

Relativistic Invariance of the NREFT Three-Particle Quantization Condition

Fabian Müller

in collaboration with

Jin-Yi Pang, Akaki Rusetsky,

Jia-Jun Wu

LATTICE22

08.08.2022



Outline

- Introduction
- Problems in non-covariant framework
 - Parametrization of interactions
 - Non-covariance of three-particle propagator
- Relativistic invariant formulation
 - Construction
 - Three-particle quantization condition
- Test of invariance with synthetic data
- Conclusion and Outlook

infinite volume (IV):

- bound states,
elastic scattering
- resonance matrix
elements : $\langle \pi\pi\pi | H_W | K \rangle$

finite volume (FV):

- two- and three-particle
energy levels
- matrix elements between
eigenstates of box

■ NREFT Quantization condition:

FV energy spectrum \longleftrightarrow IV observables

- use QC to fit parameters of EFT Lagrangian to spectrum
- use parameters to calculate IV observables

■ Lorentz invariance **broken** in finite volume

Lorentz invariance constraints the form of interactions

Example: $2 \rightarrow 2$ scattering at tree level

underlying model: invariant

$$\#dof. = 3 \times (2+2) - \textcolor{orange}{10} = 2$$

on shell 4-vector
Poincare inv.

$$\rightarrow p, \cos \theta$$

$$K_r^{(2)} = -32\pi \textcolor{blue}{a}_0 + \frac{1}{2} \tilde{r}_0 p^2 + \mathcal{O}(p^4)$$

NREFT: non-invariant

$$\#dof. = 3 \times (2+2) - \textcolor{orange}{6} = 6$$

3-vector
total momentum cons. + rotation inv.

$$\rightarrow \mathbf{P}^2, \mathbf{p}^2, \mathbf{q}^2, \mathbf{Pp}, \mathbf{Pq}, \mathbf{pq}$$

$$K_{nr}^{(2)} = \textcolor{blue}{c}_0 + \textcolor{blue}{c}_1(\mathbf{p}^2 + \mathbf{q}^2) + \textcolor{blue}{c}_2 \mathbf{P}^2 + \mathcal{O}(p^4)$$

redundant / frame-depended parameters!

or inconvenient matching:

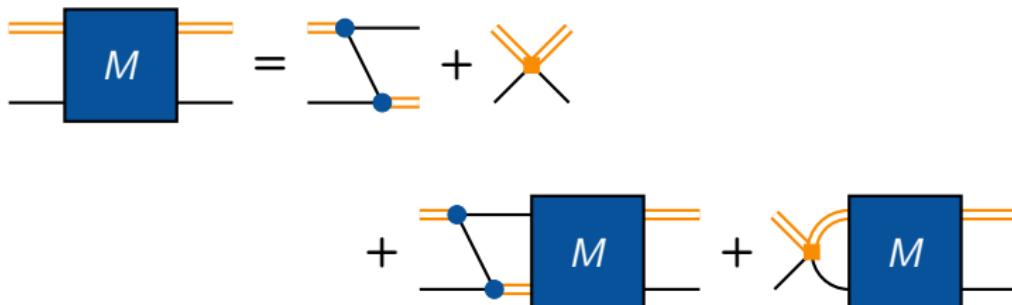
$$\prod_{i=1}^4 \sqrt{2w(\mathbf{p}_i)} K_{nr}^{(2)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = K_r^{(2)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4)$$

non-covariant factors $\sqrt{2w(\mathbf{p}_i)}$ mixing different orders

Problems

Non-Covariance of Three-Particle Propagator

Particle-dimer amplitude obeys Faddeev equation:



$$M(p, q; P) = Z(p, q; P) + \int \frac{d^3 k}{(2\pi)^3 2w(k)} Z(p, k; P) \tau(k; P) M(k, q; P)$$

Problems

Non-Covariance of Three-Particle Propagator

Particle-dimer amplitude obeys Faddeev equation:



$$M(p, q; P) = Z(p, q; P) + \int \frac{d^3 k}{(2\pi)^3 2w(k)} Z(p, k; P) \tau(k; P) M(k, q; P)$$

Kernel Z contains three-particle propagator:

not Lorentz invariant

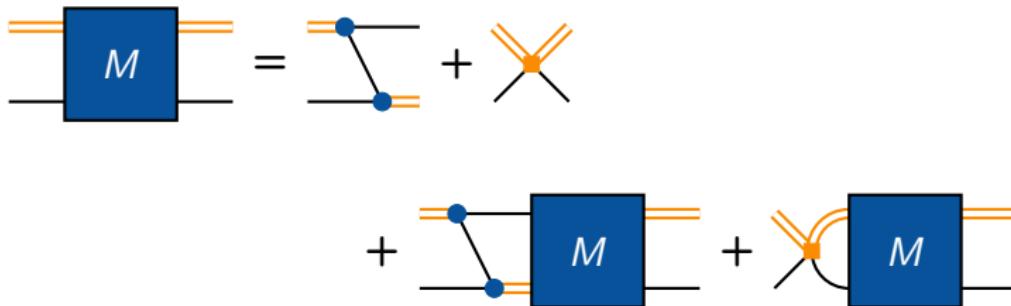
A Feynman diagram showing a blue dot connected to a black line, which then splits into two grey lines. This diagram is followed by a symbol \sim and a fraction.

$$\sim \frac{1}{2w(\mathbf{P} - \mathbf{p} - \mathbf{q})(w(\mathbf{P} - \mathbf{p} - \mathbf{q}) + w(\mathbf{p}) + w(\mathbf{q}) - P^0 - i\varepsilon)}$$

Problems

Non-Covariance of Three-Particle Propagator

Particle-dimer amplitude obeys Faddeev equation:



$$M(p, q; P) = Z(p, q; P) + \int \frac{d^3 k}{(2\pi)^3 2w(k)} Z(p, k; P) \tau(k; P) M(k, q; P)$$

Amplitude M is not Lorentz invariant!

⇒ IV observables not the same in different frames

Attempt:add *anti-particle-contribution*

$$\frac{1}{2w(\mathbf{K})(-w(\mathbf{K}) + w(\mathbf{p}) + w(\mathbf{q}) - P^0)}$$

$$\frac{1}{2w(\mathbf{K})(w(\mathbf{K}) + w(\mathbf{p}) + w(\mathbf{q}) - P^0)} \rightarrow \frac{1}{K^2 - m^2}$$

$$K^\mu = (P - p - q)^\mu$$

Problem:additional poles in *anti-particle-contribution*

⇒ loss of decoupling of low- and high-momentum regimes

⇒ breaking of unitarity (IV) & spurious subthreshold poles (FV)

cure by adjusting cutoff ⇒ cutoff not removable anymore

Relativistic Invariant Formulation

Construction

we need:

- Lorentz-invariant matching condition → covariant NREFT

$$\phi^\dagger(i\partial_t - M + \dots)\phi \rightarrow \phi^\dagger 2w(i\partial_t - w)\phi, \quad w = \sqrt{m^2 - \nabla^2}$$

\swarrow new normalization → cancels $(2w(p_i))^{1/2}$ in matching

- formally invariant propagator → arbitrary frame v^μ

$$\phi^\dagger 2w(i\partial_t - w)\phi \rightarrow \phi^\dagger 2w_v(i(v \cdot \partial) - w_v)\phi, \quad w_v = \sqrt{m^2 + \partial_\perp^2}$$
$$\partial_\perp^\mu = \partial^\mu - v^\mu(v \cdot \partial)$$

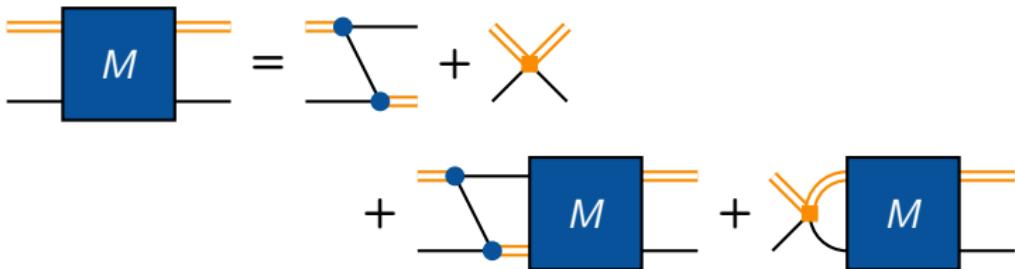
$$D_v(k) = \frac{1}{2w_v(k)(w_v(k) - v \cdot k - i\varepsilon)}, \quad w_v(k) = \sqrt{m^2 - k^2 + (v \cdot k)^2}$$

- propagator formally Lorentz-invariant
- antiparticle pole absent (integrated out in EFT)

Relativistic Invariant Formulation

Construction

Particle-dimer scattering:



$$\boxed{\Lambda^2 + k^2 - (\nu k)^2 \geq 0}$$

$$M(p, q) = Z(p, q) + \int \frac{d^4 k}{(2\pi)^4 i} Z(p, k) \frac{\tau((P - k)^2)}{2w_\nu(k)(w_\nu(k) - \nu k)} M(k, q)$$

Lorentz invariance:

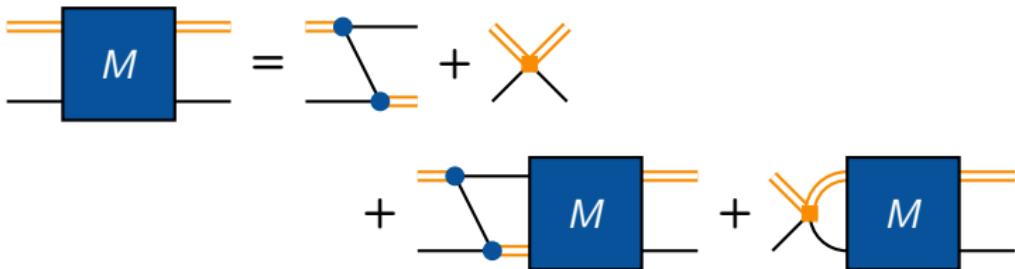
boost into the frame where $\nu k = \nu' k' = k'_0$, $w_\nu(k) = w(\mathbf{k}')$

$$M(p, q) = Z(p, q) + \int \frac{d^3 \mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} Z(p, k) \tau((P - k)^2) M(k, q)$$
$$k^2 = k'^2 = m^2$$

Relativistic Invariant Formulation

Construction

Particle-dimer scattering:



$$\boxed{\Lambda^2 + k^2 - (\nu k)^2 \geq 0}$$

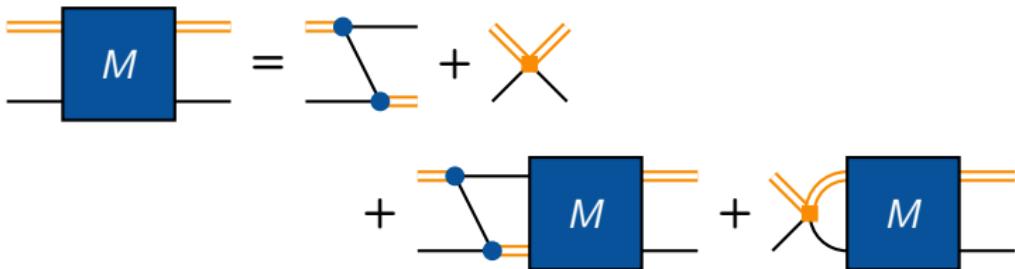
$$M(p, q) = Z(p, q) + \int_{\Lambda_v(k)}^{\Lambda_v(k)} \frac{d^4 k}{(2\pi)^4 i} Z(p, k) \frac{\tau((P - k)^2)}{2w_v(k)(w_v(k) - \nu k)} M(k, q)$$

Lorentz invariance:

boost into the frame where $\nu k = \nu' k' = k'_0$, $w_v(k) = w(\mathbf{k}')$

$$M(p, q) = Z(p, q) + \int_{\Lambda_v(k)}^{\Lambda_v(k)} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} Z(p, k) \tau((P - k)^2) M(k, q)$$
$$k^2 = k'^2 = m^2$$

Particle-dimer scattering:



$$\boxed{\Lambda^2 + k^2 - (\nu k)^2 \geq 0}$$

$$M(p, q) = Z(p, q) + \int_{\Lambda_\nu(k)} \frac{d^4 k'}{(2\pi)^4 i} Z(p, k) \frac{\tau((P - k)^2)}{2w(\mathbf{k}')(w(\mathbf{k}') - k'_0)} M(k, q)$$

Lorentz invariance:

boost into the frame where $\nu k = \nu' k' = k'_0$, $w_\nu(k) = w(\mathbf{k}')$

$$M(p, q) = Z(p, q) + \int_{\Lambda_\nu(k)} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} Z(p, k) \tau((P - k)^2) M(k, q)$$

$$k^2 = k'^2 = m^2$$

Relativistic Invariant Formulation

Three-Particle Quantization Condition

$$\int_{\Lambda_v(k)} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} \longrightarrow \int_{\Lambda_v(k)} \frac{d^3 \mathbf{k}}{(2\pi)^3 2w(\mathbf{k})}$$

Perform discretization in the lab frame and set $v^\mu = P^\mu$:

$$M_L(p, q) = Z(p, q) + \frac{1}{L^3} \sum_{\mathbf{k}} \frac{\Lambda_v}{2w(\mathbf{k})} Z(p, k) \tau_L((P - k)^2) M_L(k, q)$$

$\boxed{\Lambda^2 + m^2 - (vk)^2 \geq 0}$

Quantization Condition

$$\det(A) = 0$$

$$A_{\mathbf{pq}} = L^3 2w(\mathbf{p}) \delta_{\mathbf{pq}} \tau_L^{-1}((P - p)^2) - Z(p, q)$$

$$p, q \text{ on-shell and obey } \Lambda^2 + m^2 - (vk)^2 \geq 0$$

Test of Invariance

Setup:

consider EFT at LO in S -wave \Rightarrow 2 LECs:

- $\tau(s) = \frac{16\pi\sqrt{s}}{-a^{-1} - 8\pi\sqrt{s}J(s) - ip}$ (FV: $ip \rightarrow \frac{2}{\sqrt{\pi}L\gamma}Z_{00}^d(1; q_0^2)$)
- $K^{(3)} = \frac{H_0(\Lambda)}{\Lambda^2} \subset Z(p, k)$

Procedure:

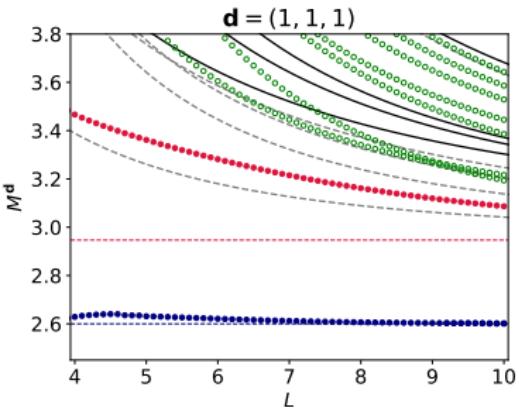
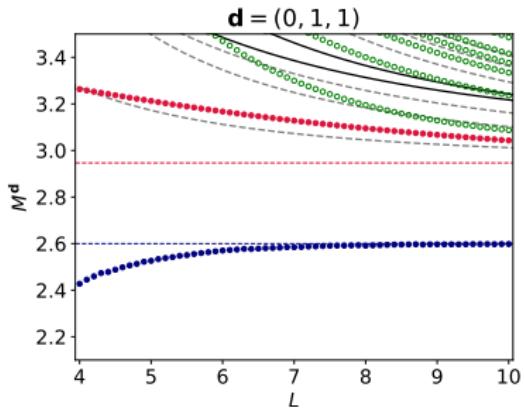
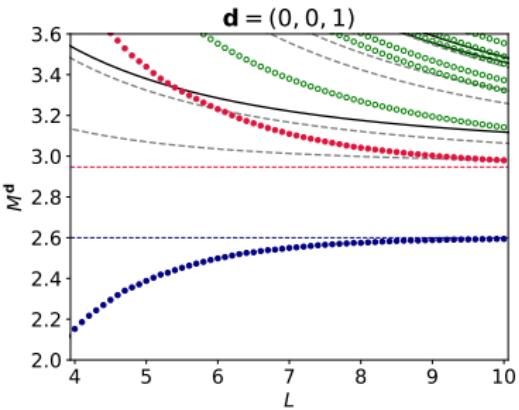
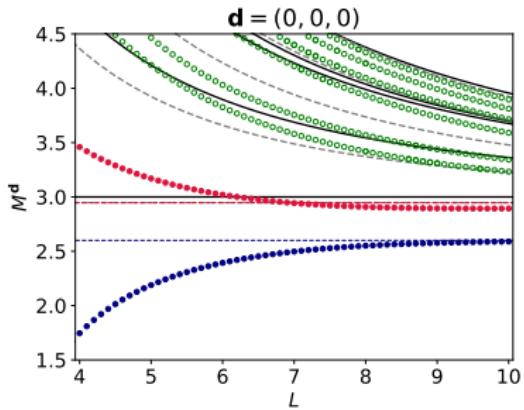
- IV ($m = 1$) in lab frame only:
 - fix $\Lambda = 3$ and $a = 5$
 - fix bound state at $E_1 = 2.6 \Rightarrow H_0 = -0.118$
 - find shallow bound state at $E_2 = 2.9467$ ($E_{PD} = 2.9473$)

- FV:

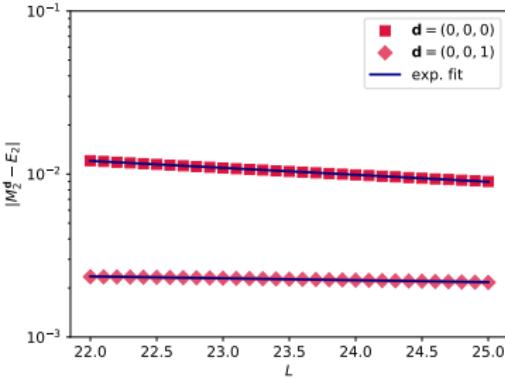
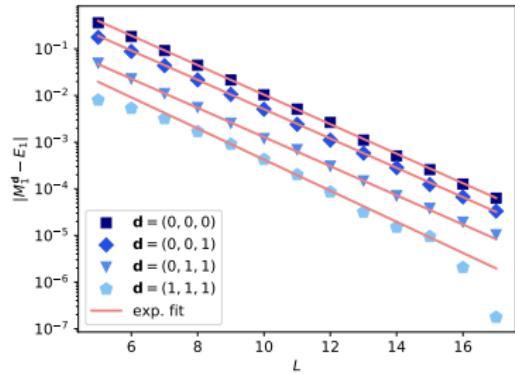
Calculate spectra in various frames $\mathbf{P} = 2\pi/L\mathbf{d}$

exp. decay of bound state energy \Rightarrow Lorentz invariance

Test of Invariance



Test of Invariance



- deep bound state:
exponential decay with same slope
- shallow bound state:
exponential decay but different slope

slope not fixed (Lorentz invariance broken in a finite volume)

Conclusion and Outlook

Conclusion:

- proposed a manifestly relativistically invariant formulation:
 - no frame depended three-particle couplings
⇒ allows to analyze data from different moving frames
 - IV observables invariant
- no violation of unitarity or spurious poles
- tested invariance using a toy model

Outlook:

- relativistically invariant formulation of three particle decays:
three particle Lellouche-Lüscher equation at higher orders

NREFT:

particle number conservation \Rightarrow separation of n -particle sectors

Modified matching:

$$K_{nr}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = K_r(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = -32\pi a_0 + \frac{1}{2}\tilde{r}_0 p^2 + \dots$$

Two-particle sector: (tree level)

$$\mathcal{L}^{(2)} = C_0 \phi^\dagger \phi^\dagger \phi \phi + C_2 \left[(w_\mu \phi)^\dagger (w^\mu \phi)^\dagger \phi \phi - m^2 \phi^\dagger \phi^\dagger \phi \phi + \text{h.c.} \right] + \dots$$

$\longleftarrow w^\mu = v^\mu w_v + i\partial_\perp^\mu$

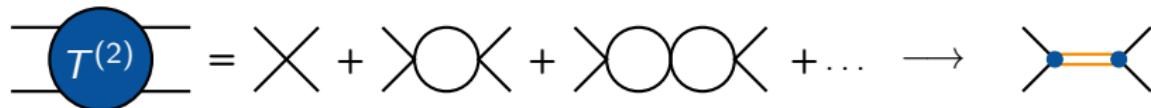
for **on-shell** momenta:

$$w_v(k) = \sqrt{m^2 - k^2 + (v \cdot k)^2} = (v \cdot k)$$

$$\Rightarrow \tilde{k}^\mu = w^\mu(k) = v^\mu w_v(k) + k_\perp^\mu = v^\mu (v \cdot k) + k_\perp^\mu = k^\mu$$

$$\Rightarrow K_{nr}^{(2)} = 4C_0 + 4C_2(s - 4m^2) + \dots \quad \longleftrightarrow \quad -32\pi a_0 + \frac{1}{2}\tilde{r}_0 p^2$$

Two-particle scattering:



$$T_S^{(2)}(P) = \frac{1}{(K_S^{(2)})^{-1} - I/2} = \frac{16\pi\sqrt{s}}{p \cot \delta_0(s) - i p(s)}$$

v^μ -indep. function of $s = P^2$ only

Introduction of dimers:

describe two-particle sector introducing dimer field T

- for S -wave: $T \sim \phi\phi$

$$\mathcal{L}_S^{(2)} = T^\dagger \left[f_0 \phi\phi + f_2 \left((w_\mu \phi)(w^\mu \phi) - m^2 \phi\phi \right) + \dots \right] + \text{h.c.}$$

- For higher partial waves:

$$\mathcal{L}_L^{(2)} = T_{\mu_1, \dots, \mu_L}^\dagger O^{\mu_1, \dots, \mu_L} + \text{h.c.}$$

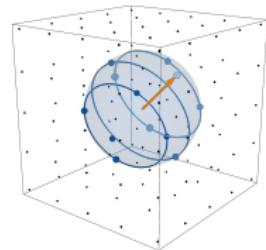
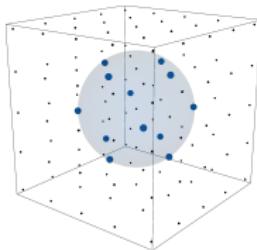
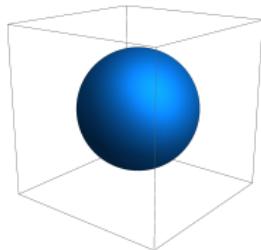
rel cov. two-particle operators in L -rep. of $\text{SO}(3)$

Partial diagonalization:

$$A_{\mathbf{pq}} = L^3 2w(\mathbf{p}) \delta_{\mathbf{pq}} \tau_L^{-1}((P - p)^2) - Z(p, q; P)$$

$$\begin{array}{ccc} \text{IV} & \longrightarrow & \text{FV} \\ \text{O}(3) & \longrightarrow & O_h \end{array} \quad \begin{array}{c} \text{moving frame} \\ g \in O_h : g\mathbf{P} = \mathbf{P} \end{array}$$

shells:



similar to partial wave expansion:

$$A_{\rho\sigma}^\Gamma(\mathbf{r}, \mathbf{s}) = \delta_{\rho\sigma} \delta_{rs} L^3 2w(\mathbf{s}) \tau_L^{-1}(\mathbf{s}) - \frac{\sqrt{\nu(r)\nu(s)}}{G} Z_{\rho\sigma}^\Gamma(\mathbf{r}, \mathbf{s})$$

$$Z_{\rho\sigma}^\Gamma(\mathbf{r}, \mathbf{s}) = \sum_g (T_{\sigma\rho}^\Gamma(g))^* Z(g p_0(r), q_0(s))$$