Investigating Unitarity Violation With Chiral Perturbation Theory

ZACK DRAPER | UNVERSITY OF WASHINGTON | LATTICE 2022

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Introduction

- Will present work from the following two papers
    - "Applicability of the two-particle quantization condition to partially-quenched theories"[1]
    - "$\pi\pi$ scattering in partially-quenched twisted-mass chiral perturbation theory"[2]
- Will focus largely on [2] using results derived from [1]
    - $I = 0$ $\pi\pi$ scattering presents challenges the $I = 1$ and $I = 2$ channels do not
    - Will demonstrate existing results contain previously unknown errors

Collaboration	Ref.	N_f	$a_0^0 M_\pi$	$\ell_{\pi\pi}^0$
ETM 16C	[335]	2	0.198(9)(6)	30(8)(6)

[3]

[1] Z. T. Draper and S. R. Sharpe, Phys. Rev. D 104, 034510 (2021), 2107.09742.
[2] Z. T. Draper and S. R. Sharpe, Phys. Rev. D 105, 034508 (2022), 2111.13975.
[3] Y. Aoki et al., FLAG review 2021, arXiv:2111.09849.

Isospin-0 Scattering in TMQCD

- ETMC has calculated $\pi\pi$ scattering for $I = 2, I = 1,$ and $I = 0$
  - Calculated using a twisted mass to get $\mathcal{O}(a)$ improvement
- $I = 0$ scattering most difficult numerically
  - Fully disconnected quark diagrams contribute
  - Isospin breaking from mass twist mixes $I = 0$ and $I = 2,\ I_3 = 0$ channels[1]
- L. Liu et al. use partial quenching to prevent isospin mixing[2]
  - In doing so, introduce unitarity violation to calculations
  - Lüscher's quantization condition, used to extract scattering lengths, is derived assuming a unitary theory

[1] M. I. Buchoff, J.-W. Chen, and A. Walker-Loud, Phys. Rev. D 79, 074503 (2009), 0810.2464.
[2] L. Liu et al., Phys. Rev. D 96, 054516 (2017), 1612.02061.

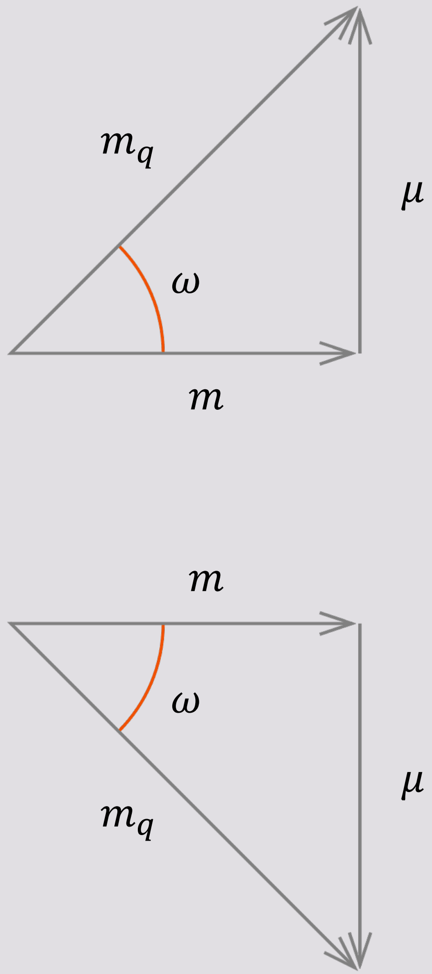
Understanding Errors with χPT

- Our goal is to use Chiral Perturbation Theory (χPT) to investigate potential $I = 0$ errors
- χPT can be made to incorporate
  - discretization errors
  - finite-volume effects
  - unphysical quark masses
  - unitarity violation
- Want a partially quenched (PQ) twisted mass (TM) version of χPT to match work of L. Liu et al.

L. Liu et al., Phys. Rev. D 96, 054516 (2017), 1612.02061.

Formulating PQTMχPT

- Leading-order Lagrangian looks like that of leading-order χPT
  - $\mathcal{L}_{\text{LO}} = \frac{f^2}{4}\text{str}\left(\partial_\mu \Sigma\, \partial_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\text{str}\left(2B_0(M^\dagger \Sigma + \Sigma^\dagger M)\right)$
- The mass twist treats up and down quarks differently
  - $M_0 = \text{diag}\left(m_q, m_q\right) \rightarrow M = m_q e^{i\tau_3 \omega} \equiv m + i\mu\tau_3$
- Hadronic fields are contained in $\Sigma = \exp\left(i\sqrt{2}\Pi/f\right)$
- Partial quenching introduces ghosts so that $\Pi = \begin{bmatrix} \phi & \eta_1 \\ \eta_2 & \tilde{\phi} \end{bmatrix}$
  - ϕ contains quark-quark NG bosons
  - $\tilde{\phi}$ contains ghost-ghost NG bosons
  - η_1, η_2 contain quark-ghost NG fermions



Partial Quenching to Prevent Isospin Mixing

- L. Liu et al. introduce additional valence and ghost u and d quarks
  - $Q^{\mathrm{tr}} = \left(u_S, d_S, u_V, d_V, \tilde{u}_V, \tilde{d}_V\right)$
- Valence and ghost quarks come with twist opposite to sea quarks
  - (u_S, d_S) comes with twist τ_3
  - $(u_V, d_V), \left(\tilde{u}_V, \tilde{d}_V\right)$ comes with twist $-\tau_3$
- $SU(4\mid 2)$ flavor symmetry broken by twisted mass
  - Left with $SU(2)_+ \times SU(2)_- \times U(1)$
  - For fields of positive twist, exact $SU(2)_+$ will prevent isospin mixing

L. Liu et al., Phys. Rev. D 96, 054516 (2017), 1612.02061.

$PQTM\chi PT$ Setup

- We can now formulate the LO Lagrangian using the standard power counting $m_q \sim p^2 \sim a^2$
  - $\mathcal{L}_{\text{LO}} = \frac{f^2}{4}\text{str}\left(\partial_\mu \Sigma\, \partial_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\text{str}\left(\chi^\dagger \Sigma + \Sigma^\dagger \chi\right) + \mathcal{V}_{a^2}$
  - $\mathcal{V}_{a^2} = -\hat{a}^2 W_6' \text{str}\left(\Sigma + \Sigma^\dagger\right)^2 - \hat{a}^2 W_7' \text{str}\left(\Sigma - \Sigma^\dagger\right)^2 - \hat{a}^2 W_8' \text{str}\left(\Sigma^2 + \Sigma^{\dagger 2}\right)$
- Π now contains PGBs and PGFs of positive, negative, and mixed twist

$$\Pi = \begin{pmatrix} \frac{\pi^0_{++}}{\sqrt{2}} + \frac{\eta_4}{2} + \frac{\phi_1}{2} & \pi^+_{-+} & \pi^+_{++} & \pi^{uu}_{-+} & \omega_{15} & \omega_{16} \\ (\pi^+_{-+})^\dagger & -\frac{\pi^0_{--}}{\sqrt{2}} - \frac{\eta_4}{2} + \frac{\phi_1}{2} & (\pi^{dd}_{-+})^\dagger & (\pi^+_{--})^\dagger & \omega_{25} & \omega_{26} \\ (\pi^+_{++})^\dagger & \pi^{dd}_{-+} & -\frac{\pi^0_{++}}{\sqrt{2}} + \frac{\eta_4}{2} + \frac{\phi_1}{2} & (\pi^+_{+-})^\dagger & \omega_{35} & \omega_{36} \\ (\pi^{uu}_{-+})^\dagger & \pi^+_{--} & \pi^+_{+-} & \frac{\pi^0_{--}}{\sqrt{2}} - \frac{\eta_4}{2} + \frac{\phi_1}{2} & \omega_{45} & \omega_{46} \\ \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \frac{\phi_0}{\sqrt{2}} + \phi_1 & \phi_{56} \\ \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \phi_{56}^\dagger & -\frac{\phi_0}{\sqrt{2}} + \phi_1 \end{pmatrix}$$

Z. T. Draper and S. R. Sharpe, Phys. Rev. D 105, 034508 (2022), 2111.13975.

$PQTM\chi PT$ Tree-Level Results

- $I = 2, I = 1$, and $I = 0$ scattering calculated first at tree-level
  - $\mathcal{A}_{I=2}^{\mathrm{LO}} = \frac{1}{f^2}\left(-s + 2M_{\mathrm{OS}}^2\right) - \frac{64}{f^4}\hat{a}^2 W_7' - \frac{32}{f^4}\hat{a}^2 W_8'$
  - $\mathcal{A}_{I=1}^{\mathrm{LO}} = \frac{t-u}{f^2}$
  - $\mathcal{A}_{I=0}^{\mathrm{LO}} = \frac{1}{f^2}\left(2s - M_{\mathrm{OS}}^2\right) - \frac{160}{f^4}\hat{a}^2 W_7' - \frac{80}{f^4}\hat{a}^2 W_8'$

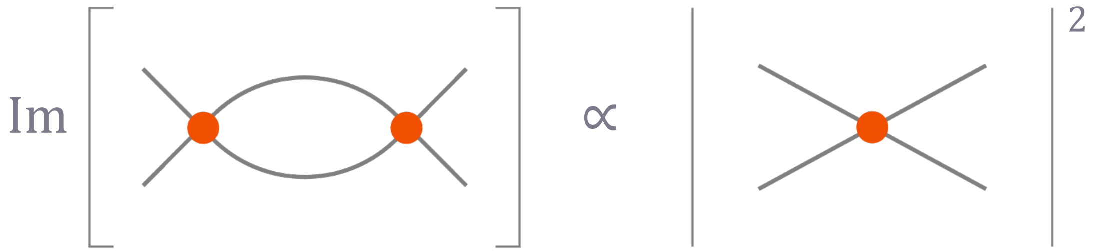
$$M_{\mathrm{OS}}^2 \equiv 2B_0 m_q - \frac{16}{f^2}\hat{a}^2 W_8'$$

$$\left(M_{SS}^{\pm}\right)^2 \equiv 2B_0 m_q$$

- $I = 0$ amplitude contains unaccounted for $\mathcal{O}(a^2)$ discretization errors
  - Could introduce $\mathcal{O}(100\%)$ errors into calculation since $a^2 \sim M_{\mathrm{OS}}^2$
- Lattice calculation of $\mathcal{A}_{I=2}^{\mathrm{LO}}$ would allow discretization errors to be subtracted
  - $\mathcal{A}_{I=0}^{\mathrm{LO}} - \frac{5}{2}\mathcal{A}_{I=2}^{\mathrm{LO}} = \frac{1}{f^2}\left(\frac{9}{2}s - 6M_{\mathrm{OS}}^2\right)$

Lüscher and S-Channel Unitarity

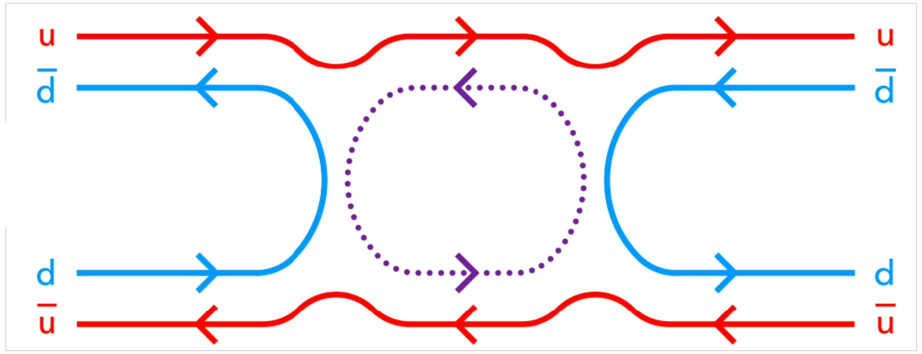
- We turn our attention from discretization errors to unitarity violation
    - Unitarity can be violated when external and loop quarks differ due to partial quenching
- Lüscher’s quantization condition is derived using s-channel unitarity
- The imaginary part of loop must be proportional to the squared magnitude of the tree-level amplitude
    - Proportionality constant related to the phase space available
- Multiple ways to check for possible violation
    - Quark line diagram analysis
    - Explicit computation



Z. T. Draper and S. R. Sharpe, Phys. Rev. D 104, 034510 (2021), 2107.09742.

Quark Line Diagrams: $I = 0$

- Many diagrams contribute to $I = 0$ scattering
  - A single diagram with different external quarks and loop quarks is enough to violate unitarity
- Consider diagram contributing to s-channel at one-loop

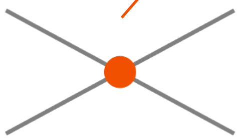


- d and $\bar{d}$ annihilate to produce valence, sea, or ghost quarks
  - Unless contributions cancel, process will violate unitarity

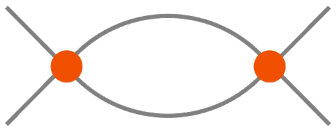
$PQTM\chi PT$ S-Channel Loops: $I = 0$

- An explicit calculation of the $I = 0$ amplitude confirms s-channel unitarity is violated
- The imaginary part of the loop is not proportional to the tree-level amplitude squared

$$\mathcal{A}^{\mathrm{LO}}_{I=0} = \frac{1}{f^2}\left(2s - M^2_{\mathrm{OS}}\right) - 10w'_7 - 5w'_8 \quad \text{where} \quad w'_k = \frac{16}{f^4}\hat{a}^2 W'_k$$



$$\begin{aligned}\mathcal{A}^{(s1)}_{\mathrm{OS},I=0} = {} & \frac{3s^2}{8f^4} J\left(s, M^{\pm\,2}_{\mathrm{SS}}, M^{\pm\,2}_{\mathrm{SS}}\right) + \frac{1}{8f^4} J\left(s, M^2_{\mathrm{OS}}, M^2_{\mathrm{OS}}\right)\Big(8M^2_{\mathrm{OS}}\left(8f^2 w'_7 + 26M^2_{\mathrm{OS}} + 5s\right) \\ & - 8M^{\pm\,2}_{\mathrm{SS}}\left(-8f^2 w'_7 + 44M^2_{\mathrm{OS}} + 7s\right) - 112 f^2 s w'_7 + 256 f^4 w'^{\,2}_7 + 160 M^{\pm\,4}_{\mathrm{SS}} + 13s^2\Big) \\ & - \frac{3}{f^4}\left(-3M^{\pm\,2}_{\mathrm{SS}} + 3M^2_{\mathrm{OS}} + M^{0\,2}_{\mathrm{SS}}\right)^2 J\left(s, M^2_{\mathrm{OS}}, M^{0\,2}_{\mathrm{SS}}\right) \\ & + \frac{3}{2f^4}\left(3M^{\pm\,2}_{\mathrm{SS}} - 2\left(M^2_{\mathrm{OS}} + M^{0\,2}_{\mathrm{SS}}\right)\right)^2 J\left(s, M^{0\,2}_{\mathrm{SS}}, M^{0\,2}_{\mathrm{SS}}\right) \\ & + \frac{5s}{12f^4} I\left(M^{\pm\,2}_{\mathrm{SS}}\right) + \frac{1}{36f^4} I\left(M^2_{\mathrm{OS}}\right)\left(-336 f^2 w'_7 - 168 M^{\pm\,2}_{\mathrm{SS}} + 120 M^2_{\mathrm{OS}} + 65s\right)\end{aligned}$$



What goes wrong?

- Consider an operator $\mathcal{O}$ that annihilates a pair of pions
  - $\mathcal{O}(t) = \pi^+(t)\pi^-(t)$ where $\pi^+(t) = \sum_{\vec{x}} \overline{d}\gamma_5 u(\vec{x},t)$ and $\pi^-(t) = \sum_{\vec{x}} \overline{u}\gamma_5 d(\vec{x},t)$
- For a unitary theory, can extract finite volume energies using the following
  - $\langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle = \sum_n c_n e^{-E_n \tau}$
- Useful to consider a ratio of correlation functions
  - $R(\tau) = \frac{\langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle}{\langle \pi(\tau)\pi(0)\rangle^2} = Z e^{-\delta E_0|\tau|} + \text{excited– state contributions}$
  - Allows for the extraction of $\delta E_0 = E_0 - 2M_\pi$

Lüscher's Quantization Condition

- Links finite-volume spectrum and infinite-volume scattering amplitude
- Liu et al. uses the threshold expansion for a unitary theory
    - $\delta E_0 = -\frac{4\pi a_0}{M_\pi L^3}\left[1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2} + \mathcal{O}(L^{-3})\right]$ where c_i are known constants
    - Allows for extraction of scattering length, a_0

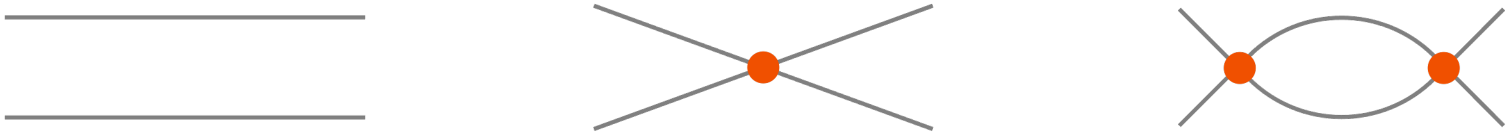
M. Lüscher, Commun.Math.Phys. 105, 153 (1986).

Computational Approach

- Neither of these two parts of Lüscher’s formalism hold when s-channel unitarity is violated
  - $R(\tau) = \frac{\langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle}{\langle \pi(\tau)\pi(0)\rangle^2} = Ze^{-\delta E_0|\tau|} + \cdots = Z\left[1 - |\tau|\delta E_0 + \frac{\tau^2}{2}(\delta E_0)^2\right] + \cdots$
  - $\delta E_0 = -\frac{4\pi a_0}{M_\pi L^3}\left[1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2} + \mathcal{O}(L^{-3})\right]$

- Contributions to threshold expansion come from three diagrams

(a) (b) (c)



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  - $\delta E_0 = -\frac{4\pi a_0}{M_\pi L^3}\left[1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2} + \mathcal{O}(L^{-3})\right]$

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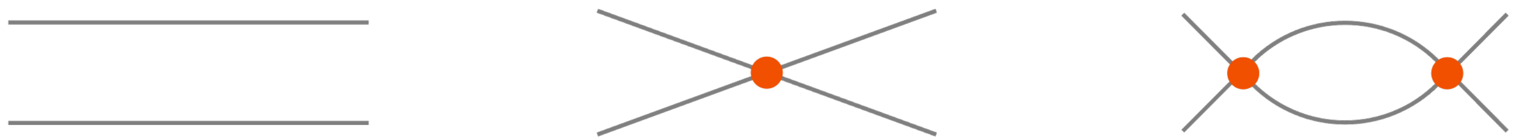

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  - $\delta E_0 = -\frac{4\pi a_0}{M_\pi L^3}\left[\boxed{1} + \boxed{c_1 \frac{a_0}{L}} + c_2 \frac{a_0^2}{L^2} + \mathcal{O}(L^{-3})\right]$
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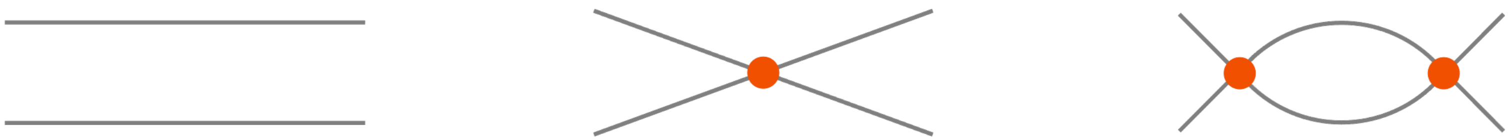

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Lüscher When S-Channel Unitarity Is Violated

- Can quantify errors that arise from applying Lüscher's formalism when not applicable
- Errors arise from two steps:
  - Extracting the energy spectrum from correlators
    - $R(\tau) = Z\left[1 - |\tau|\delta E_0 + \frac{\tau^2}{2}(\delta E_0)^2\right] + \cdots$ leads to $\lesssim 5\%$ error
  - Extracting scattering amplitude from energy spectrum
    - $\delta E_0 = -\frac{4\pi a_0}{M_\pi L^3}\left[1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2} + \mathcal{O}(L^{-3})\right]$ leads to $\sim 25\%$ error

Uncontrolled Errors

- Applying PQTMχPT to study of $I = 0$ $\pi\pi$ scattering we found:
    - $\sim 100\%$ uncontrolled error arising from discretization effects
        - A method of subtracting off these discretization errors
    - $\sim 25\%$ error arising from unitarity violation invalidating use of Lüscher's formalism
- One might hope to control $\sim 25\%$ error
    - Could forgo Lüscher's formalism and fit finite-volume correlators directly to PQTMχPT
    - Light states appearing in loops make analysis intractable
        - Intermediate OS π^0 and sea quark π^0 are lighter than two OS pions
        - Two sea quark π^0s also lighter than two OS pions
- Leaves us with an essentially irreducible $\sim 25\%$ systematic error

$I = 0$ $\pi\pi$ Scattering in FLAG Report

- Reported in the 2021 Flavor Lattice Averaging Group (FLAG) report
- Uncertainty is underestimated

Collaboration	Ref.	N_f	publication status	chiral extrapolation	cont. extrapolation	finite volume	$a_0^0 M_\pi$	$\ell^0_{\pi\pi}$
Fu 17	[391]	2+1	A	■	○	★	0.217(9)(5)	45.6(7.6)(3.8)
Fu 13	[334]	2+1	A	■	■	★	0.214(4)(7)	43.2(3.5)(5.6)
Fu 11	[392]	2+1	A	■	■	★	0.186(2)	18.7(1.2)
Mai 19	[389]	2	P	■	■	○	0.2132(9)	
ETM 16C	[335]	2	A	★	■	★	0.198(9)(6)	30(8)(6)
Caprini 11	[332]						0.2198(46)(16)(64)	
Colangelo 01	[323]						$0.220(5)_{\mathrm{tot}}$	

Y. Aoki et al., FLAG review 2021, arXiv:2111.09849.

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