Scattering from generalised ϕ^4

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Three-particle quantization

- Three approaches can be used in in lattice QCD to study scattering process of three particle
  - Relativistic fields theory (RFT) [M. T. Hansen , S. R. Sharpe (2014)]
  - Non-relativistic effective field theory (NREFT) [H. W. Hammer , J. Y. Pang , A. Rusetsky (2017)] is algebraically equivalent to FVU. A separate check of NREFT is thus not needed.
  - Finite-volume unitarity (FVU) [M. Mai, M. Döring (2017)]
- We are testing the RFT and FVU approach.

- There have been studies of systems with no resonances e.g.:
  [T. D. Blanton et al.(2021)] [T. D. Blanton et al.(2019)]
  [Hadron Spectrum Collaboration M. T. Hansen et al. (2020)]
  [M. Fischer at al. (2020)]
  [A. Alexandru et al.(2020)]
  [C. Culver et al. (2019)]
  [B. Hörz and A. Hanlon (2019)]
- Our aim is to study whether the formalism is usable for resonances in a toy model
  [GWQCD Collaboration, M. Mai et al. (2021)]

model 3-particle resonance

- We use complex field
- masses $3m_0 < m_1 < 5m_0$

$$S = \int dx \sum_{i=0,1} \left[\frac{1}{2}\partial_\mu\varphi_i^\dagger\partial_\mu\varphi_i + \frac{1}{2}m_i\varphi_i^\dagger\varphi_i + \lambda_i(\varphi_i^\dagger\varphi_i)^2\right] + \frac{g}{2}\varphi_1^\dagger\varphi_0^3 + h.c.$$

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- At $g = 0$
  - $U(1) \quad U(1)$ global symmetry $\varphi_0 \to e^{i\theta} \varphi_0 \quad \varphi_1 \to e^{i\theta'} \varphi_1$
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- With $g > 0$
  - φ_1 becomes unstable
  - residual global symmetry is $\varphi_0 \to e^{i\theta}\varphi_0$ together with $\varphi_1 \to e^{i3\theta}\varphi_1$
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- The residual symmetry avoid mixing $\varphi_0 \leftrightarrow \varphi_0^3$
- The model is most likely trivial. However, with small but finite lattice spacing the model effectively describes an interacting continuum field theory.

- Lattice discretization with $\partial_\mu \phi(x) = \phi(x+\mu) - \phi(x)$, $m_i = \frac{1-2\lambda_i}{\kappa_i} - 8$, $\hat{\lambda}_i = \frac{\lambda_i}{4\kappa_i^2}$, $\hat{g} = \frac{1}{4\sqrt{\kappa_0 \kappa_1^3}}$ and $\varphi_i = \sqrt{2\kappa_i}\phi_i$

$$S = \sum_x \sum_{i=0,1} \left[\kappa_i \sum_\mu \phi_i^\dagger(x)\phi_i(x+\mu) + \hat{\lambda}_i(\phi_i^\dagger(x)\phi_0(x) - 1) + \phi_i^\dagger \phi_i(x) + h.c. \right] + \frac{\hat{g}}{2}\phi_1^\dagger \phi_0^3 + h.c.$$

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- The limit $\hat{\lambda} \to \infty$ restrict the filed to satisfy

$$\phi_i^\dagger(x)\phi_i(x) = 1$$

and thus the fields can be described with a phase $\phi_i = e^{i\theta_i}$

- The action becomes

$$S = \sum_x \sum_{i=0,1} \left[\kappa_i \sum_\mu \phi_i^\dagger(x)\phi_i(x+\mu) + h.c. \right] + \frac{\hat{g}}{2}\phi_1^\dagger \phi_0^3 + h.c.$$

Monte Carlo simulation

- Metropolis-Hastings algorithm to generate ensembles
  - We did not implement more advanced algorithm such Cluster Algorithm [U. Wol (1989)] since we are simulating at values of m_0 not too small and it is difficult to parallelize.
- Implementation with Kokkos to have a performance portable implementation [H. C. Edwards, C. R. Trott, D. Sunderland (2014)]

Spectrum $g = 0$

- Single particle energy level

$$\tilde{\phi}_i^\dagger(t)\tilde{\phi}_i(0) \qquad |A_{\phi_i \to 0}| \left(e^{-E_1 t} - e^{-E_1 (T-t)}\right)$$

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- Two particle energy level

$$\tilde{\phi}_0^\dagger(t)^2\tilde{\phi}_0(0)^2 \qquad |A_{2\phi_0 \to 0}| \left(e^{-E_2 t} - e^{-E_2(T-t)}\right) + |A_{\phi_0 \to \phi_0}| e^{-E_1 T}$$

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- Three particle energy level

$$\tilde{\phi}_0^\dagger(t)^3\tilde{\phi}_0(0)^3 \qquad |A_{3\phi_0 \to 0}| \left(e^{-E_3 t} - e^{-E_3(T-t)}\right)$$
$$+ |A_{2\phi_0 \to \phi_0}| e^{-E_1 T} \left(e^{-t(E_2 - E_1) + e^{-(T-t)(E_2 - E_1)}}\right)$$

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$$+|A_{2\phi_0\to\phi_0}|e^{-E_1 T}\left(e^{-t(E_2-E_1)} + e^{-(T-t)(E_2-E_1)}\right)$$

- We simulate at T large enough to neglect finite volume pollution (T=64)

- FVU

$$\det \left[B + C - 2L^3 E_{\mathbf{p}} \left(\tilde{K}_2^{-1} - \Sigma_2^L \right) \right] = 0$$

- RFT

$$\det \left[K_{df,3} + L^3 \left(\tilde{F}/3 - \tilde{F}(K_2^{-1} + \tilde{F} + \tilde{G})^{-1} \tilde{F} \right)^{-1} \right] = 0$$

- Both depend on the spectator moment
- B, Σ_2^L, $\tilde{F}$ and $\tilde{G}$ can be computed from the finite volume spectrum
- $\tilde{K}_2$ and K_2 related to the two-body scattering amplitude
- $C(s)$ and $K_{df,3}$ the three body forces, infinite-volume quantity related to the scattering amplitude

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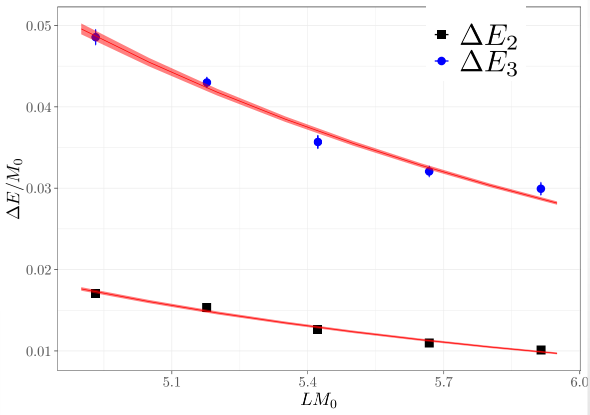
- $C(s)$ and $K_{df,3}$ the three body forces, infinite-volume quantity related to the scattering amplitude
- Isotropic approximation:
  - s-wave dominance
  - Three body force independent from the spectator momentum
- truncation of the momentum space.

Two-particle phase shift $\qquad \frac{q^*}{M_0} \cot\delta = \frac{1}{a_0 M_0}$

Three-particle force $\qquad K_3 = P_0$

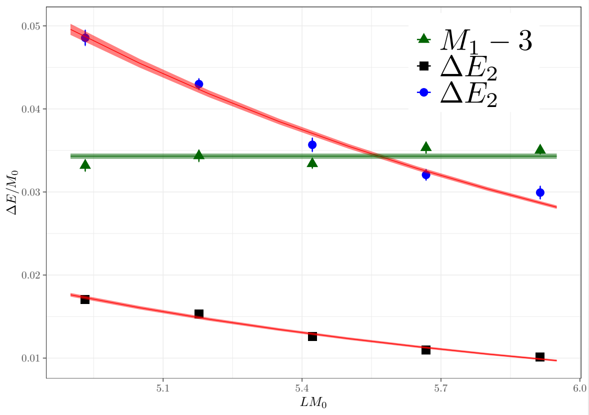


- fit:

$$\chi^2/d.o.f. = 1.8$$
$$P_0 = 1351(500)$$
$$a_0 M_0 = -0.1514(20)$$

- $a_0 M_0$ compatible with the only two-particle fit

Two-particle phase shift	$\frac{q^*}{M_0} \cot \delta = \frac{1}{a_0 M_0}$
Three-particle force	$K_3 = P_0$
One heavy-particle ϕ_1 mas	M_1=const



- fit:

$$\chi^2/d.o.f. = 1.8$$
$$P_0 = 1351(500)$$
$$a_0 M_0 = -0.1514(20)$$
$$M_1/M_0 = 3.0343(3)$$

- $a_0 M_0$ compatible with the only two-particle fit

Spectrum $g > 0$

- Single particle energy level

$$\tilde{\phi}_0^\dagger(t)\tilde{\phi}_0(0) \qquad |A_{\phi_0 \to 0}|e^{-M_0 t}$$

- Two particle energy level

$$\tilde{\phi}_0^\dagger(t)^2\tilde{\phi}_0(0)^2 \qquad |A_{2\phi_0 \to 0}|e^{-E_2 t}$$

- Three particle operator with the same quantum number $\tilde{\phi}_0^3$ and $\tilde{\phi}_1$ thus we measure the matrix

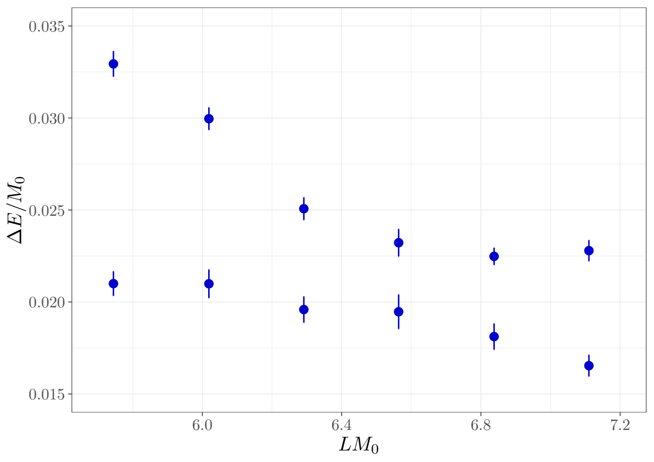
$$c(t) = \begin{pmatrix} \tilde{\phi}_0^\dagger(t)^3\tilde{\phi}_0(0)^3 & \tilde{\phi}_0^\dagger(t)^3\tilde{\phi}_1(0) \\ \tilde{\phi}_1^\dagger(t)\tilde{\phi}_0(0)^3 & \tilde{\phi}_1^\dagger(t)\tilde{\phi}_1(0) \end{pmatrix}$$

  and we solve the GEVP [B. Blossier, M. Della Morte, G von Hippel, T. Mendes, R. Sommer (2009)]

$$C(t)v_n = \lambda(t)C(t_0)v_n$$

- The eigenvalues are $\lambda_n \quad e^{-(t-t_0)E_3^n}$

- Simulation parameters: $\kappa_0 = 0.14771$, $\kappa_1 = 0.131062$, $g = 10$
- $T = 64$, $L = 21 - 26$
- Three-particle spectrum



- We try two Ansatz for the three-body force K_3
  - Resonance

$$K_3 = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$$

  - No-resonance $c_0 = 0$ and ϕ_1 a stable particle

$$K_3 = c$$

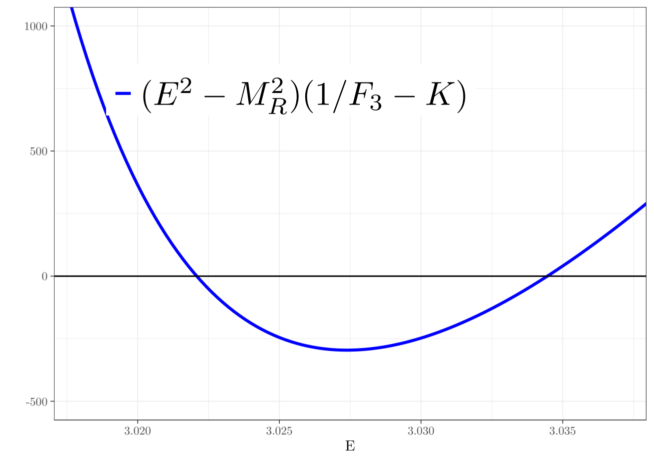
- In order to change smoothly between the two Ansatz we multiply the quantization condition by the pole of K_3, e.g. in RFT

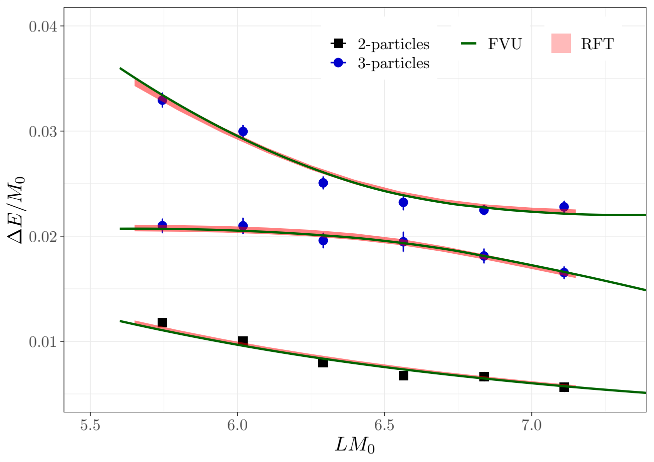
$$(E^2 - m_1^2)\left(1/F_3^{iso} + K_{df,3}^{iso}\right) = 0$$

so

$$\begin{cases} c_0 > 0 & = \quad E = m_1 \text{ is not a solution} \\ c_0 = 0 & = \quad E = m_1 \text{ is a solution} \end{cases}$$

- Solutions of the quantization condition



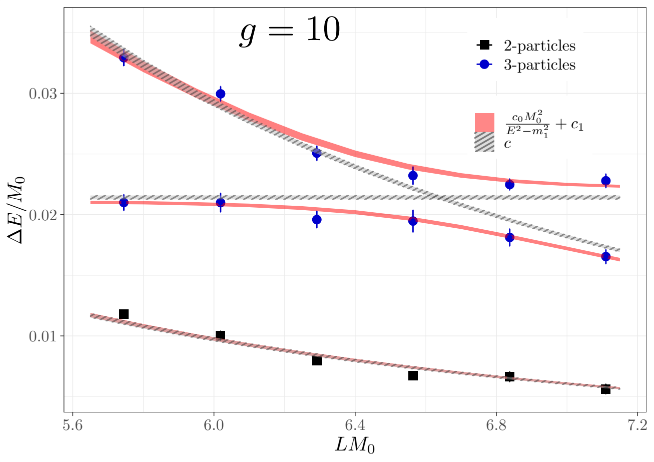


Two-particle	$\frac{q^*}{M_0}\cot\delta = \frac{1}{a_0 M_0}$
Three-particle	$K^{iso}_{df,3} = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$

- fit:

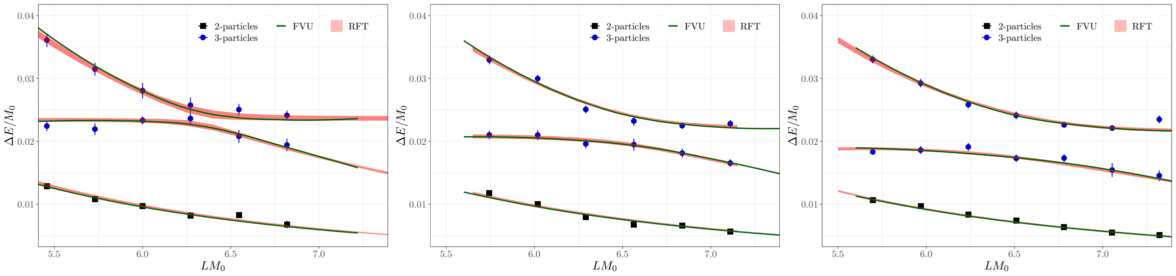
$$\chi^2/d.o.f. \quad 1.5$$

- Very similar prediction of the energy levels between FVU and RFT



- We also try explicitly the fit with $c_0 = 0$, no resonance and a stable particle ϕ_1 $\chi^2/d.o.f$ 3.1
- $c_0 > 0$ reproduce better the data $\chi^2/d.o.f.$ 1.5

- We simulate also $g = 5, 10, 20$

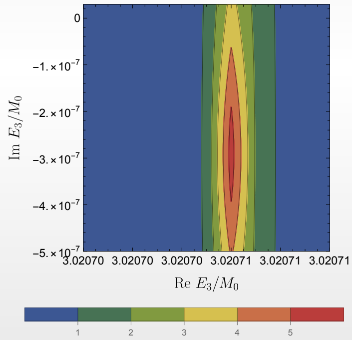


- $\chi^2/d.o.f. \quad [1, 1.5]$
- Avoid level crossing more visible with larger g

- Compute the scattering amplitude M from the three body force

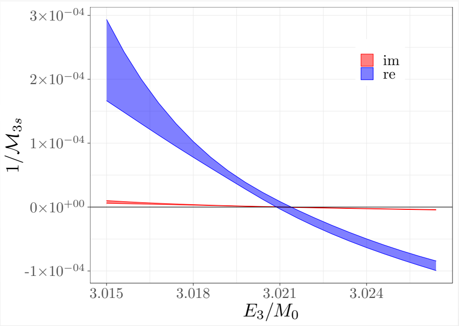
- FVU

$$T_3 = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_\ell} \frac{1}{\tilde{K}_2^{-1} - \Sigma_n} T_3$$



- RFT

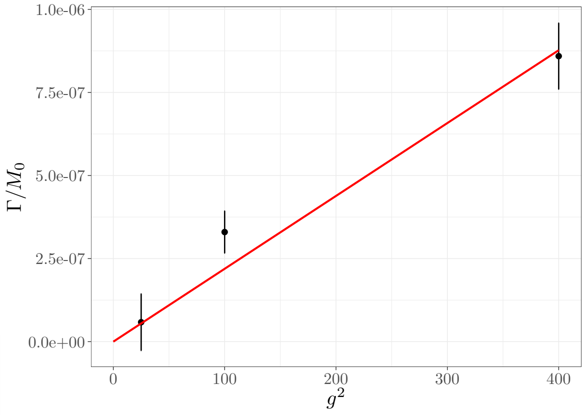
$$M_{3s}(E) = \frac{\sum_i L(\vec{p}_i) R(\vec{p}_i)}{1/K_3 + F^\infty}$$



- Fit F^∞ as a polynomial in the energy
- Find the pole as $1/K_3 + F^\infty = 0$

- Near resonance we expect the amplitude

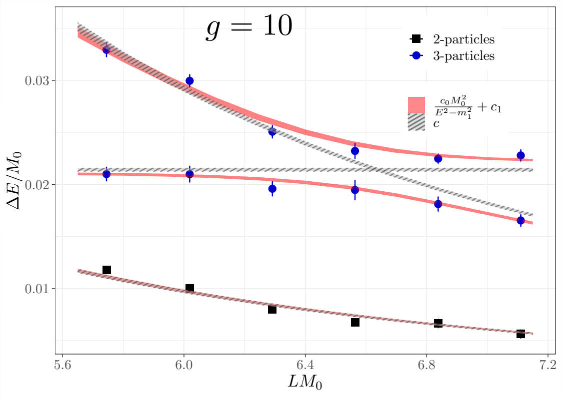
$$M_3 = \frac{c}{E - M_R + i\Gamma/2}$$



- We find a very small width $\Gamma/M_0 \quad 10^{-7}$
- $\Gamma = \frac{1}{2M_R}\frac{1}{s!}\int dQ_3 |M_{1\to 3}|^2$
- $\int dQ_3 \quad 10^{-7}$

Conclusion

- $g > 0$ evidence of a resonance and the avoided level crossing
- The three-particle quantization condition describe with good χ^2 the energy level
- Extract the pole position in the scattering amplitude M_3



Thank you for your attention

Backup slides

- Simulation parameters:

κ_0	κ_1	g
0.148522	0.134228	0
0.147957	0.131234	5
0.147710	0.131062	10
0.147145	0.131062	20

- $T = 64$, $L = 20 - 27$