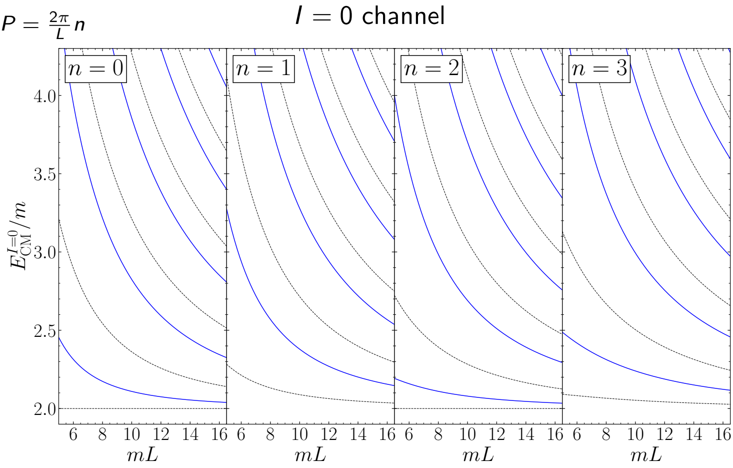
$$= (4\pi)^2\delta(p_1 - p_1')\delta(p_2 - p_2')\left[\delta_{ik}\delta_{jl}\sigma_1(s) + \delta_{ij}\delta_{kl}\sigma_2(s) + \delta_{il}\delta_{jk}\sigma_3(s)\right]$$

$$\sigma_2(\theta) = \frac{\theta(i\pi - \theta)}{(i2\pi - \theta)(i\pi + \theta)}, \quad \sigma_1(\theta) = \frac{-i2\pi}{i\pi - \theta}\sigma_2(\theta), \quad \sigma_3(\theta) = \frac{-i2\pi}{\theta}\sigma_2(\theta)$$

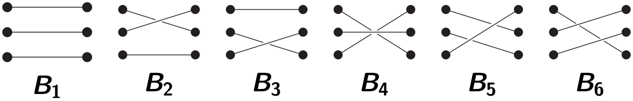
$s = 2m^2[1 + \cosh(\theta)]$

Two-particle $I = 0$ channel

$P = \frac{2\pi}{L} n$



Three-particle $I = 3$ and $I = 2$ channels



$$C_{I=3} = B_1 + B_2 + B_3 + B_4 + B_5 + B_6$$

$$C_{I=2} = \begin{pmatrix} B_1 - \frac{1}{2}B_2 + B_3 - \frac{1}{2}B_4 - \frac{1}{2}B_5 - \frac{1}{2}B_6 & \frac{\sqrt{3}}{2}\left[B_2 - B_4 + B_5 - B_6\right] \\ \frac{\sqrt{3}}{2}\left[B_2 - B_4 - B_5 + B_6\right] & B_1 + \frac{1}{2}B_2 - B_3 + \frac{1}{2}B_4 - \frac{1}{2}B_5 - \frac{1}{2}B_6 \end{pmatrix}$$

1×1 matrix if a pair of inital/final momenta are equal

No contribution if all three initial/final momenta are equal