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Finite-volume formalism on the t-channel cut

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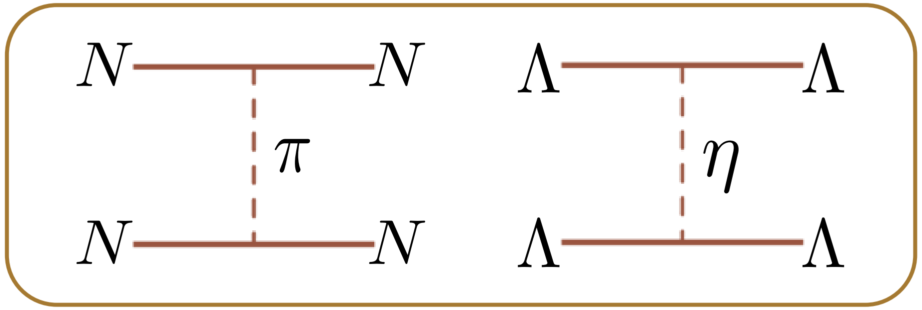
Introduction and Motivation

- Progress has been made in recent years in the study of baryon-baryon elastic scattering on the lattice, e.g. $NN \rightarrow NN$ and $\Lambda\Lambda \rightarrow \Lambda\Lambda$ *e.g.* [Green, Hanlon, Junarkar, Wittig, 2021]
- These include *t*-channel exchanges of lighter mesons:

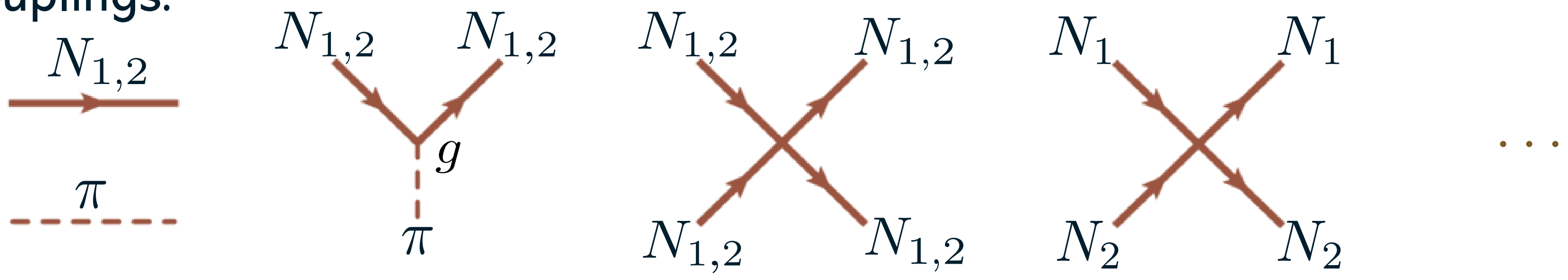


- *t*-channel exchanges lead to branch cuts just below threshold when analytically continuing angular-momentum projected amplitudes
- Potential challenges to finite-volume formalism

Our aim: adapting FV 2→2 scattering formalism so it can be applied on the *t*-channel cut

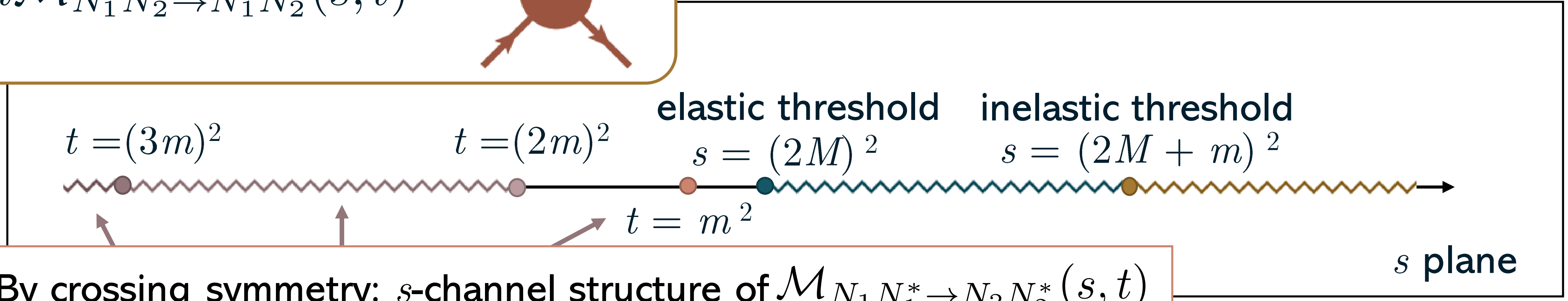
Toy model theory

- Theory with two complex scalars N_1, N_2 of mass M and lighter real scalar π of mass m with the couplings:



- Focus on $N_1 N_2 \to N_1 N_2$ amplitude:

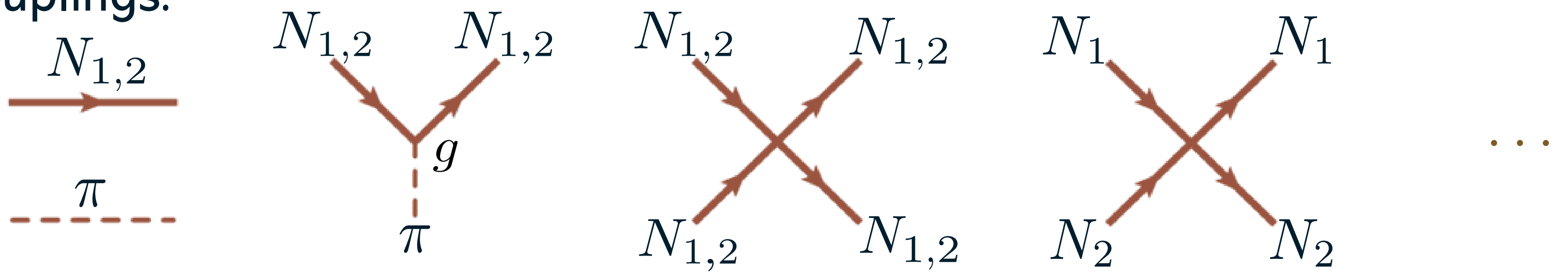
$$i\mathcal{M}_{N_1 N_2 \to N_1 N_2}(s,t) =$$



By crossing symmetry: s-channel structure of $\mathcal{M}_{N_1 N_1^* \to N_2 N_2^*}(s,t)$

Toy model theory

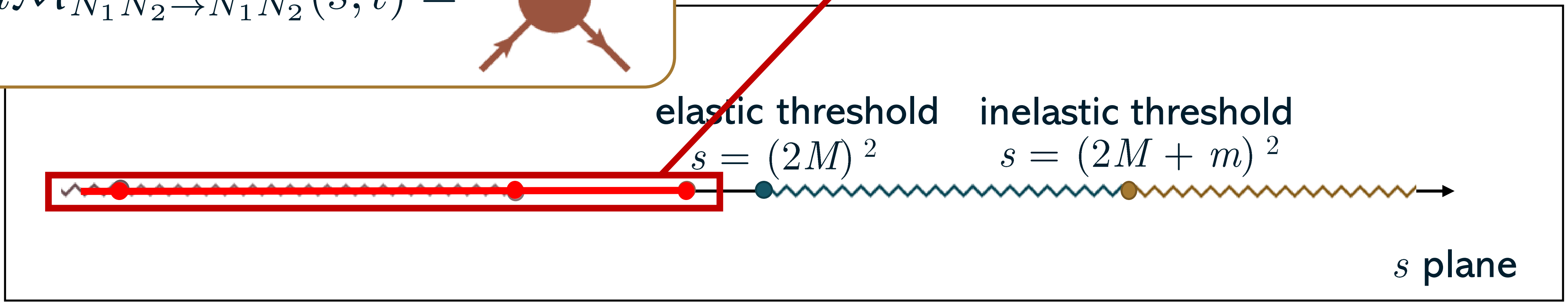
- Theory with two complex scalars N_1, N_2 of mass M and lighter real scalar π of mass m with the couplings:



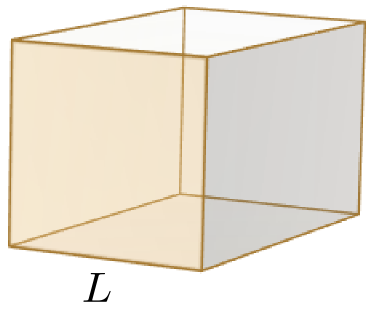
- Focus on $N_1 N_2 \to N_1 N_2$ amplitude:

$$i\mathcal{M}_{N_1 N_2 \to N_1 N_2}(s,t) =$$

AM projection: t-channel cut $s \leq (2M)^2 - m^2$



FV formalism for 2→2 scattering



Setting up theory in finite-volume:

- Periodic cubic spatial volume of side L
- L large enough to neglect $\mathcal{O}\left(e^{-mL}\right)$
- Finite time extent T and discretisation effects negligible

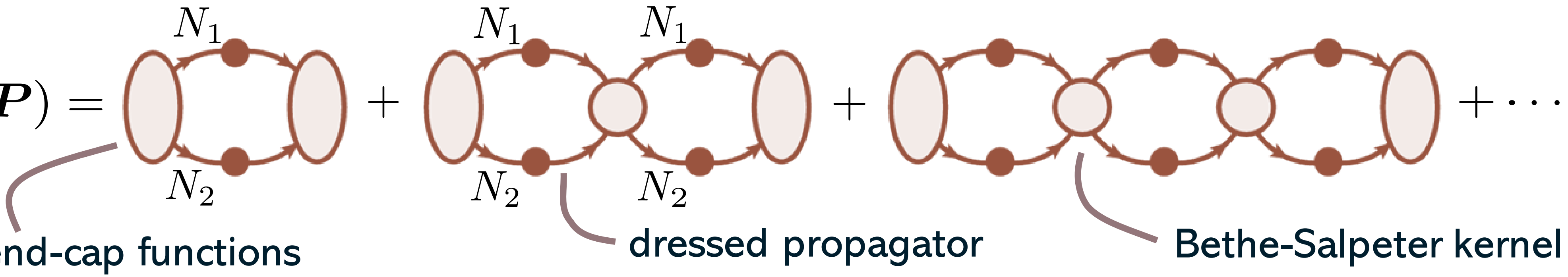
Discretised momenta: $\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{n}\,, \ \ \boldsymbol{n} \in \mathbb{Z}^3$

Discretised spectrum:

Lüscher scattering formalism: FV spectrum $E_0(L),\ E_1(L),\ \ldots$ → IV amplitude $\mathcal{M}$

Start from the finite-volume correlator: has poles at FV energy levels

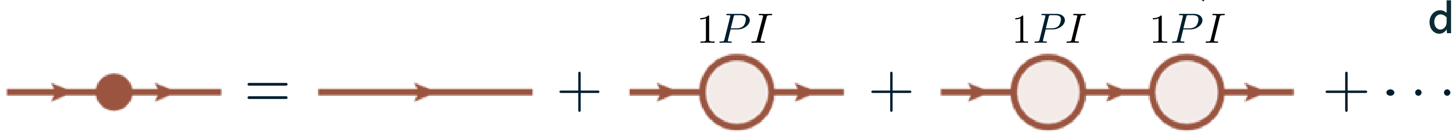
$$C_L(E, \boldsymbol{P}) =$$



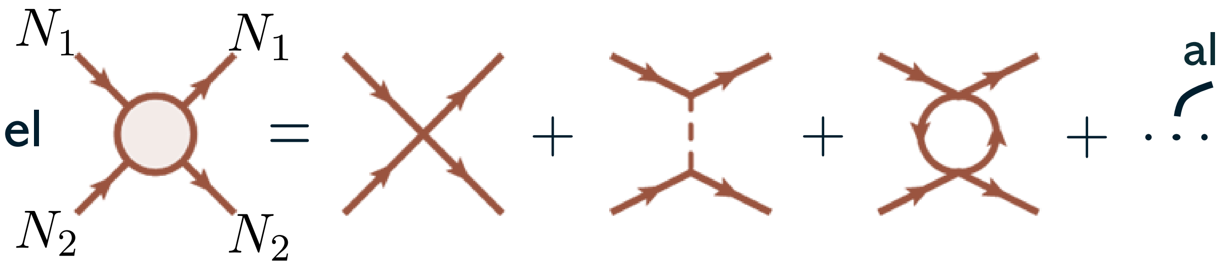
[Lüscher, 1986; Kim, Sachrajda, Sharpe 2005]

The skeleton expansion: ingredients

Dressed propagator



Bethe-Salpeter kernel



Loops!

$$\sum = \int + \left(\sum - \int\right)$$

FV loop = IV loop + Sum-integral difference

sum over discretised spatial loop momenta

$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] f(\boldsymbol{k})$$

- exponentially suppressed $\mathcal{O}\left(e^{-mL}\right)$ for smooth $f(\boldsymbol{k})$
- power-like decay in L otherwise

Intermediate states going on-shell lead to singularities.

If intermediate states cannot go on-shell:

$$\sum = \int$$

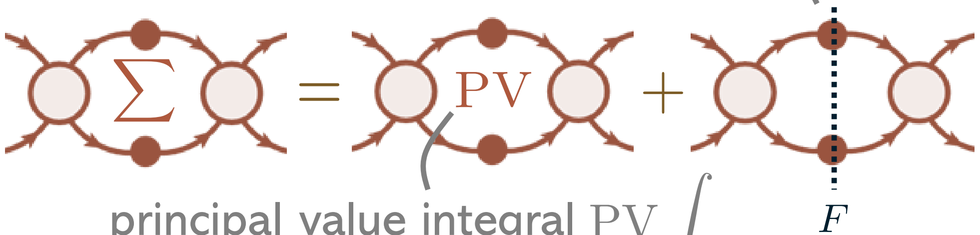
FV loop = IV loop

FV correlator in the elastic regime

Elastic regime: $(2M)^2 < s < (2M+m)^2$ — 3-particle inelastic threshold

- only 2N intermediate states can go on-shell
- BS kernel safe: replace with IV version
- cutting rule for remaining FV loops

F-cut: $\sum - \mathrm{PV}\int$



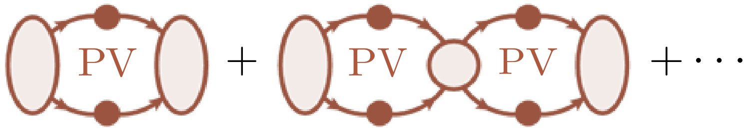
principal value integral $\mathrm{PV}\int$

Rearrange correlator: assume on-shell state dominance, project to definite AM

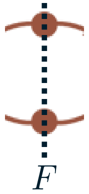
endcap overlap vectors

$$C_L(E,\boldsymbol{P}) = C_\infty^{PV}(E,\boldsymbol{P}) + iA(E,\boldsymbol{P})\frac{1}{F^{-1}(E,\boldsymbol{P};L)+\mathcal{K}(E,\boldsymbol{P})}A^\dagger(E,\boldsymbol{P})$$

IV correlator with PV prescription



matrix of known functions



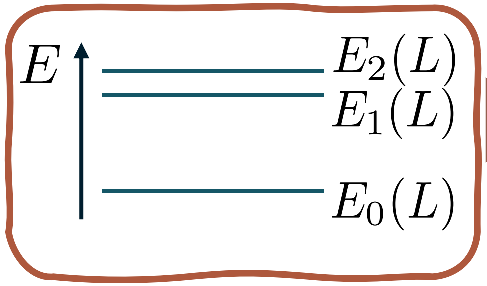
K-matrix

matrices and vectors in AM index space

Quantisation condition in the elastic regime

- Correlator poles give FV spectrum
- Lüscher quantisation condition follows

FV Spectrum



obtaining K-matrix

$$\det\left[F^{-1}(E_n, \boldsymbol{P}; L) + \mathcal{K}(E_n, \boldsymbol{P})\right] = 0$$

Lüscher Quantisation Condition

[Lüscher, 1986]

obtaining IV amplitude

$$\mathcal{M} = \frac{1}{\mathcal{K}^{-1} - i\rho}$$

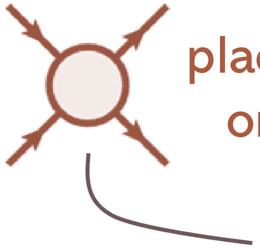
$$\rho = \frac{\sqrt{1 - 4M^2/s}}{32\pi}$$

- Works for our theory in elastic region $(2M)^2 < s < (2M+m)^2$
- Analytical continuation below threshold works above t-channel cut at $s = (2M)^2 - m^2$
- Inconsistencies on the cut – t-channel issue:
  - F-matrix produces a real value → K-matrix should be real
  - K-matrix has complex on cut!

[Green, Hanlon, Junarkar, Wittig, 2021]

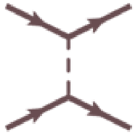
The t-channel problem

Root of the problem traced back to assumption of on-shell term dominance:

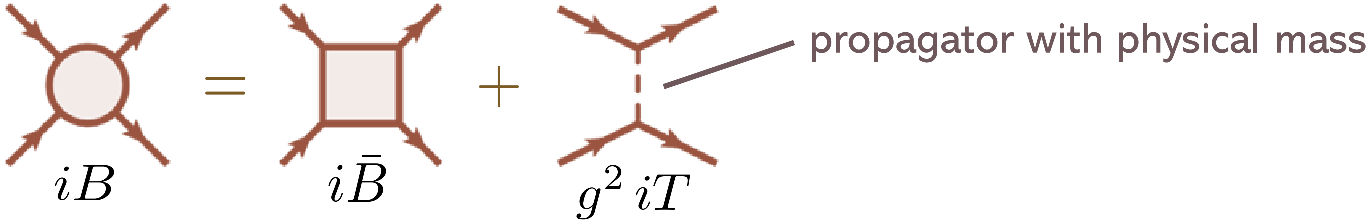


placed fully on-shell

- safe above threshold: no singular behaviour introduced
- not safe below threshold: introduces t-channel singularities

recall that BS kernel includes:  , ...

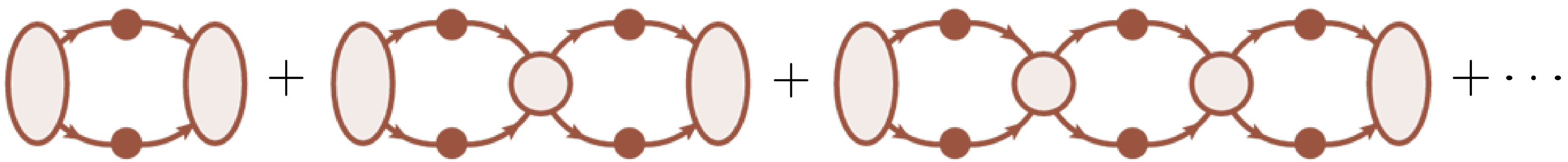
Proposed solution: first separate BS kernel

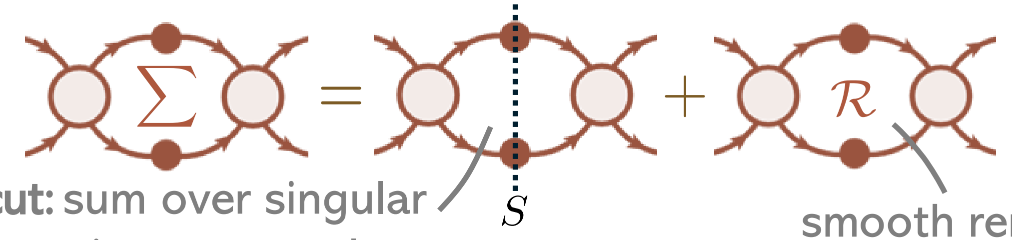


- $\bar{B}$ kernel is now safe when put on-shell for $s > (2M)^2 - (2m)^2$
- t-channel exchange is safe if kept partially off-shell
- Step back: how can we keep iT off-shell?

Cutting rule revisited

- Return to correlator

$C_L(E, \boldsymbol{P}) =$ 

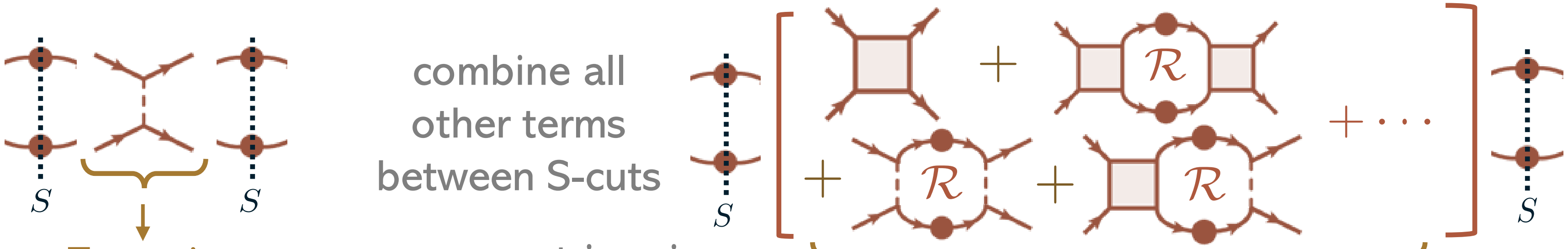
... but now keep the loop sums: 

S-cut: sum over singular term in energy only

smooth remainder term

- Project to definite AM, **keep loop momentum as an extra index**
- Rearrange correlator:

leave t-channel exchanges between S-cuts



combine all other terms between S-cuts

T matrix

matrices in [angular momentum x momentum] index space

K-bar matrix

Adapted quantisation condition

- We arrive at the form:

$$C_L(E,\boldsymbol{P}) = iA(E,\boldsymbol{P})\frac{1}{S^{-1}(E,\boldsymbol{P};L)+g^2T(E,\boldsymbol{P})+\bar{K}(E,\boldsymbol{P})}A^\dagger(E,\boldsymbol{P})+\mathcal{R}(E,\boldsymbol{P})$$

endcap overlap vectors

smooth remainder

matrix of known functions from the S-cut

also known functions*

K-bar matrix safe on-shell: put on-shell and leave momentum index redundant

- Quantisation condition follows:

$$\det\left[S^{-1}(E_n,\boldsymbol{P};L)+g^2T(E_n,\boldsymbol{P})+\bar{K}(E_n,\boldsymbol{P})\right]=0$$

- So can find K-bar matrix from FV spectrum…

but need to relate it to the amplitude!

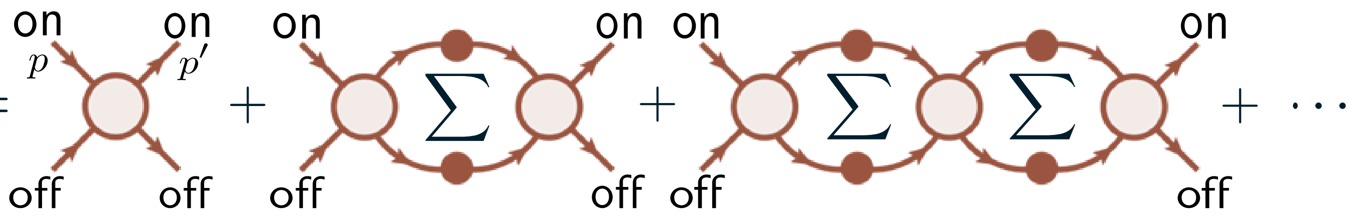
∗ s-wave components of the T-matrix:

$$T_{\boldsymbol{k}^*,\ell=0;\boldsymbol{k}'^*,\ell'=0}(E,\boldsymbol{P}) = -\frac{1}{|\boldsymbol{k}^*||\boldsymbol{k}'^*|}\log\left[\frac{2\omega_{\boldsymbol{k}^*}\omega_{\boldsymbol{k}'^*}-2|\boldsymbol{k}^*||\boldsymbol{k}'^*|-2M^2+m^2-i\epsilon}{2\omega_{\boldsymbol{k}^*}\omega_{\boldsymbol{k}'^*}+2|\boldsymbol{k}^*||\boldsymbol{k}'^*|-2M^2+m^2-i\epsilon}\right]$$

From FV to the amplitude

- Define the FV "partially off-shell amplitude" :

$$i\mathcal{M}_L\,(p, P-p; p', P-p') =$$



$p = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$ $\quad p' = (\omega_{\boldsymbol{p}'}, \boldsymbol{p}')$

- AM-projected version can be expressed in terms of T and K-bar matrices:

$$i\mathcal{M}_L = (i\bar{K} + ig^2T)\frac{1}{1+S(\bar{K}+g^2T)} = (i\bar{K} + ig^2T) + (i\bar{K} + ig^2T)iSi\mathcal{M}_L$$

- Related to IV scattering amplitude:

$$\mathcal{M}(P) = \lim_{\text{on-shell}} \lim_{L\to\infty} \mathcal{M}_L(p, P-p; p', P-p')$$

Summary

- Introduced theory to model t-channel cut
- Checked Lüscher formalism holds in elastic regime, but fails when analytically continued to the cut
- Identified the origin of the issue and set up an adapted method to derive a modified quantisation condition
- Tied quantisation condition objects back to the scattering amplitude

Outlook

- Finalise checks and generalisation of formalism for generic 2→2 scattering, including spin and prepare publication
- Perform mock data analysis (e.g. HADSPEC) using formalism
- Apply method to LQCD results with ground state on cut (e.g. $NN \to NN$ and $\Lambda\Lambda \to \Lambda\Lambda$)
- Explore potential connection to dispersive methods

Questions

….thank you for listening!