

Scattering from generalised ϕ^4

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Three-particle quantization

- Three approaches can be used in lattice QCD to study scattering process of three particle
 - Relativistic fields theory (RFT) [[M. T. Hansen , S. R. Sharpe \(2014\)](#)]
 - Non-relativistic effective field theory (NREFT) [[H. W. Hammer , J. Y. Pang , A. Rusetsky \(2017\)](#)] is algebraically equivalent to FVU. A separate check of NREFT is thus not needed.
 - Finite-volume unitarity (FVU) [[M. Mai, M. Döring \(2017\)](#)]
- We are testing the RFT and FVU approach.

- There have been studies of systems with no resonances e.g.:
 - [T. D. Blanton et al.(2021)] [T. D. Blanton et al.(2019)]
 - [Hadron Spectrum Collaboration M. T. Hansen et al. (2020)]
 - [M. Fischer at al. (2020)]
 - [A. Alexandru et al.(2020)]
 - [C. Culver et al. (2019)]
 - [B. Hörz and A. Hanlon (2019)]
- Our aim is to study whether the formalism is usable for resonances in a toy model
 - [GWQCD Collaboration, M. Mai et al. (2021)]

model 3-particle resonance

- We use complex field
- masses $3m_0 < m_1 < 5m_0$

$$S = \int dx \sum_{i=0,1} \left[\frac{1}{2} \partial_\mu \varphi_i^\dagger \partial_\mu \varphi_i + \frac{1}{2} m_i \varphi_i^\dagger \varphi_i + \lambda_i (\varphi_i^\dagger \varphi_i)^2 \right] + \frac{g}{2} \varphi_1^\dagger \varphi_0^3 + h.c.$$

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- At $g = 0$
 - $U(1) \otimes U(1)$ global symmetry $\varphi_0 \rightarrow e^{i\theta} \varphi_0 \otimes \varphi_1 \rightarrow e^{i\theta'} \varphi_1$
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- With $g > 0$
 - φ_1 becomes unstable
 - residual global symmetry is $\varphi_0 \rightarrow e^{i\theta} \varphi_0$ together with $\varphi_1 \rightarrow e^{i3\theta} \varphi_1$
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- The model is most likely trivial. However, with small but finite lattice spacing the model effectively describes an interacting continuum field theory.

- Lattice discretization with $\partial_\mu \phi(x) = \phi(x + \mu) - \phi(x)$, $m_i = \frac{1-2\lambda_i}{\kappa_i} - 8$, $\hat{\lambda}_i = \frac{\lambda_i}{4\kappa_i^2}$,
 $\hat{g} = \frac{1}{4\sqrt{\kappa_0 \kappa_1^3}}$ and $\varphi_i = \sqrt{2\kappa_i} \phi_i$

$$S = \sum_x \sum_{i=0,1} \left[\kappa_i \sum_\mu \phi_i^\dagger(x) \phi_i(x + \mu) + \hat{\lambda}_i (\phi_i^\dagger(x) \phi_0(x) - 1) + \phi_i^\dagger \phi_i(x) + h.c. \right]$$
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$$+ \frac{\hat{g}}{2} \phi_1^\dagger \phi_0^3 + h.c.$$

- The limit $\hat{\lambda} \rightarrow \infty$ restrict the field to satisfy

$$\phi_i^\dagger(x) \phi_i(x) = 1$$

and thus the fields can be described with a phase $\phi_i = e^{i\theta_i}$

- The action becomes

$$S = \sum_x \sum_{i=0,1} \left[\kappa_i \sum_\mu \phi_i^\dagger(x) \phi_i(x + \mu) + h.c. \right] + \frac{\hat{g}}{2} \phi_1^\dagger \phi_0^3 + h.c.$$

Monte Carlo simulation

- Metropolis-Hastings algorithm to generate ensembles
 - We did not implement more advanced algorithm such Cluster Algorithm [[U. Wolff \(1989\)](#)] since we are simulating at values of m_0 not too small and it is difficult to parallelize.
- Implementation with Kokkos to have a performance portable implementation [[H. C. Edwards, C. R. Trott, D. Sunderland \(2014\)](#)]

Spectrum $g = 0$

- Single particle energy level

$$\langle \tilde{\phi}_i^\dagger(t) \tilde{\phi}_i(0) \rangle \sim |A_{\phi_i \rightarrow 0}| \left(e^{-E_1 t} - e^{-E_1(T-t)} \right)$$

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- Two particle energy level

$$\langle \tilde{\phi}_0^\dagger(t)^2 \tilde{\phi}_0(0)^2 \rangle \sim |A_{2\phi_0 \rightarrow 0}| (e^{-E_2 t} - e^{-E_2(T-t)}) + |A_{\phi_0 \rightarrow \phi_0}| e^{-E_1 T}$$

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- Three particle energy level

$$\begin{aligned} \langle \tilde{\phi}_0^\dagger(t)^3 \tilde{\phi}_0(0)^3 \rangle \sim & |A_{3\phi_0 \rightarrow 0}| (e^{-E_3 t} - e^{-E_3(T-t)}) \\ & + |A_{2\phi_0 \rightarrow \phi_0}| e^{-E_1 T} (e^{-t(E_2-E_1)} + e^{-(T-t)(E_2-E_1)}) \end{aligned}$$

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$$+ |A_{2\phi_0 \rightarrow \phi_0}| e^{-E_1 T} \left(e^{-t(E_2-E_1)} + e^{-(T-t)(E_2-E_1)} \right)$$

- We simulate at T large enough to neglect finite volume pollution ($T=64$)

• FVU

$$\det \left[B + C - 2L^3 E_{\mathbf{P}} (\tilde{K}_2^{-1} - \Sigma_2^L) \right] = 0$$

• RFT

$$\det \left[K_{df,3} + L^3 \left(\tilde{F}/3 - \tilde{F}(K_2^{-1} + \tilde{F} + \tilde{G})^{-1} \tilde{F} \right)^{-1} \right] = 0$$

- Both depend on the spectator moment
- B , Σ_2^L , \tilde{F} and \tilde{G} can be computed from the finite volume spectrum
- \tilde{K}_2 and K_2 related to the two-body scattering amplitude
- $C(s)$ and $K_{df,3}$ the three body forces, infinite-volume quantity related to the scattering amplitude

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- $C(s)$ and $K_{df,3}$ the three body forces, infinite-volume quantity related to the scattering amplitude
- Isotropic approximation:
 - s-wave dominance
 - Three body force independent from the spectator momentum
- truncation of the momentum space.

Two-particle phase shift

Three-particle force

$$\frac{q^*}{M_0} \cot \delta = \frac{1}{a_0 M_0}$$

$$K_3 = P_0$$

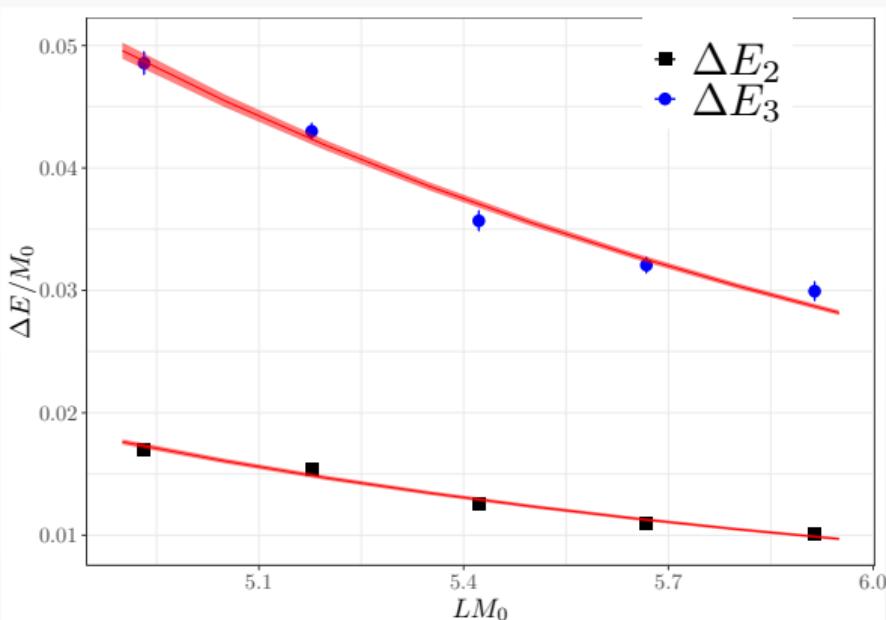
- fit:

$$\chi^2/d.o.f. = 1.8$$

$$P_0 = 1351(500)$$

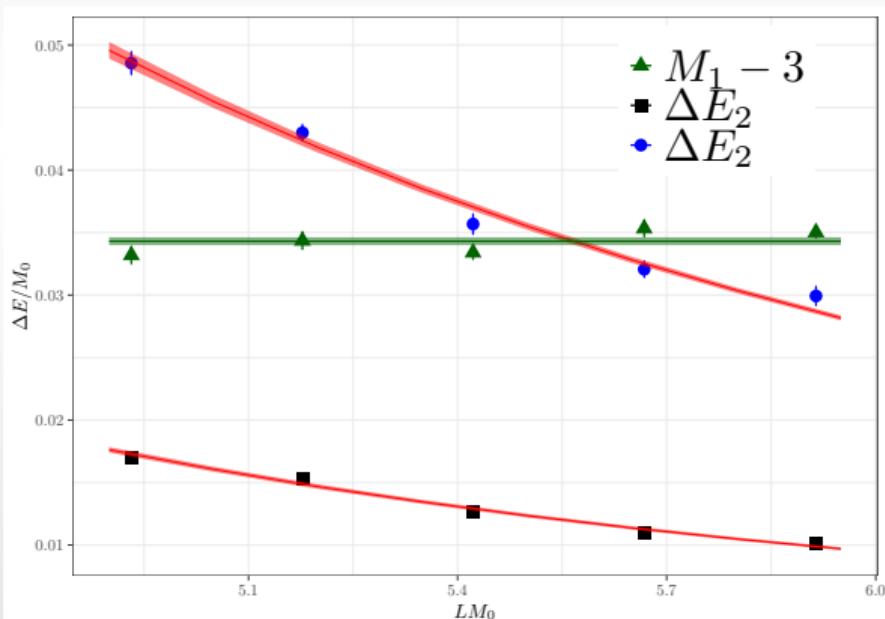
$$a_0 M_0 = -0.1514(20)$$

- $a_0 M_0$ compatible with the only two-particle fit



Two-particle phase shift

Three-particle force

One heavy-particle ϕ_1 mas

$$\frac{q^*}{M_0} \cot \delta = \frac{1}{a_0 M_0}$$

$$K_3 = P_0$$

$$M_1 = \text{const}$$

- fit:

$$\chi^2/d.o.f. = 1.8$$

$$P_0 = 1351(500)$$

$$a_0 M_0 = -0.1514(20)$$

$$M_1/M_0 = 3.0343(3)$$

- $a_0 M_0$ compatible with the only two-particle fit

Spectrum $g > 0$

- Single particle energy level

$$\langle \tilde{\phi}_0^\dagger(t) \tilde{\phi}_0(0) \rangle \sim |A_{\phi_0 \rightarrow 0}| e^{-M_0 t}$$

- Two particle energy level

$$\langle \tilde{\phi}_0^\dagger(t)^2 \tilde{\phi}_0(0)^2 \rangle \sim |A_{2\phi_0 \rightarrow 0}| e^{-E_2 t}$$

- Three particle operator with the same quantum number $\tilde{\phi}_0^3$ and $\tilde{\phi}_1$ thus we measure the matrix

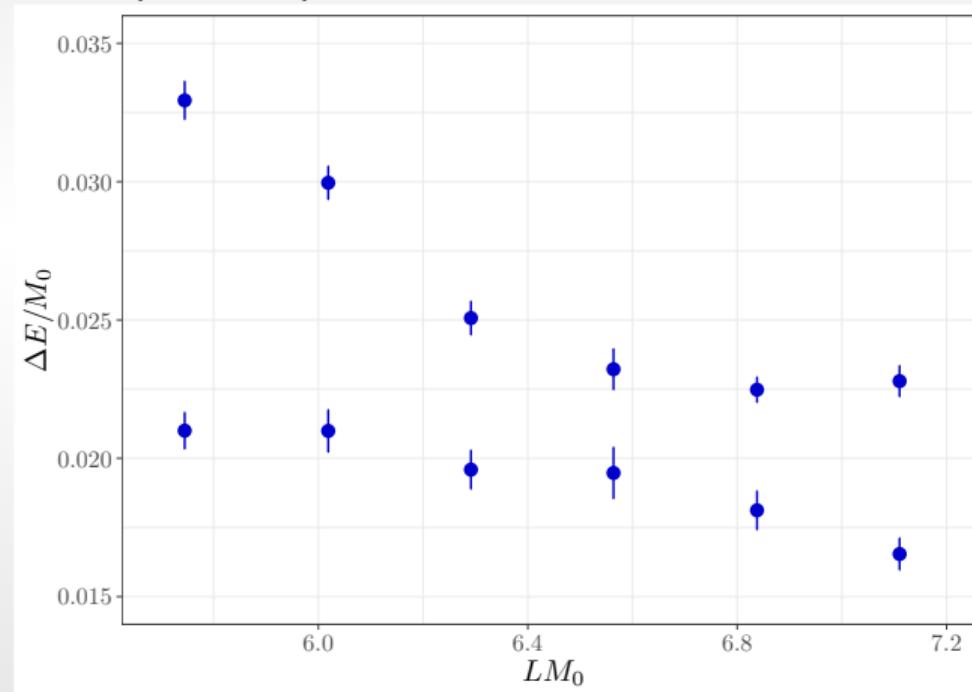
$$c(t) = \begin{pmatrix} \langle \tilde{\phi}_0^\dagger(t)^3 \tilde{\phi}_0(0)^3 \rangle & \langle \tilde{\phi}_0^\dagger(t)^3 \tilde{\phi}_1(0) \rangle \\ \langle \tilde{\phi}_1^\dagger(t) \tilde{\phi}_0(0)^3 \rangle & \langle \tilde{\phi}_1^\dagger(t) \tilde{\phi}_1(0) \rangle \end{pmatrix}$$

and we solve the GEVP [B. Blossier, M. Della Morte, G von Hippel, T. Mendes, R. Sommer (2009)]

$$C(t)v_n = \lambda(t)C(t_0)v_n$$

- The eigenvalues are $\lambda_n \sim e^{-(t-t_0)E_3^n}$

- Simulation parameters: $\kappa_0 = 0.14771$, $\kappa_1 = 0.131062$, $g = 10$
- $T = 64$, $L = 21 - 26$
- Three-particle spectrum



- We try two Ansatz for the three-body force K_3

- Resonance

$$K_3 = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$$

- No-resonance $c_0 = 0$ and ϕ_1 a stable particle

$$K_3 = c$$

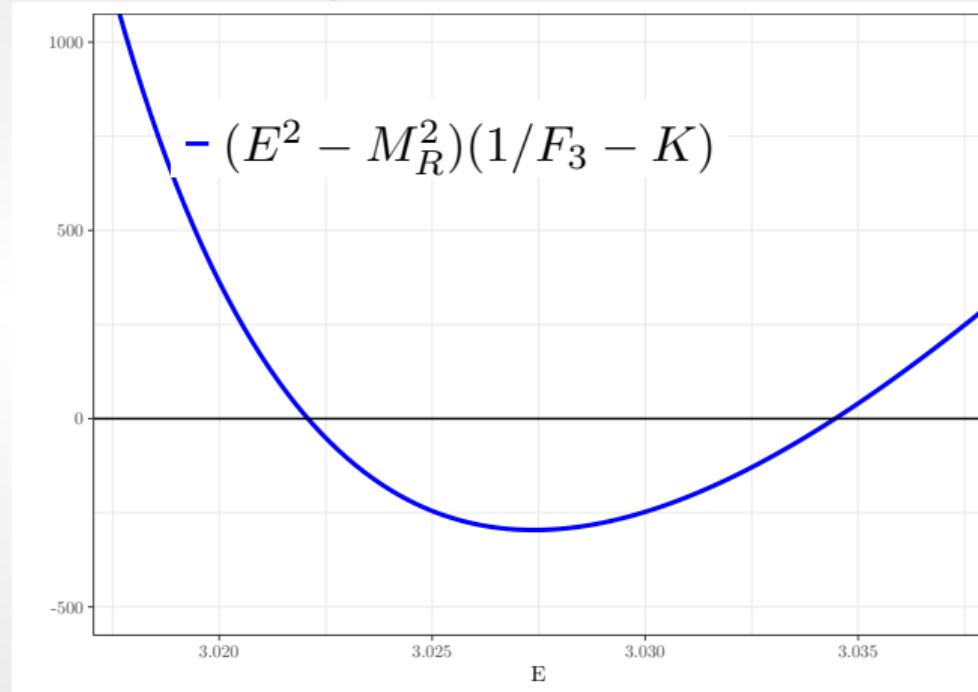
- In order to change smoothly between the two Ansatz we multiply the quantization condition by the pole of K_3 , e.g. in RFT

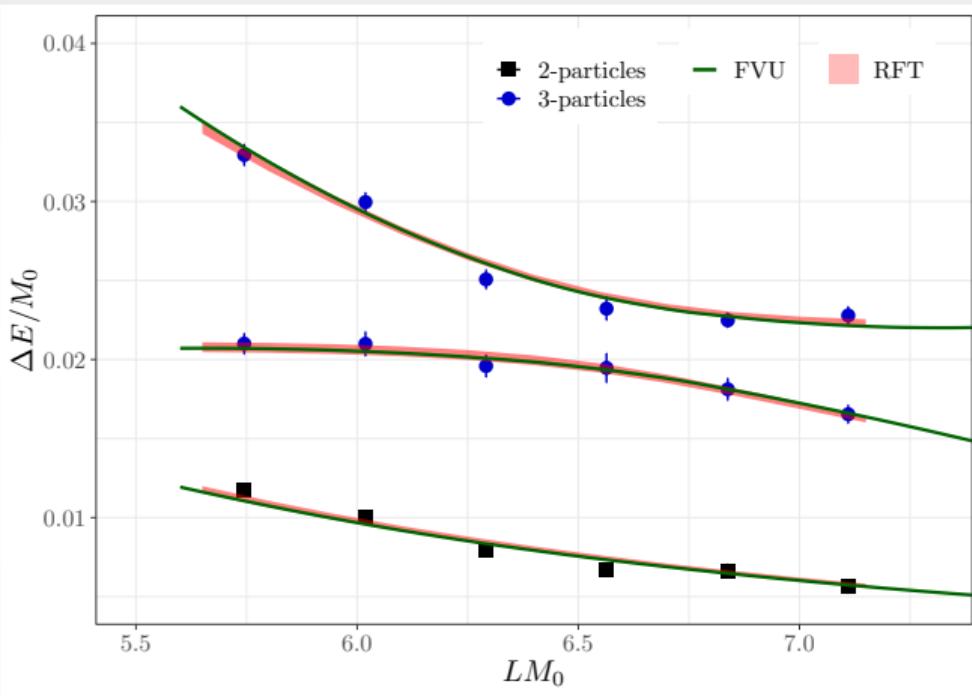
$$(E^2 - m_1^2) \left(1/F_3^{iso} + K_{df,3}^{iso} \right) = 0$$

so

$$\begin{cases} c_0 > 0 \implies E = m_1 \text{ is not a solution} \\ c_0 = 0 \implies E = m_1 \text{ is a solution} \end{cases}$$

- Solutions of the quantization condition





Two-particle

$$\frac{q^*}{M_0} \cot \delta = \frac{1}{a_0 M_0}$$

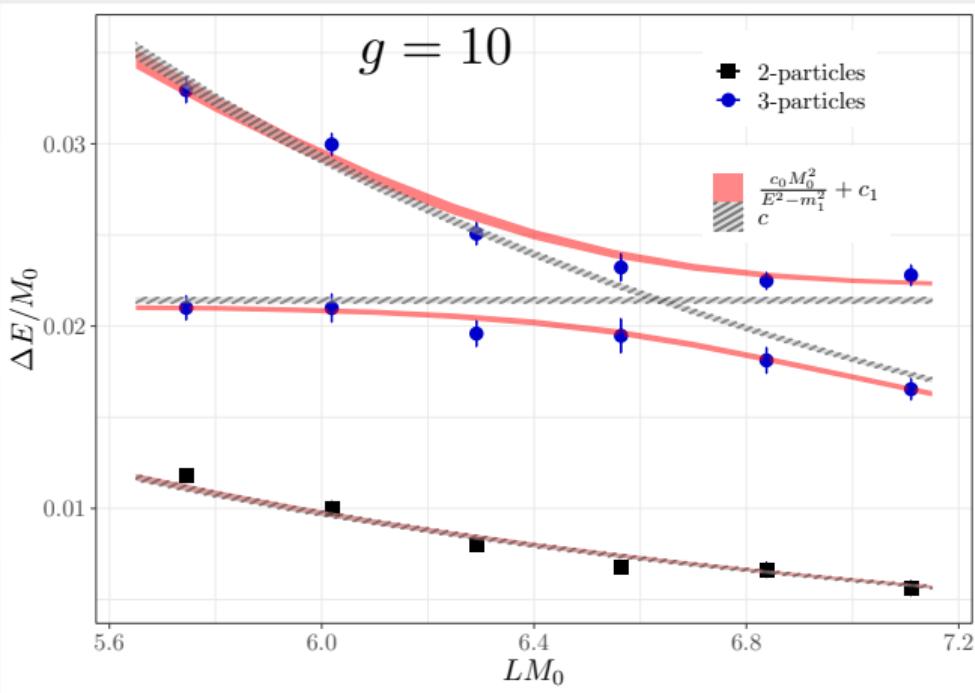
Three-particle

$$K_{df,3}^{iso} = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$$

- fit:

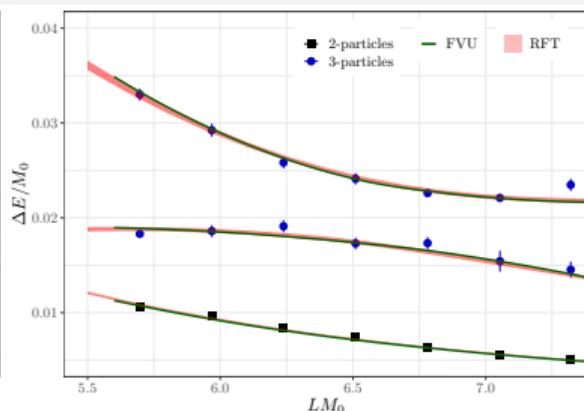
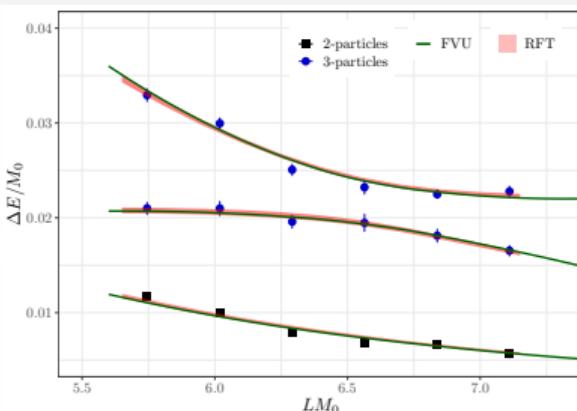
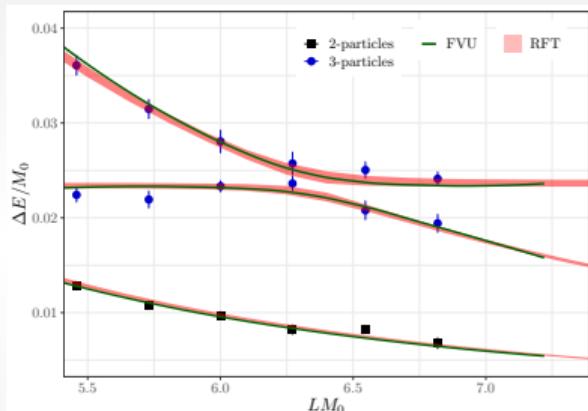
$$\chi^2/d.o.f. \sim 1.5$$

- Very similar prediction of the energy levels between FVU and RFT



- We also try explicitly the fit with $c_0 = 0$, no resonance and a stable particle ϕ_1 $\chi^2/d.o.f \sim 3.1$
- $c_0 > 0$ reproduce better the data $\chi^2/d.o.f. \sim 1.5$

- We simulate also $g = 5, 10, 20$

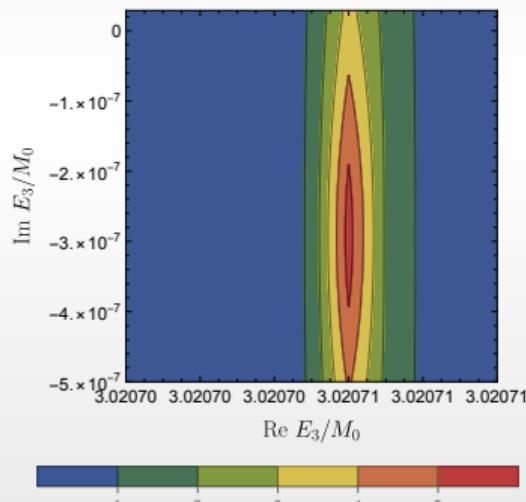


- $\chi^2/d.o.f. \in [1, 1.5]$
- Avoid level crossing more visible with larger g

- Compute the scattering amplitude \mathcal{M} from the three body force

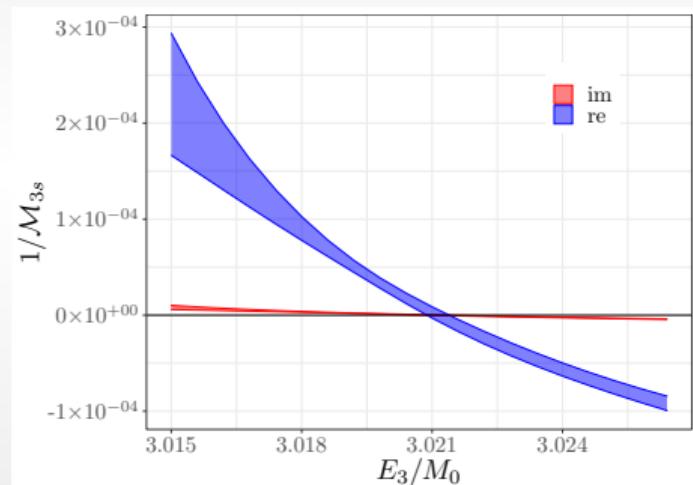
- FVU

$$\mathcal{T}_3 = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_\ell} \frac{1}{\tilde{K}_2^{-1} - \Sigma_n} \mathcal{T}_3$$



- RFT

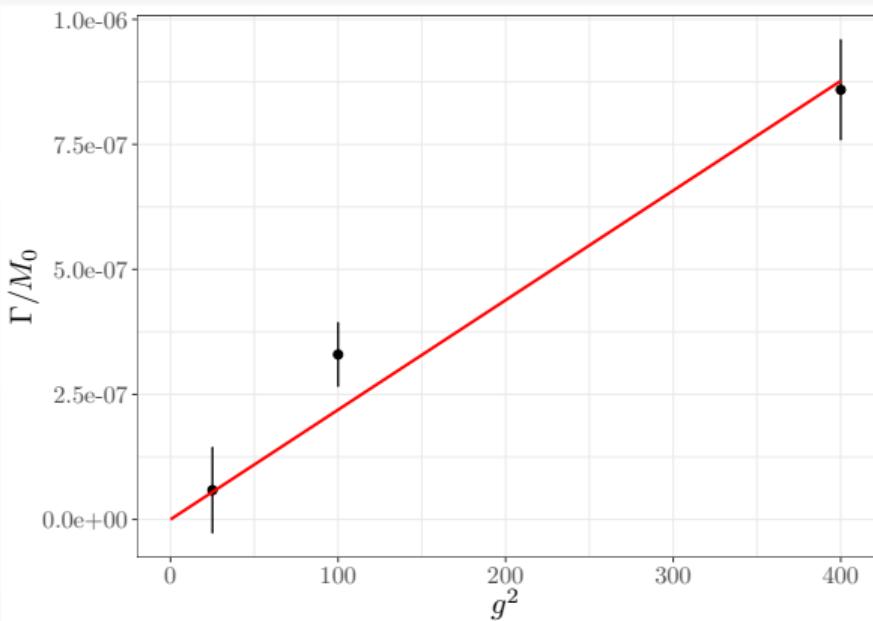
$$\mathcal{M}_{3s}(E) = \frac{\sum_i \mathcal{L}(\vec{p}_i) \mathcal{R}(\vec{p}_i)}{1/K_3 + F^\infty}$$



- Fit F^∞ as a polynomial in the energy
- Find the pole as $1/K_3 + F^\infty = 0$

- Near resonance we expect the amplitude

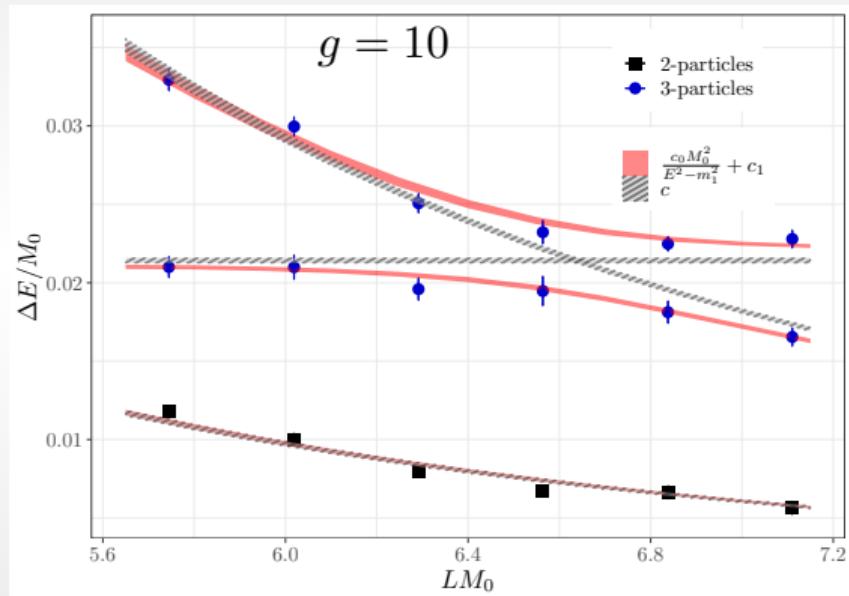
$$\mathcal{M}_3 = \frac{c}{E - M_R + i\Gamma/2}$$



- We find a very small width $\Gamma/M_0 \sim 10^{-7}$
- $\Gamma = \frac{1}{2M_R} \frac{1}{s!} \int dQ_3 |\mathcal{M}_{1 \rightarrow 3}|^2$
- $\int dQ_3 \sim 10^{-7}$

Conclusion

- $g > 0$ evidence of a resonance and the avoided level crossing
- The three-particle quantization condition describe with good χ^2 the energy level
- Extract the pole position in the scattering amplitude \mathcal{M}_3



Introduction
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$g = 0$
ooo

$g > 0$
oooooooo

Pole of \mathcal{M}_3
oo

Conclusion
oo●

Thank you for your attention

Backup slides

- Simulation parameters:

κ_0	κ_1	g
0.148522	0.134228	0
0.147957	0.131234	5
0.147710	0.131062	10
0.147145	0.131062	20

- $T = 64$, $L = 20 - 27$