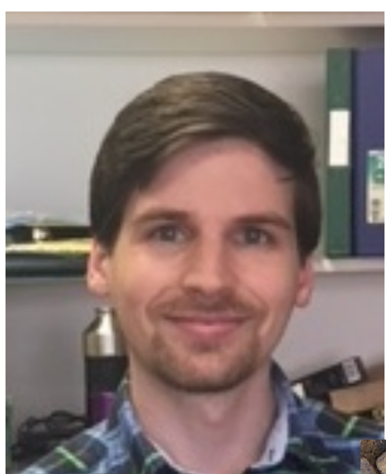
$\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ interactions from the lattice

Steve Sharpe
University of Washington



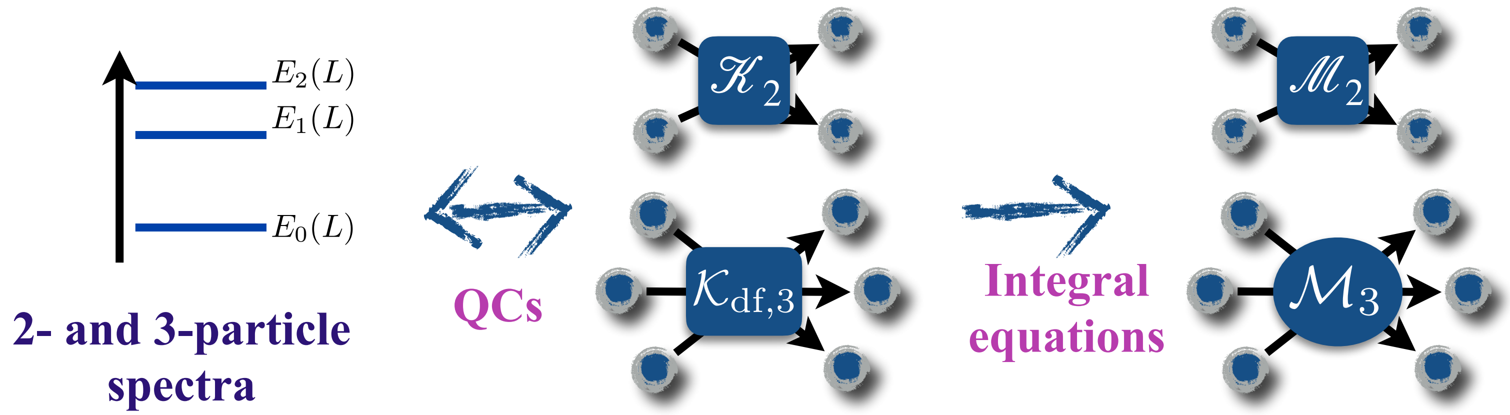
Based on work with Tyler Blanton, Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, & Fernando Romero-López:

2111.12734 [hep-lat] (JHEP)

& in preparation

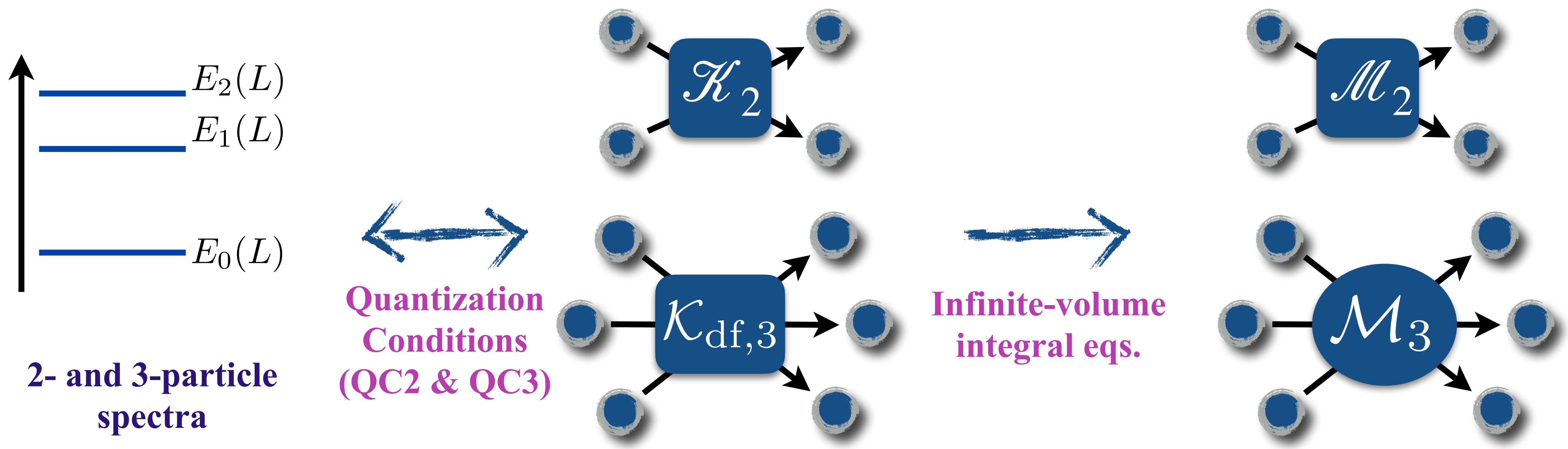
Overall program

[References in backup slides]



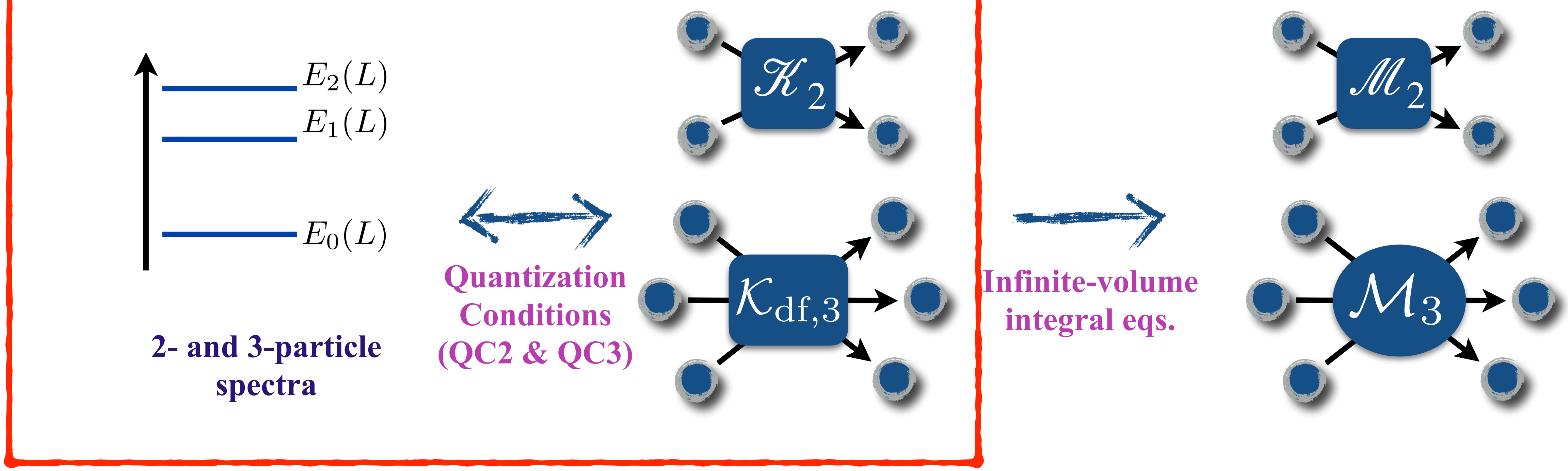
- Use finite-volume 2- and 3-particle spectra, obtained with lattice QCD, to determine 2- and 3-particle scattering amps
  - Formalism exists for arbitrary choices of spinless particles
  - Implemented for 3 identical scalars ($3\pi^+$, $3K^+$) & $3\pi(I=1)$ & ϕ^4 theory
  - Many systems of interest involve nondegenerate particles, e.g. $\pi\pi N$
  - First step in this direction is to consider "2+1 systems" $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$
    - Dominant s-wave interactions are mildly repulsive, so no resonances in 2-particle subchannels or overall system
  - Use RFT formalism

Workflow



- $\mathcal{K}_{\mathrm{df},3}$ is a real, infinite-volume (but scheme-dependent) K matrix that is smooth aside from possible 3-particle resonance poles; integral equations ensure unitarity of $\mathcal{M}_3$
- Parametrize $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$ in an "effective-range-like expansion" about threshold and determine parameters by fitting spectrum
- With multiple frames and waves, there is not a 1-to-1 relation between energies and phase shifts, so a global fit is required

Workflow



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New features for 2+1 systems

[Blanton, SRS, 2105.12904 (PRD)]

[Blanton, Romero-López, SRS, 2111.12734 (JHEP)]; https://github.com/ferolo2/QC3_release

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

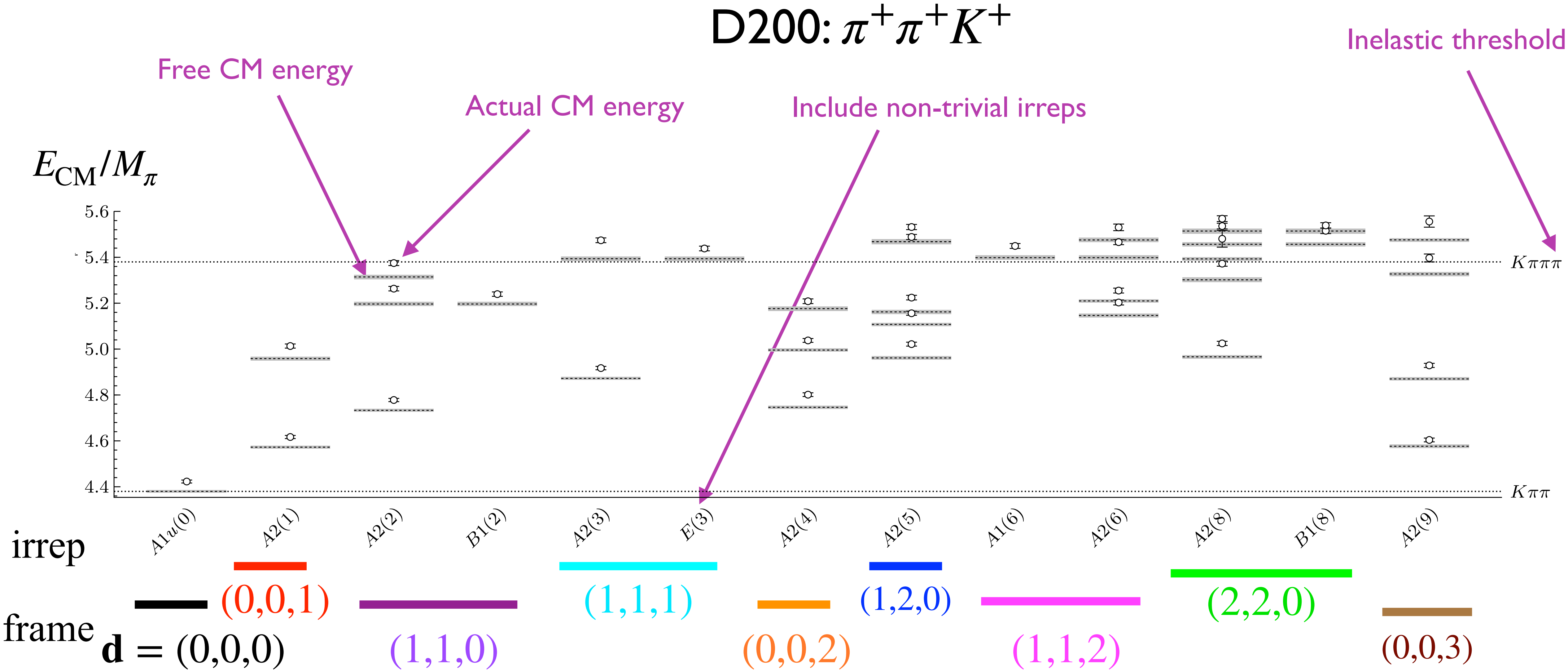
- QC3 involves matrices with an additional spectator-flavor index: $k\ell mi$
  - E.g., for $\pi^+\pi^+K^+$, spectator is π^+ ($i=1 \Rightarrow \pi^+K^+$ scattering) or K^+ ($i=2 \Rightarrow \pi^+\pi^+$ scattering)
  - All partial waves contribute to π^+K^+ scattering ($i=1$), while only even waves contribute to $\pi^+\pi^+$ scattering ($i=2$)
  - In practice, we set $\ell_{\max}=1$, in order to avoid too many parameters, particularly in $\mathcal{K}_{\mathrm{df},3}$
- Cut-off function H, must be chosen to avoid left-hand cuts, which occur when $s_2=|m_1^2-m_2^2|$ in subchannel with particles of masses m_1 & m_2
- Python implementation of QC3 available on GitHub

LQCD details

- Use similar methods as for $3\pi^+$ and $3K^+$ [Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]
  - CLS ensembles D200 & N203 (open BC in time): $a \approx 0.064\,\mathrm{fm}$
  - Use stochastic LapH & contraction tricks to obtain multiple levels for $2\pi^+$, π^+K^+, $2K^+$, $\pi^+\pi^+K^+$ & $\pi^+K^+K^+$ in frames with up to $d^2 = 3$ ($\mathbf{P} = (2\pi/L)\mathbf{d}$), projected onto irreps of corresponding finite-volume little groups
  - Fit to correlator ratios to directly obtain shifts from free energies in "lab frame", ΔE_{lab}

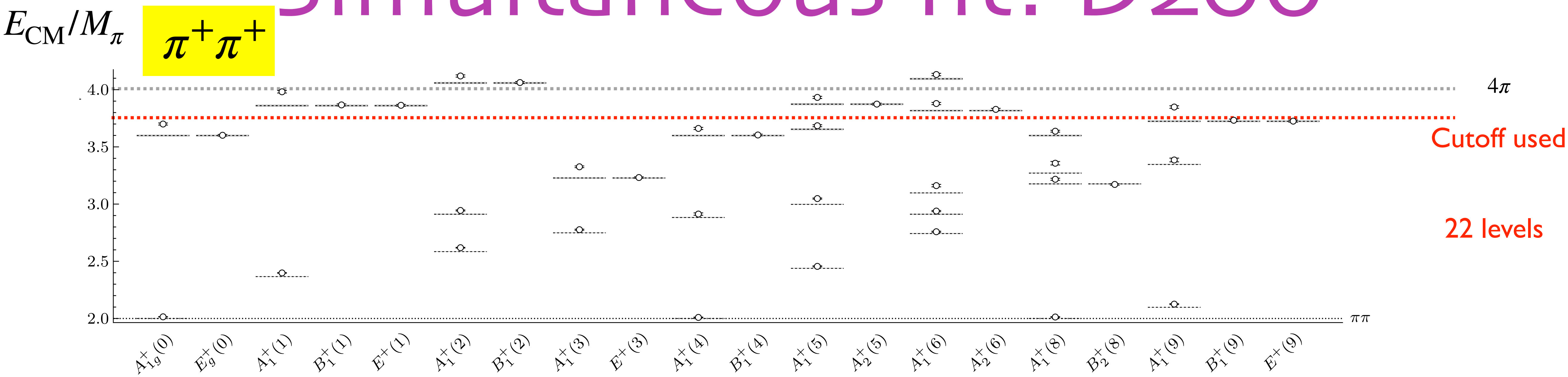
	$(L/a)^3 \times (T/a)$	M_π [MeV]	M_K [MeV]	N_{cfg}	Bin size	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	1	5.4
D200	$64^3 \times 128$	200	480	2000	3	4.2

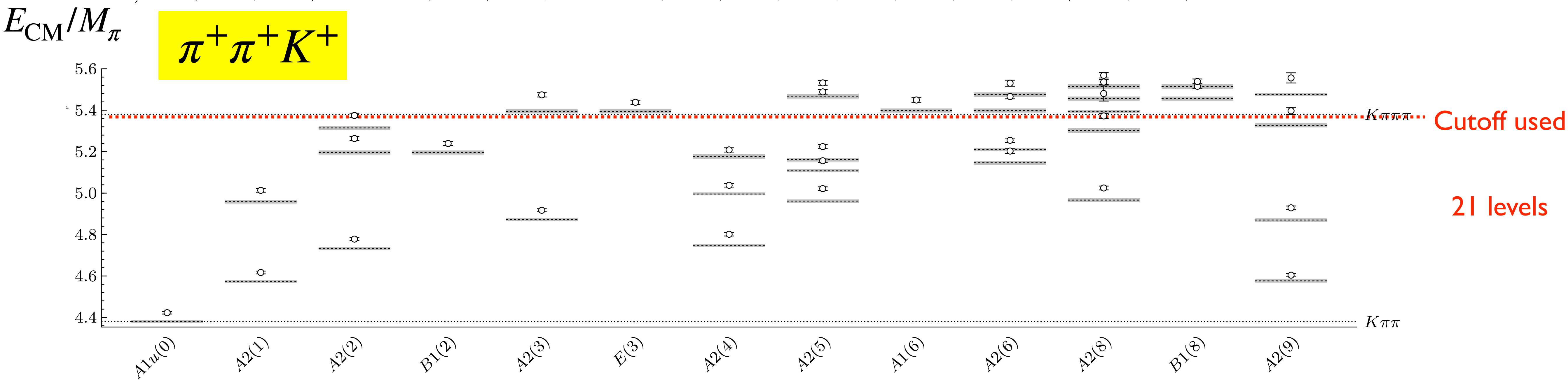
Example of levels

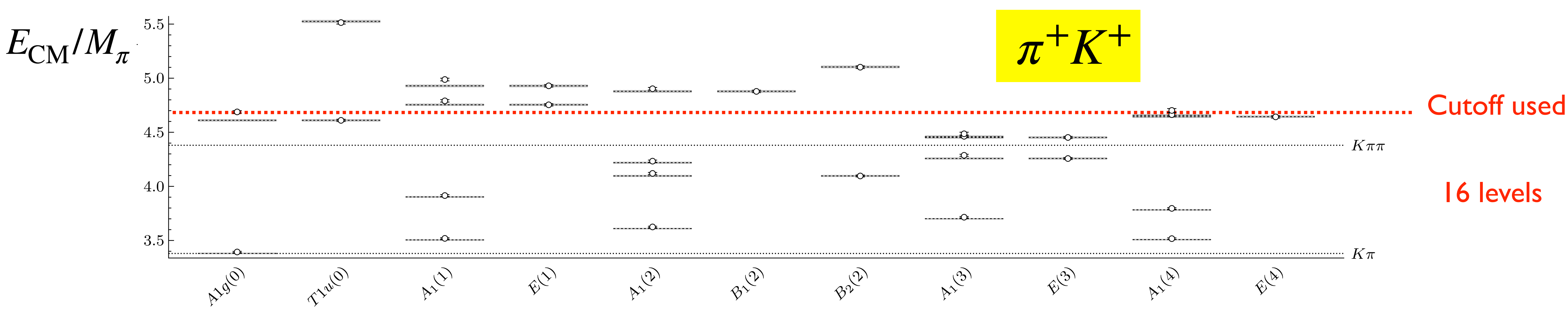


- Previously converted energy shifts $\Delta E_{\rm lab}$ to $E_{\rm CM}$ using M_π & M_K determined at rest on given jackknife sample, and with continuum dispersion relation, and then fit QCs to $E_{\rm CM}$
  - Increases errors in data, leading to fits that seem better than they really are
- Here fit QCs directly to $\Delta E_{\rm lab}$

Simultaneous fit: D200







Threshold expansion for $\mathcal{K}_2$

- Work to linear order in q^2 expansion, dropping $\mathcal{O}(q^4)$ terms
  - Implies that we keep s- and p-wave terms in π^+K^+ channel, but only s waves for identical pairs
  - Previously we found that d-wave terms were needed for a good description of $2\pi^+$ and $2K^+$ levels, but here that would lead to too many parameters
  - Thus we expect (and find) poorer global fits
- Use forms with Adler zero rather than effective-range expansions, since we have previously found the former to provide better fits for $2\pi^+/3\pi^+$ system

For $2\pi^+$ (& similarly $2K^+$): $$q \cot \delta_0^{\pi\pi} = \frac{M^2\sqrt{s_2}}{s_2 - 2z^2M^2}(B_0^{\pi\pi} + B_1^{\pi\pi}q^2)$$

For π^+K^+ s-wave: $$q \cot \delta_0^{\pi K} = \frac{M_\pi^2\sqrt{s_2}}{s_2 - M_\pi^2 - M_K^2}(B_0^{\pi K} + B_1^{\pi K}q^2)$$

For π^+K^+ p-wave: $$q^3 \cot \delta_1^{\pi K} = \frac{M_\pi^3\sqrt{s_2}}{M_\pi + M_K}\frac{1}{P_0^{\pi K}}$$

Threshold expansion for $\mathscr{K}_{\mathrm{df},3}$

[Blanton, SRS, 2105.12904 (PRD)]

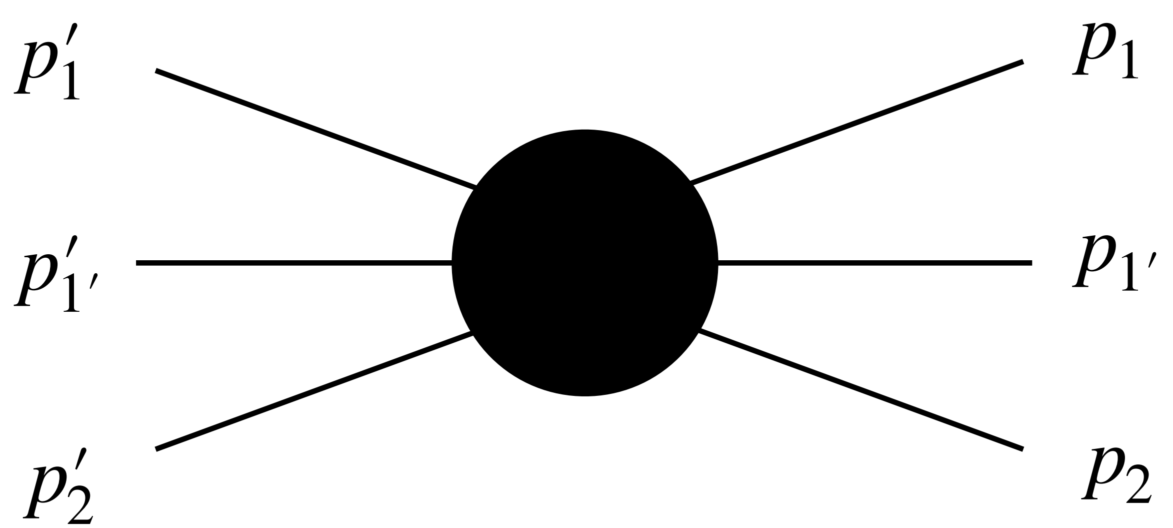
- Work to linear order, dropping $\mathcal{O}(\Delta^2)$ terms
- 4 terms allowed by symmetries (Lorentz, $1 \leftrightarrow 1'$, time-reversal, parity)
  - Only $\mathscr{K}^E$ couples to nontrivial irreps: contains J=0,1 while other terms only have J=0

$$\boxed{\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{B,1}\Delta_2^S + \mathcal{K}_{\mathrm{df},3}^{E,1}\widetilde{t}_{22}}$$

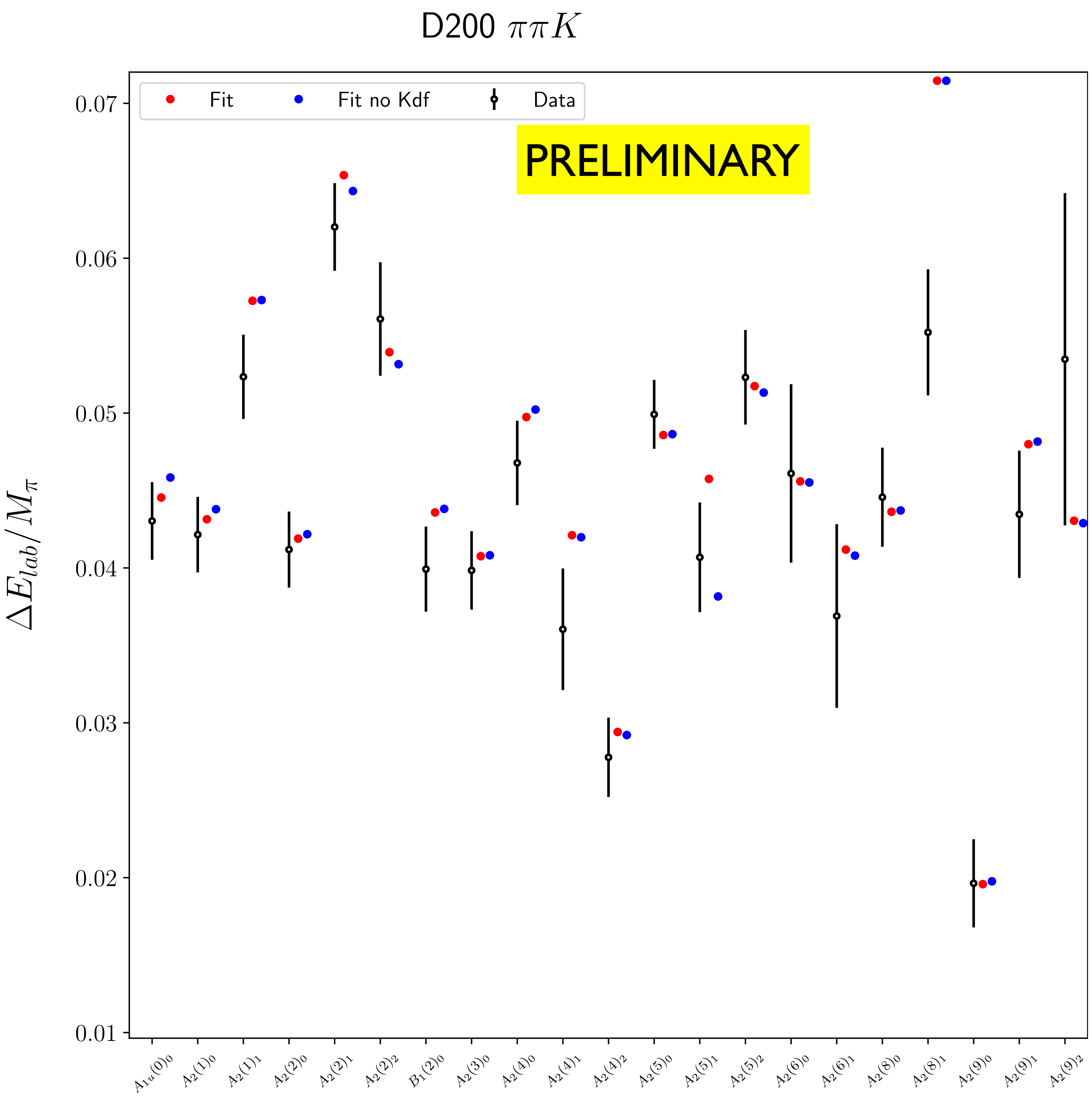
$$\Delta = \frac{s-M}{M^2}, \qquad s = (p_1 + p_{1'} + p_2)^2 = P^2,$$

$$\Delta_2^S = \Delta_2 + \Delta_2', \quad \Delta_2 = \frac{(p_1 + p_{1'})^2 - 4m_1^2}{M^2}, \quad \Delta_2' = \frac{(p_1' + p_{1'}')^2 - 4m_1^2}{M^2},$$

$$\widetilde{t}_{22} = \frac{t_{22}}{M^2} = \frac{(p_2 - p_2')^2}{M^2}, \qquad M = 2m_1 + m_2 .$$



Preliminary results

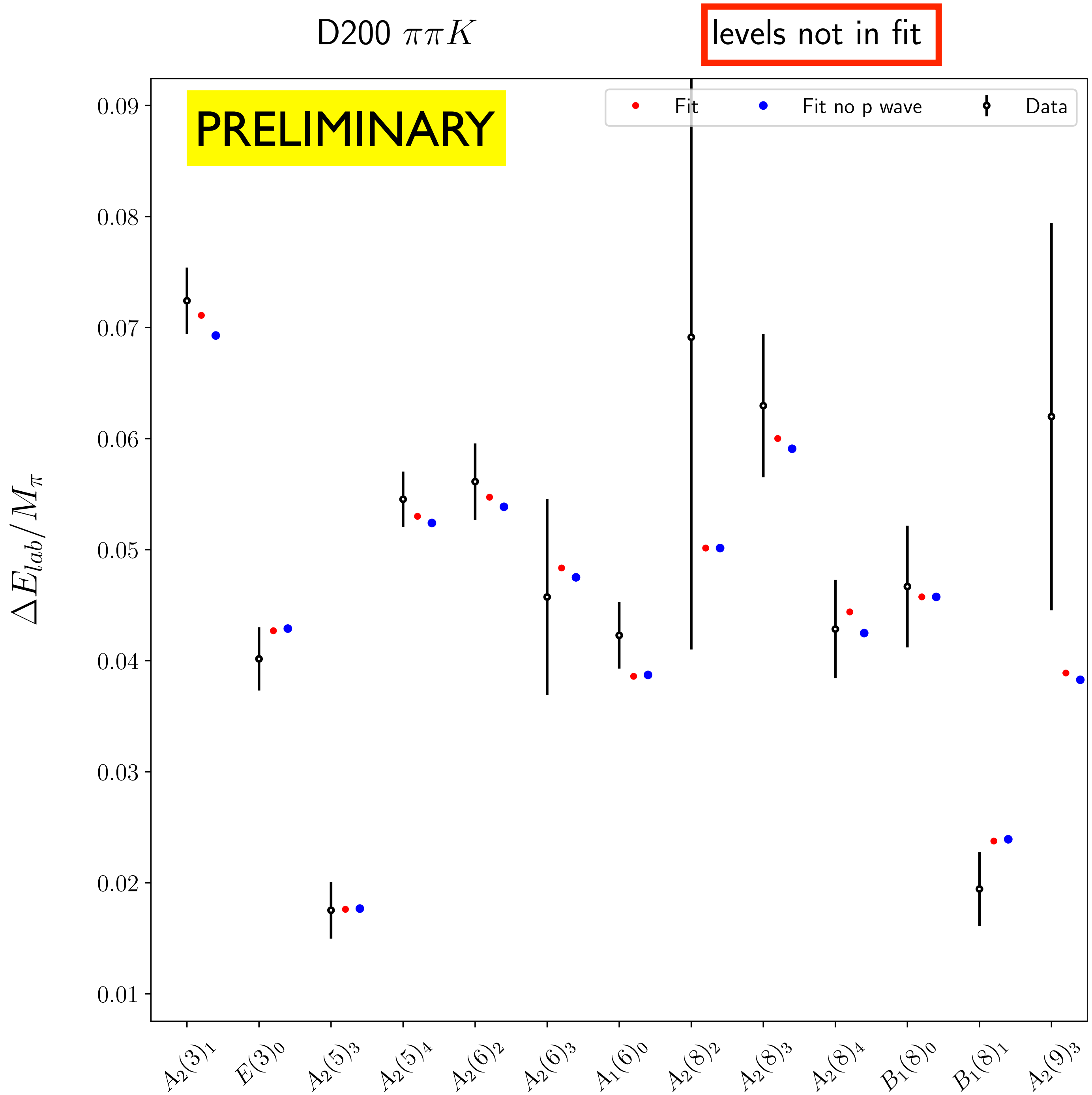


$B_0^{\pi\pi}$	-11.5(6)
$B_1^{\pi\pi}$	-2.5(4)
$z_{\pi\pi}^2$	1 (fixed)
$B_0^{\pi K}$	-12.9(4)
$B_1^{\pi K}$	-2.8(3)
$z_{\pi K}^2$	1 (fixed)
$P_0^{\pi K}$	0.0007(6)
$\mathcal{K}_0$	190(80)
$\mathcal{K}_1$	-690(340)
$\mathcal{K}_B$	160(650)
$\mathcal{K}_E$	170(420)

Significance of nonzero $\mathscr{K}_{\text{df},3}$ is 3.2σ

9-parameter fit to 59 $\pi^+\pi^+$, π^+K^+, $\pi^+\pi^+K^+$ levels on D200: $\chi^2/\text{dof} = 112/50$

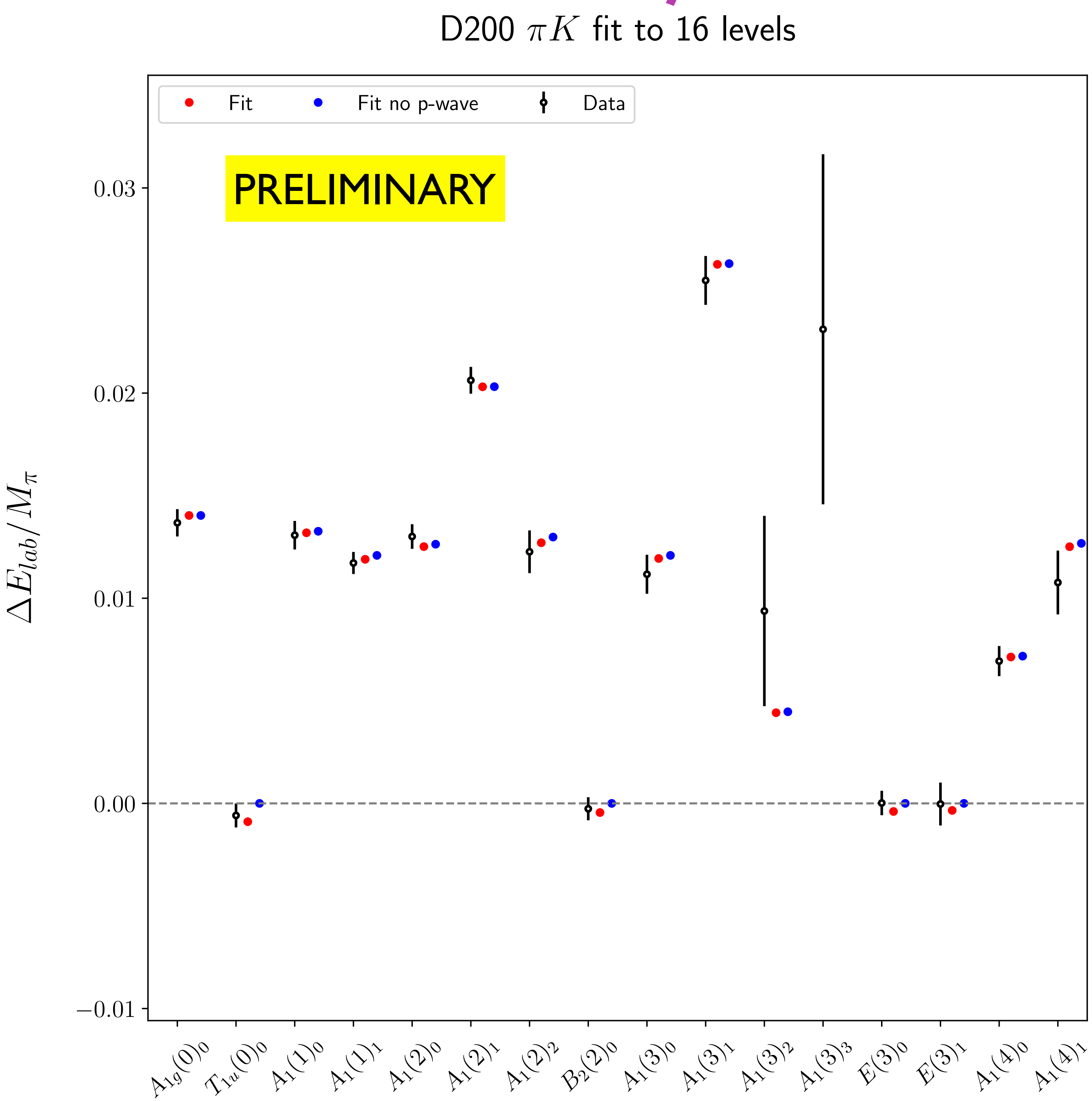
Preliminary results



Fit predicts levels above inelastic threshold

9-parameter fit to 59 $\pi^+\pi^+$, π^+K^+, $\pi^+\pi^+K^+$ levels on D200: $\chi^2/\mathrm{dof} = 112/50$

Preliminary results

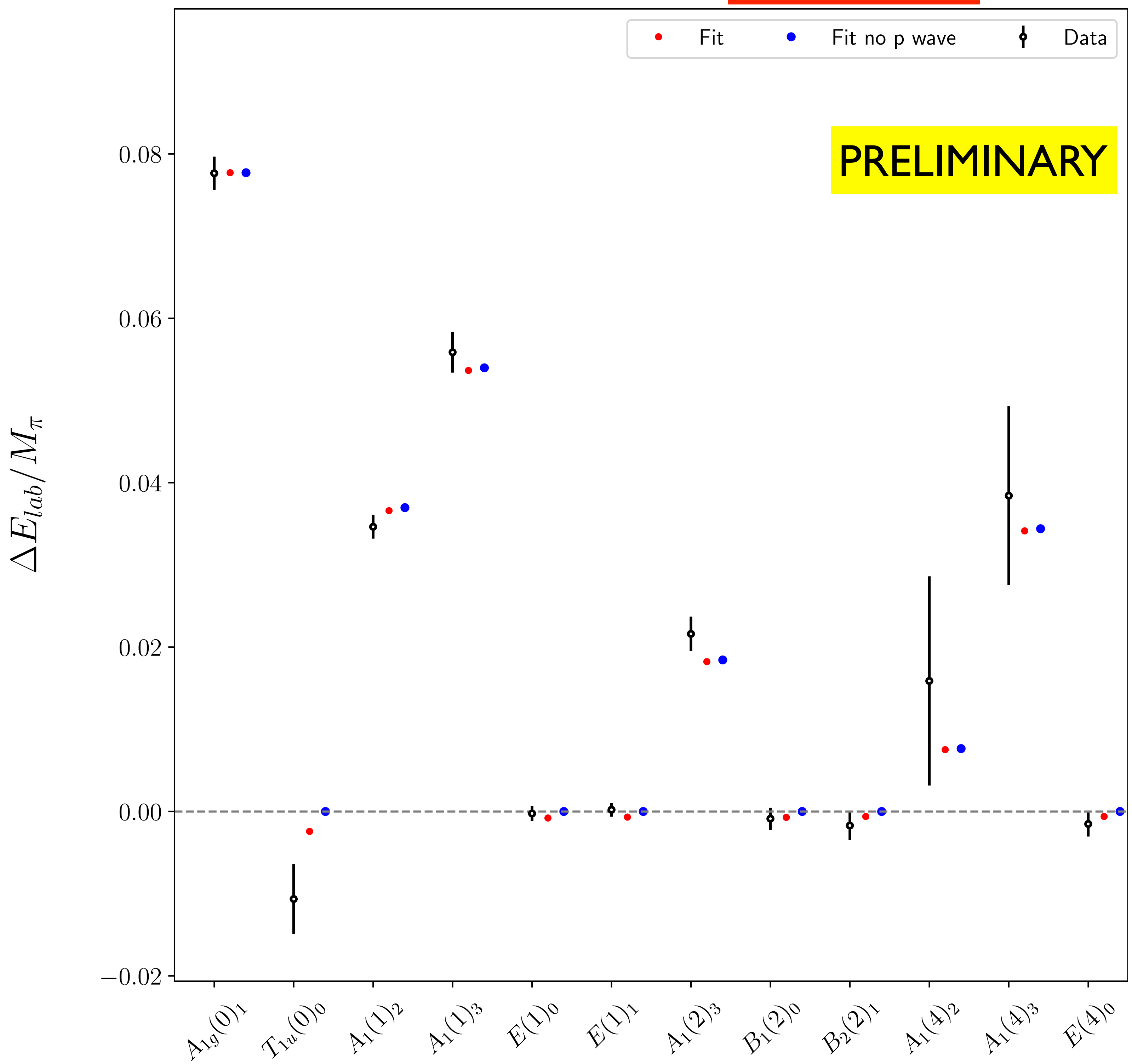


Significance of attractive p-wave interaction is 1.7σ

3-parameter fit to 16 π^+K^+ levels on D200: $\chi^2/\mathrm{dof} = 15.5/13$

Preliminary results

D200 πK fit to 16 levels; levels not in fit



Fit predicts levels above inelastic threshold

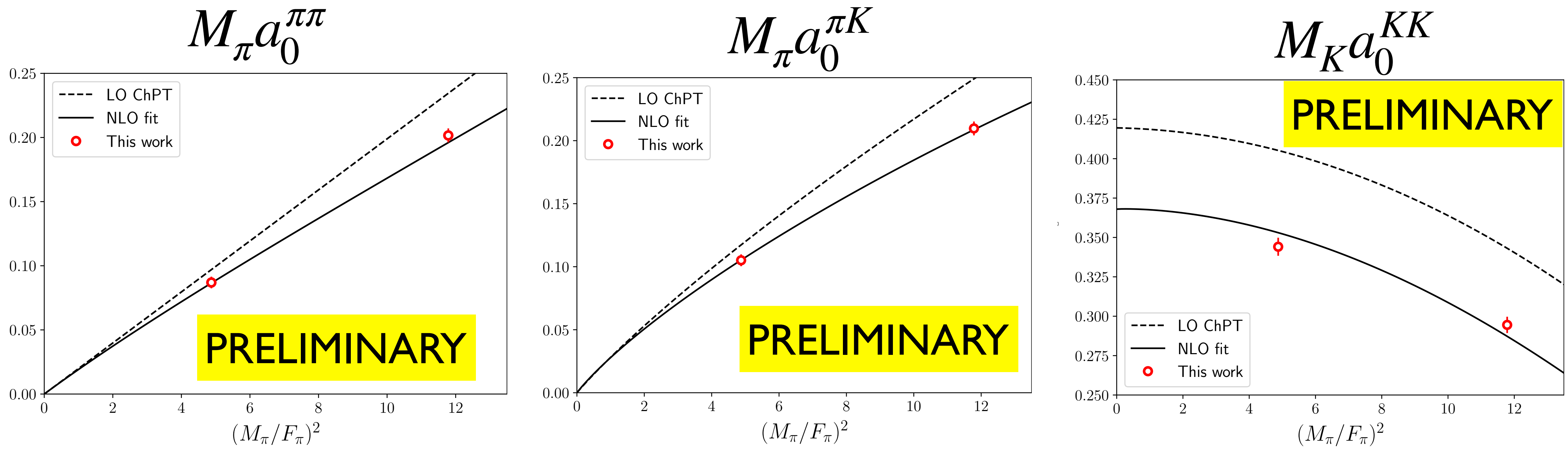
3-parameter fit to 16 π^+K^+ levels on D200: $\chi^2/\text{dof} = 15.5/13$

Conclusions from fits

- Working at linear order (s- + p-wave) in threshold expansions gives a reasonable description of energy levels, although significantly worse than that we obtained for identical particles when including d waves
- Fits continue to predict levels with good accuracy above inelastic threshold, indicating that the threshold expansions are not breaking down
- Simultaneous fits to 2- and 3-particle spectra lead to somewhat smaller errors in 2-particle parameters compared to fits to 2-particle spectra alone
- We obtain 1-2% precision in s-wave scattering lengths for all channels
- We find attractive p-wave $\pi^+ K^+$ scattering length, but only with 1-2σ significance
    - Opposite sign to (very weakly) repulsive experimental result
- We find nonzero $\mathscr{K}_{\mathrm{df},3}$ with 3-5σ significance for $\pi^+\pi^+K^+$ & $K^+K^+\pi^+$
    - To do better would require a reduction in the ~10% errors in ΔE_{lab}

Comparison with ChPT

Simultaneous fits to $a_0^{\pi\pi}, a_0^{\pi K}, a_0^{KK}$ using NLO SU(3) ChPT



Determine LECs (at scale $4\pi F_\pi$)

χ^2/dof	$L_{\pi\pi}$	L_5
$5.5/4$	-0.00089(3)	-0.00050(8)

Consistent with FLAG result

$\mathcal{K}^B$ & $\mathcal{K}^E$ vs ChPT form

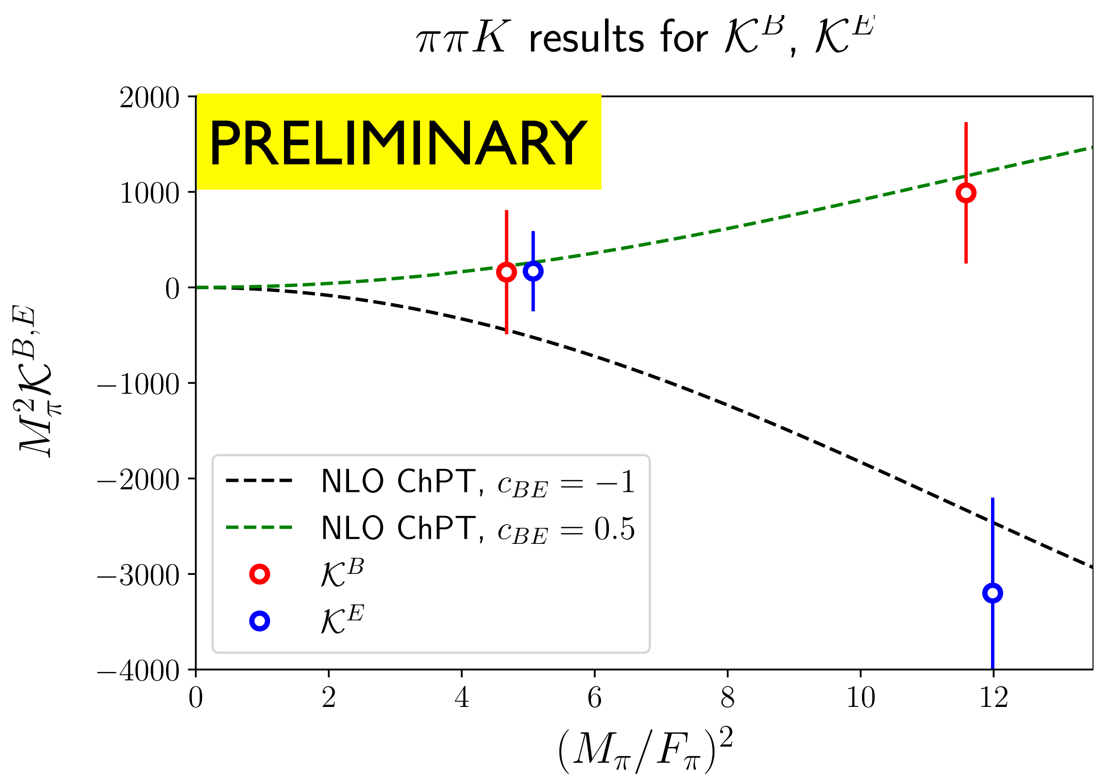


Figure 7: Results for $M_\pi^2 \mathcal{K}^{B,E}(\pi\pi K)$ compared to generic NLO ChPT forms.

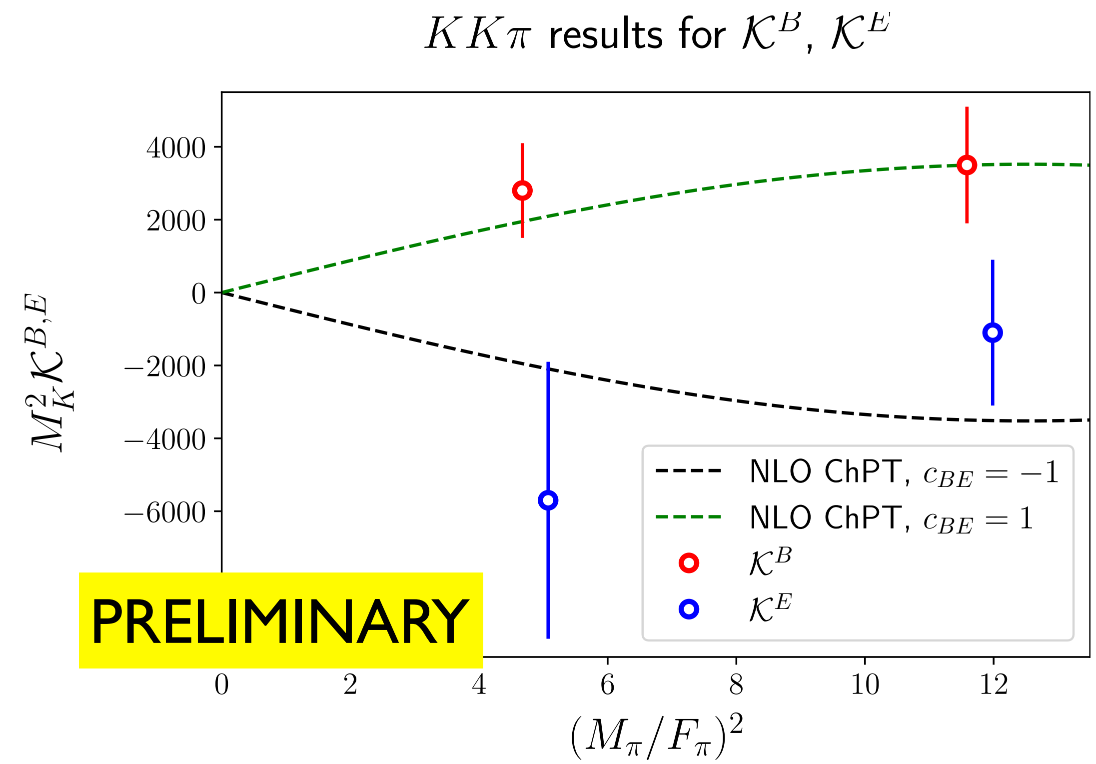


Figure 10: Results for $M_K^2 \mathcal{K}^{B,E}(KK\pi)$ compared to generic NLO ChPT forms.

Generic NLO ChPT forms:

$$M_\pi^2 \mathcal{K}_{\mathrm{df},3}^{B,E}(\pi\pi K) = c_{BE} r_\pi^4 r_K^2 \,, \qquad M_\pi^2 \mathcal{K}_{\mathrm{df},3}^{B,E}(KK\pi) = c_{BE} r_K^4 r_\pi^2 \,.$$

$$r_\pi = \frac{M_\pi}{F_\pi} \quad \text{and} \quad r_K = \frac{M_K}{F_K} \,,$$

Summary & Outlook

- (First step of) 3-particle formalism successfully applied to 2+1 systems
  - We encountered no problems with unphysical solutions in applying the QC3
  - Enlarged matrices in QC3 require small clusters to perform fits in ~days
- Determining 3-particle interaction ($\mathscr{K}_{\mathrm{df},3}$) remains challenging
- Current & future work
  - Analysis should be extended to chiral behavior of effective range
  - Solving integral equations (second step of formalism)
  - Physical-point ensembles [See plenary talk by Fernando Romero-López]
  - Extend to systems with 2- and 3-particle resonant behavior (e.g. $3\pi(I=0,1)$, $\pi\pi N$)
  - Dreaming of 3 neutrons (formalism in progress [Draper, Hansen, Romero-López, SRS])

Thanks
Any questions?

Backup slides

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

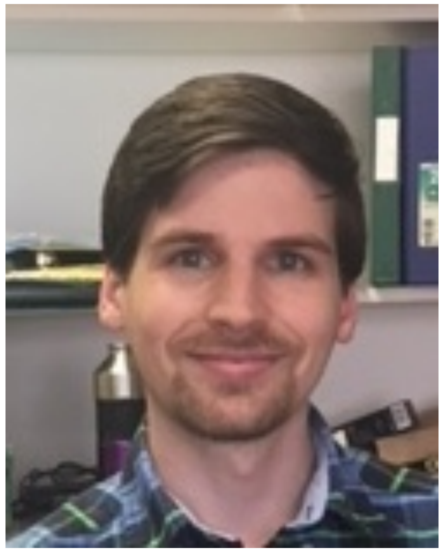
"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory,"

arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

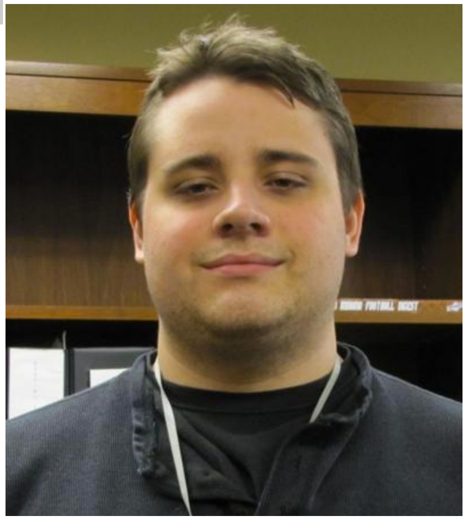
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



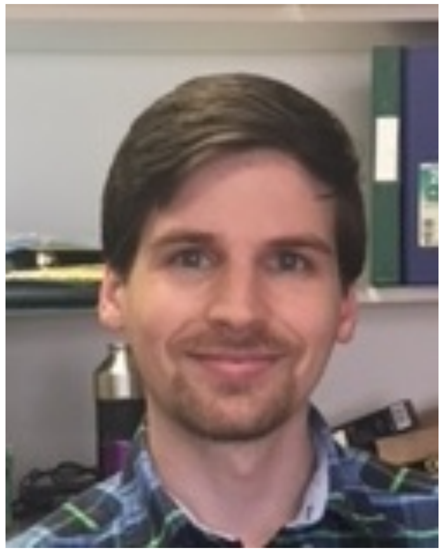
Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)



<u>Tyler Blanton & SRS:</u>

“Alternative derivation of the relativistic three-particle quantization condition,”

arXiv:2007.16188 (PRD) [BS20a]

“Equivalence of relativistic three-particle quantization conditions,”

arXiv:2007.16190 (PRD) [BS20b]

“Relativistic three-particle quantization condition for nondegenerate scalars,”

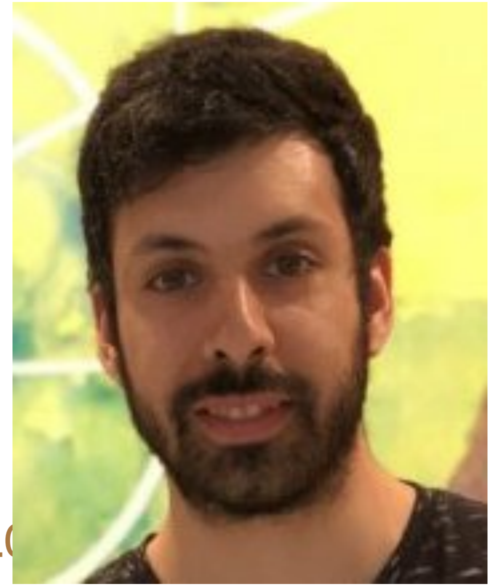
arXiv:2011.05520 (PRD)

“Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems,” arXiv:2105.12904 (PRD)

<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

“$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,” arXiv:2106.05590 (JHEP)

“$\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ interactions from lattice QCD,” in progress



Other work

★ **Implementing RFT integral equations**

- A. Jackura et al., 2010.09820 [Solving s-wave RFT integral equations in presence of bound states]
- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]

★ **Reviews**

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ **Numerical simulations in scalar theories**

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, [2- & 3-body interactions in φ^4 theory]

Other work

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176 , JHEP [Formalism & examples]
- M. Döring et al., 1802.03362 , PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111 , PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807 , JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP, [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
- J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, 2205.11316 , [Resonance form factors from finite-volu correlation functions with the external field method]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749,PRD [applying FVU approach to $3\pi^+$spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144 [determining $3\pi^+$ interaction from LQCD spectrum]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]