

Static Energy in $(2 + 1 + 1)$ -Flavor Lattice QCD: Scale Setting and Charm Effects

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Based on [N. Brambilla, et al., arXiv/2206.03156[hep-lat]]

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Motivation

- Static energy $E_0(r)$ studied on the lattice since earliest times (**confinement**)¹, defined as ground state of Wilson loop.
- High accuracy, numerically cheap observable on the lattice.
- Of major importance for **scale setting** in the **past** – **still now?**
- Wilson loops and related correlators contribute in most **effective field theories** for heavy quarks and quarkonia.
- **No scheme change** required between lattice and continuum.
- To date, extraordinarily **detailed lattice studies** in pure gauge theory ($N_f = 0$) or $(2+1)$ -flavor QCD² $\Rightarrow \Lambda_{\overline{MS}}^{N_f=0,3}$.
- So far, dearth of results in $N_f = (2+1+1)$ -flavor lattice QCD.

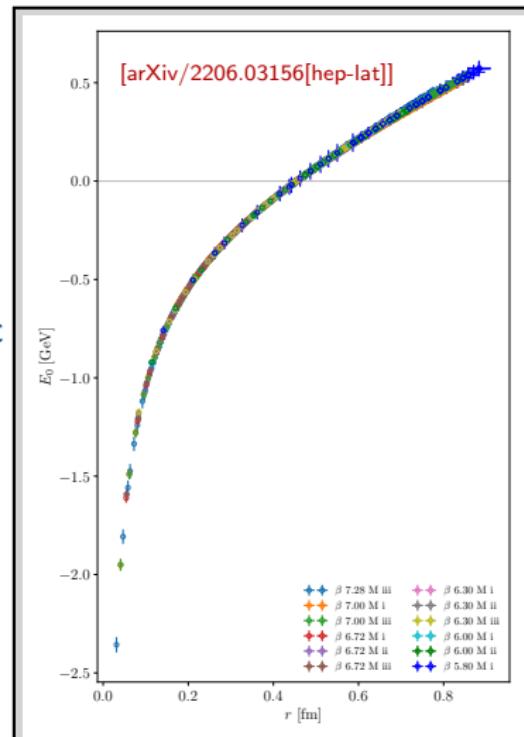
¹[K. Wilson, PRD 10 (1974) 2445]

²[A. Bazavov, et al., PRD 100 (2019) 11, 114511]



Summary

- We calculate the static energy in **($2 + 1 + 1$)-flavor QCD** covering $r \approx 0.03 - 0.9$ fm.
- We **self-consistently convert** the distance r/a and the static energy aE_0 **to physical units** using scales obtained from E_0 .
- We resolve dependence of the scales and the effective string tension on **sea quark masses**.
- We study in detail **effects of the charm sea** for $r \gtrless 1/m_c$.



Lattice setup

- (2 + 1 + 1)-flavor QCD **ensembles with HISQ** ³ and 1-loop Symanzik gauge action generated by MILC collaboration ⁴.
- **Three light quark masses** ($m_l/m_s \approx 1/27, 1/10, 1/5$), physical strange and charm sea.
- **Six lattice spacings** via f_{p4s} scale ($a_{f_{p4s}} \approx 0.03 - 0.15$ fm).
- We compute the **Wilson line correlator in Coulomb gauge**, with or without one step of HYP smearing ⁵.

$$W(\mathbf{r}, \tau, a) = \prod_{u=0}^{\tau/a-1} U_4(\mathbf{r}, ua, a),$$
$$C(\mathbf{r}, \tau, a) = \left\langle \frac{1}{N_\sigma^3} \sum_{\mathbf{x}} \sum_{\mathbf{y}=R(\mathbf{r})} \frac{1}{N_c N_r} \text{tr} \left[W^\dagger(\mathbf{x} + \mathbf{y}, \tau, a) W(\mathbf{x}, \tau, a) \right] \right\rangle,$$

³[E. Follana, et al., PRD 75 (2007) 054502]

⁴[A. Bazavov, et al., PRD 98 (2018) 7, 074512]

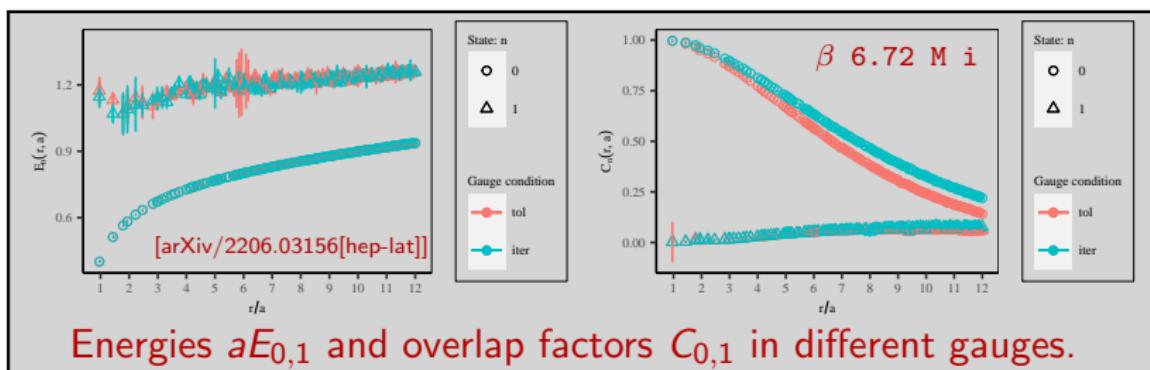
⁵[A. Hasenfratz, F. Knechtli, PRD 64 (2001) 034504]



Spectrum of Wilson line correlators

- **Improved action** \Rightarrow spectral representation holds at $\tau/a \geq 2$.
- **Physical T_{\max} and $r_{\max} \simeq 0.6 - 0.9$ fm** implies much fewer time slices on coarse lattices (factor five variation in a).
- On coarse lattices **limitation in number of states**, N_{st} .

$$C(\mathbf{r}, \tau, a) = e^{-\tau E_0(\mathbf{r}, a)} \left(C_0(\mathbf{r}, a) + \sum_{n=1}^{N_{\text{st}}-1} C_n(\mathbf{r}, a) \prod_{m=1}^n e^{-\tau \Delta_m(\mathbf{r}, a)} \right) + \dots,$$



Fits

- Up to $N_r \sim 500$ $|r|/a$ values after averaging, cover factor 24.
- Up to $T_{\max}/a \times N_r \sim 10^4$ correlated data \Rightarrow automation!
- Fit range [i.e. $\tau_{\min, N_{\text{st}}} (r/a)$] varies with N_{st} and $|r|$

$$|r| + 0.2 \text{ fm} \leq \tau_{\min,1} \leq 0.3 \text{ fm} \quad \text{for } N_{\text{st}} = 1 \Rightarrow \text{prior values,}$$

$$\frac{2}{3}|r| + 0.1 \text{ fm} \leq \tau_{\min,2} \leq \tau_{\min,1} - 2a \quad \text{for } N_{\text{st}} = 2 \Rightarrow \text{final choice,}$$

$$\frac{1}{3}|r| \leq \tau_{\min,3} \leq \tau_{\min,2} - 2a \quad \text{for } N_{\text{st}} = 3 \Rightarrow \text{cross-check}$$

- We use Bayesian fits with **loose, linear priors** ($\geq 10\%$ width).
- Priors for excited state overlap factors with $\geq 100\%$ width: no unacceptable examination of data, robust for prior variation.
- Priors for $\Delta_1 = E_1 - E_0$ are from SU(3) pure gauge theory ⁶.

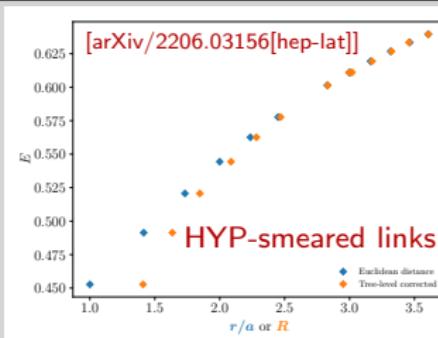
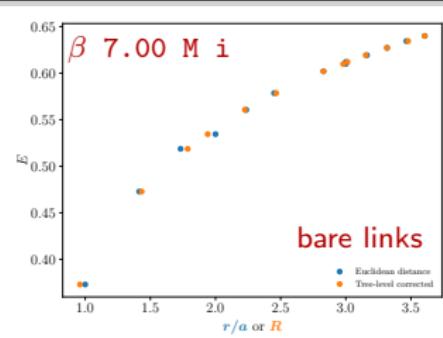
⁶[K. Juge, et al., PRL90, 161601 (2003)]



Discretization artifacts

- $E_0(\mathbf{r}, a)$ is available only at discrete distances, and depends on the direction of $\mathbf{r} \Rightarrow$ not smooth in $|\mathbf{r}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
- Distance r_I defined via tree-level ⁷ gluon propagator $D_{\mu\nu}(k)$

$$E_0^{\text{tree}}(\mathbf{r}, a) = -C_F g_0^2 \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot r} D_{44}(k) \equiv -\frac{C_F g_0^2}{4\pi} \frac{1}{r_I}.$$



Tree-level-correction (aka: -improvement) in use for $|\mathbf{r}|/a \leq 6$.

⁷Extension to one loop ongoing: [G. v. Hippel, et al., in preparation: TUM-EFT 171/22]

Determination of scales

- The scales are defined in terms of **the force** $F(r) \equiv \frac{dE_0(r)}{dr}$

$$r_i^2 F(r_i) = c_i, \quad i = 0, 1, 2$$

with $c_0 = 1.65$ ⁸, $c_1 = 1$ ⁹, and $c_2 = 1/2$ ¹⁰.

- The r_i correspond to **different regimes**. (2 + 1)-flavor QCD:

$$r_0 \approx 0.475 \text{ fm}, \quad r_1 \approx 0.3106 \text{ fm}, \quad r_2 \approx 0.145 \text{ fm},$$

- A **Cornell Ansatz** encodes the main features of $E_0(r)$ locally

$$E(R, a) = -\frac{A}{R} + B + \Sigma R,$$

- $r_i/a = \sqrt{(c_i - A)/\Sigma}$; identify Σ at large r with string tension
- Consistent definition** of $F(r, a)$ across various a is difficult

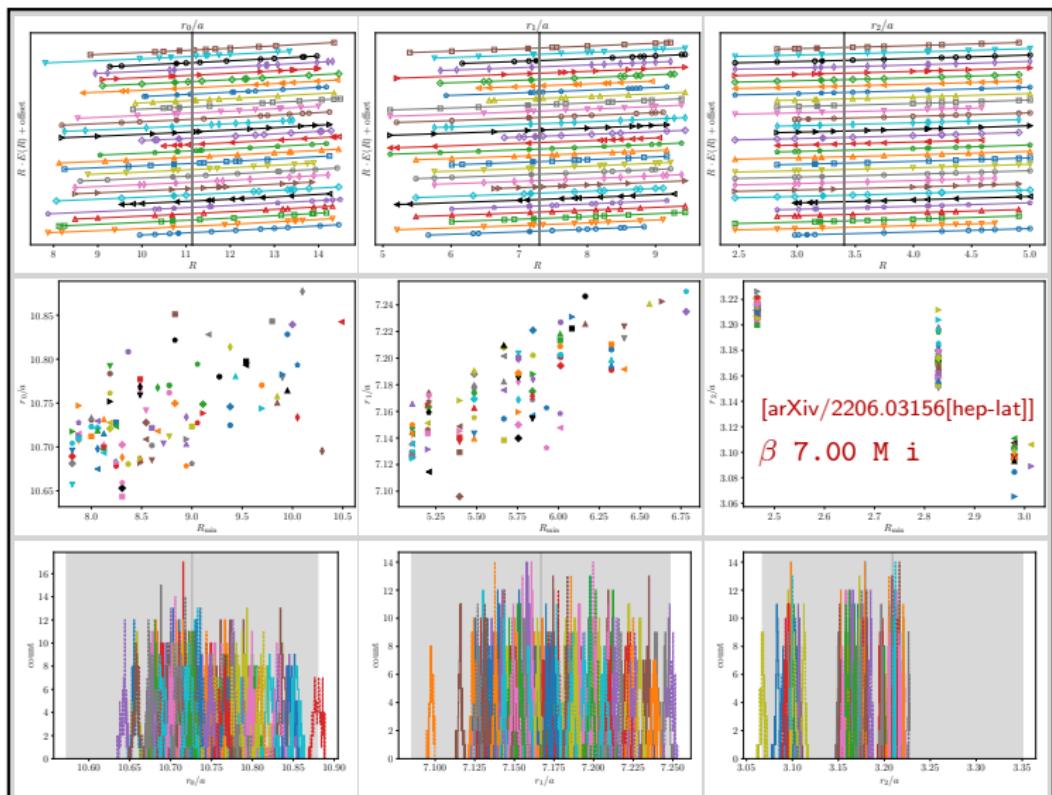
⁸[R. Sommer, NPB 411 (1994) 839]

⁹[C. Bernard, et al., PRD 62 (2000) 034503]

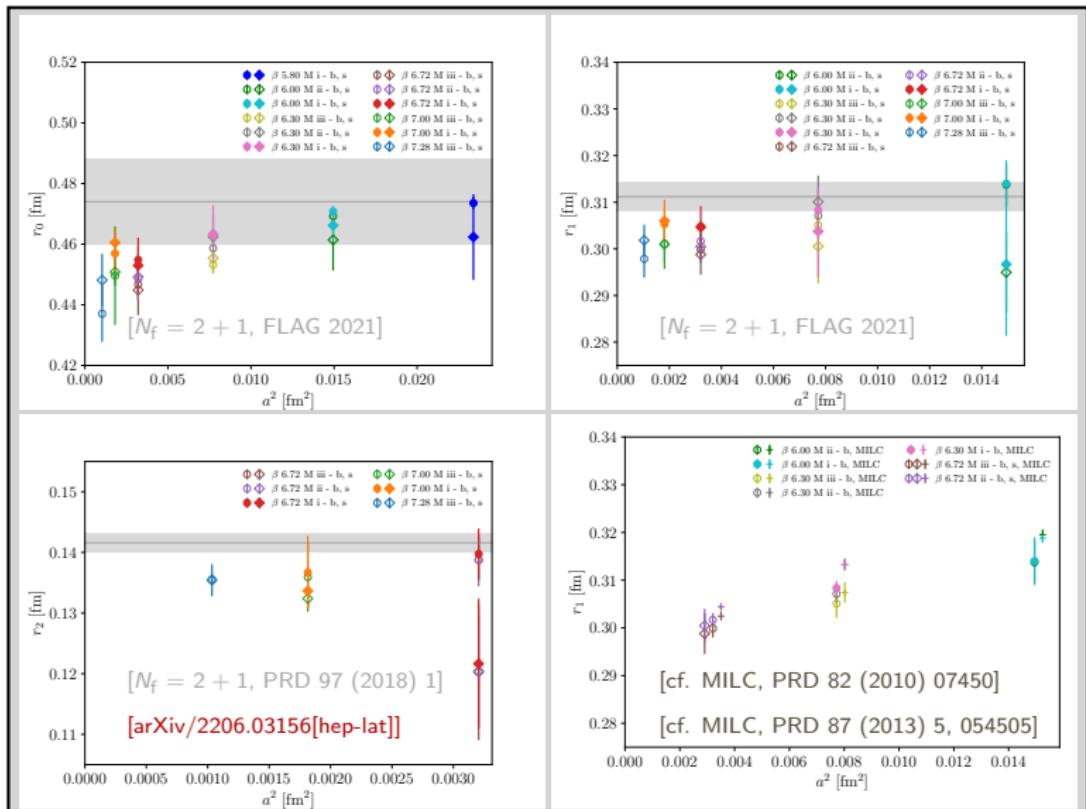
¹⁰[A. Bazavov, et al., PRD 97 (2018) 1, 014510]



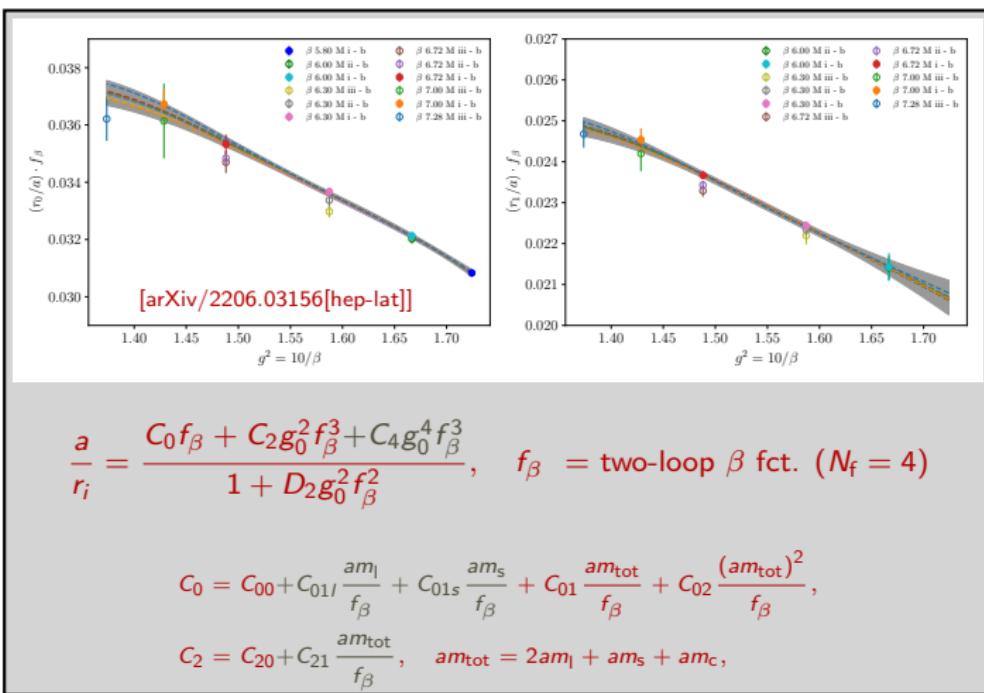
Defining the force through asymmetric, random picking



Lattice spacing, N_f , and quark mass dependence



Smoothing (via Allton-Ansatz)



Continuum extrapolation

- Using smoothed results we extrapolate to the continuum limit:
 $r_0/r_1, r_1/r_2; a_{f_{p4s}} r_0/a, a_{f_{p4s}} r_1/a; \sqrt{\sigma a^2} r_0/a$
- Lattice spacing dependence via $x = (a/r_i)^2, i = 0, 1$
- m_l dependence via $y = m_l/m_s$ (sea or tuned ⁴ m_s),
where we use PDG values ¹¹ in the continuum GMOR

$$m_l/m_s = 1/(2M_K^2/M_\pi^2 - 1) \Rightarrow \begin{cases} m_l/m_s|_{M_\pi^2 = M_{\pi^\pm}^2} &= 0.04128, \\ m_l/m_s|_{M_\pi^2 = M_{\pi^0}^2} &= 0.03851. \end{cases}$$

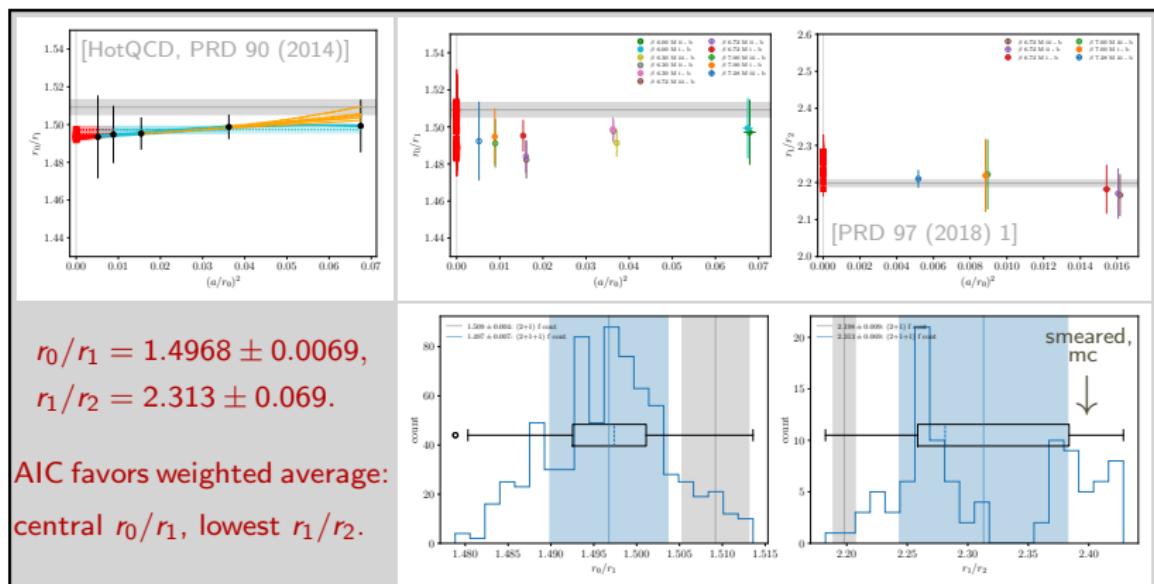
- Logs in the Symanzik expansion via $\alpha = 1$ or $\alpha_b = g_0^2/(4\pi u_0^2)$
 $\xi = \xi_0 + \alpha^2[\xi_1 x + \xi_2 x y^{1,2}] + \xi_3 x^2 + \xi_4 y \quad (l,q;l,qm;mc),$

⁴[A. Bazavov, et al., PRD 98 (2018) 7, 074512]

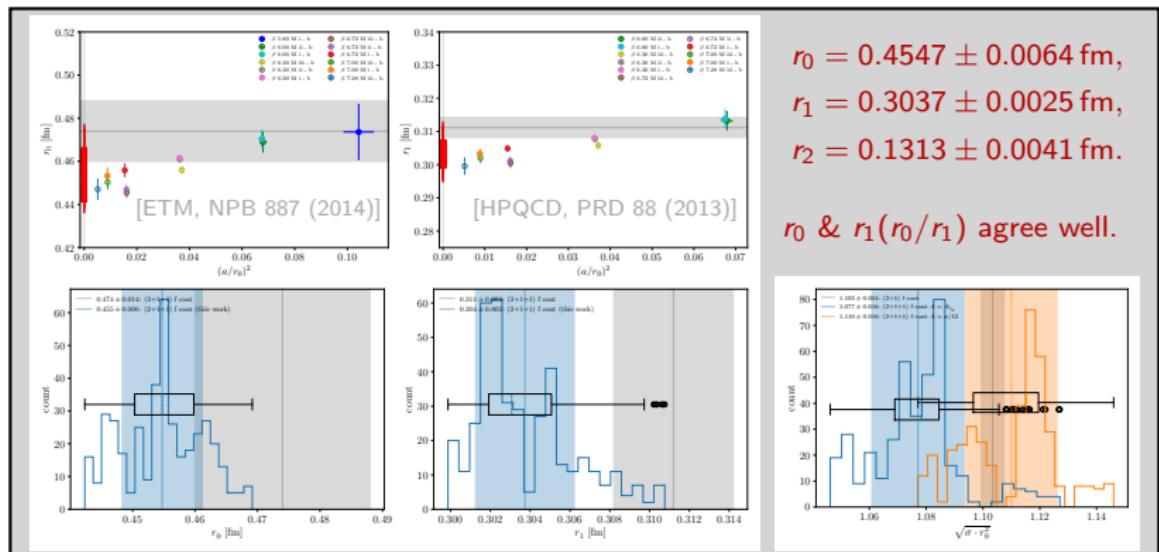
¹¹We use: $2M_K^2 = M_{K^\pm}^2 + M_{K^0}^2$ and $M_\pi^2 = M_{\pi^\pm}^2$ or $M_{\pi^0}^2$



The ratios r_0/r_1 and r_1/r_2



The scales r_0 and r_1 and the string tension

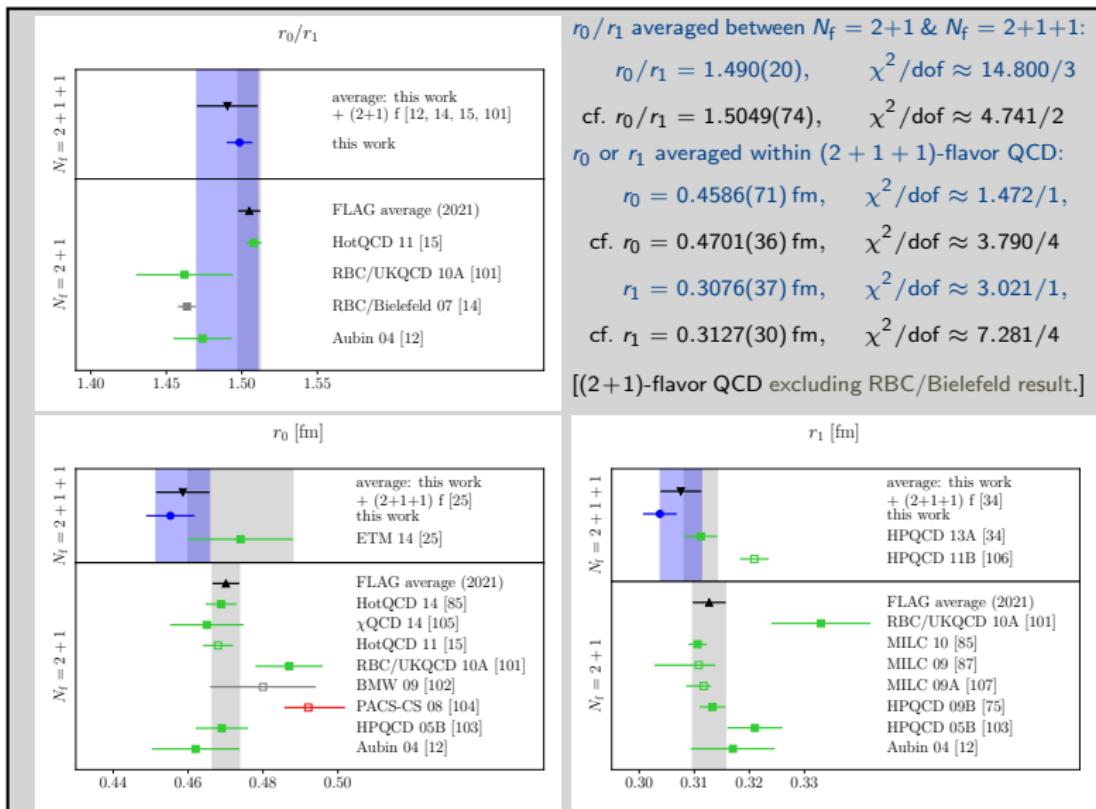


The string tension is consistent with (2 + 1)-flavor QCD ¹²:

$$\sqrt{\sigma r_0^2} = 1.077 \pm 0.016 \quad (A = A_{r_0}), \quad \sqrt{\sigma r_0^2} = 1.110 \pm 0.016 \quad (A = \pi/12).$$

¹²[RBC/Bielefeld, PRD 77 (2008) 014511]

Comparison to published results



Static energy in perturbation theory with massless sea

- The static energy is defined as

$$E_0(r) = \lim_{t \rightarrow \infty} \frac{i}{t} \ln \left\langle \text{tr } \mathcal{P} \exp \left[i g \oint_{r \times t} dz^\mu A_\mu(z) \right] \right\rangle,$$

and—for massless sea quarks—perturbatively known at N³LL

$$E_0(r) = \Lambda - \frac{C_F \alpha_s}{r} \left(1 + \# \alpha_s + \# \alpha_s^2 + \# \alpha_s^3 \ln \alpha_s + \# \alpha_s^3 + \# \alpha_s^4 \ln^2 \alpha_s + \# \alpha_s^4 \ln \alpha_s + \dots \right),$$

- Integrating the force eliminates the leading renormalon

$$E_0(r) = \int_{r^*}^r dr' F(r') + \text{const.}$$

- $E_0(r)$ depends on N_f via its coefficients $\#$ and via α_s
- α_s depends on a scale $\nu \sim 1/r$ (we use $\nu = 1/r$)



Massive charm sea

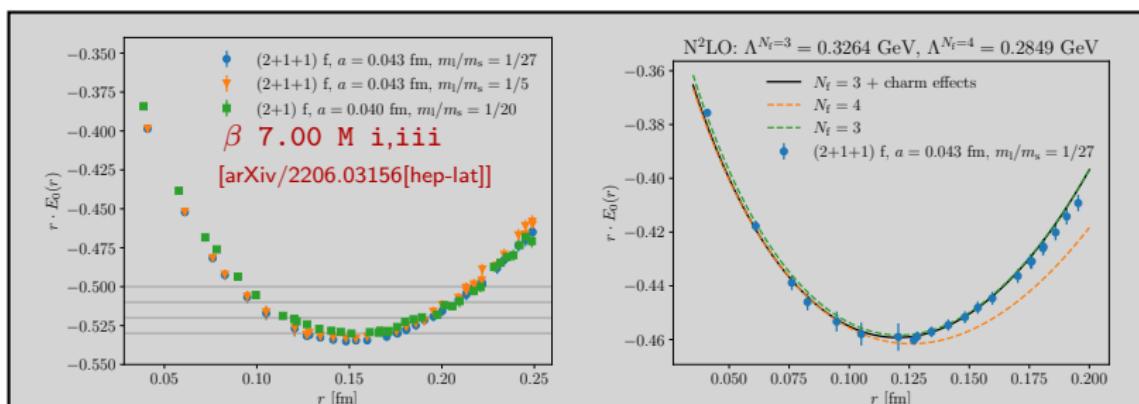
- Massive charm quark decouples for large r :

$rm_c \gg 1 \Rightarrow$ charm is effectively infinitely heavy $\Rightarrow N_f = 3$

$rm_c \ll 1 \Rightarrow$ charm is effectively massless $\Rightarrow N_f = 4$

- For $rm_c \sim 1 \Rightarrow$ massive correction $\delta V_m^{(N_f)}(r)$ known at $N^2\text{LO}$

$$E_{0,m}^{(N_f)}(r) = \int_{r^*}^r dr' F^{(N_f)}(r') + \delta V_m^{(N_f)}(r) + \text{const},$$



Conclusions

- We have computed the static energy $E_0(r)$ in $(2+1+1)$ -flavor QCD over a wide range of lattice spacings and quark masses.
- First simultaneous determination of both the scales r_0 , r_1 and their ratio r_0/r_1 , and the ratio r_1/r_2 and the string tension σ .
- r_1/a on coarse lattices is slightly lower than earlier results ¹³; procedural differences in defining r_1/a lead to larger errors.
- r_0/r_1 and $r_0\sqrt{\sigma}$ are consistent with $(2+1)$ -flavor QCD results.
- r_1/r_2 differs significantly from $(2+1)$ -flavor QCD results.
- We study the charm quark contribution across its decoupling.
- $E_0(r)$ at $r \lesssim 0.2$ fm is inconsistent with $(2+1)$ -flavor QCD, but well reproduced by perturbation theory at two-loop order.

¹³[MILC, PRD 82 (2010) 07450]; [MILC, PRD 87 (2013) 5, 054505]



Data sets

MILC (2 + 1 + 1)-flavor gauge ensembles ¹ used in this study.

our naming	$N_\sigma^3 \times N_\tau$	β	$a f_{p4s}$ (fm)	u_0	am_l	am_s	am_c	m_l/m_s	$(am_s)_{\text{tuned}}$	M_π (MeV)	#conf.
$\beta = 5.80$ M i	$32^3 \times 48$	5.80	0.15294	0.85535	0.00235	0.0647	0.831	phys	0.06852	131	1041
$\beta = 6.00$ M ii	$32^3 \times 64$	6.00	0.12224	0.86372	0.00507	0.0507	0.628	1/10 phys	0.05296	217	1000
$\beta = 6.00$ M i	$48^3 \times 64$			0.00184						132	709
$\beta = 6.30$ M iii	$32^3 \times 96$				0.0074	0.037	0.44	1/5		316	1008
$\beta = 6.30$ M ii	$48^3 \times 96$	6.30	0.08786	0.874164	0.00363	0.0363	0.43	1/10	0.03627	221	1031
$\beta = 6.30$ M i	$64^3 \times 96$			0.0012		0.432		phys		129	1074
$\beta = 6.72$ M iii	$48^3 \times 144$				0.0048	0.024	0.286	1/5		329	1017
$\beta = 6.72$ M ii	$64^3 \times 144$	6.72	0.05662	0.885773	0.0024	0.022	0.286	1/10	0.02176	234	1103
$\beta = 6.72$ M i	$96^3 \times 192$			0.0008		0.022	0.26	phys		135	1268
$\beta = 7.00$ M iii	$64^3 \times 192$				0.00316	0.0158	0.188	1/5		315	1165
$\beta = 7.00$ M i	$144^3 \times 288$	7.00	0.0426	0.892186	0.000569	0.01555	0.1827	phys	0.01564	134	478
$\beta = 7.28$ M iii	$96^3 \times 288$	7.28	0.03216	0.89779	0.00223	0.01115	0.1316	1/5	0.01129	309	821

Time and distance intervals in the full data set.

$a f_{p4s}$ (fm)	β	T_{\min}/a	T_{\max}/a	T_{\max} (fm)	r_{\max}/a	r_{\max} (fm)
0.15294	5.8	1	9	1.35	6	0.92
0.12224	6.0	1	6	0.73	6	0.73
0.08786	6.3	1	8	0.70	8	0.70
0.05662	6.72	1	10	0.57	12	0.68
0.0426	7.0	1	20	0.85	20	0.85
0.03216	7.28	1	28	0.91	24	0.78

¹[A. Bazavov, et al., PRD 98 (2018)]



Fits

- For each N_{st} , $\tau_{\min, N_{\text{st}}}(\mathbf{r}/a)$ is varied by $\pm a \Rightarrow$ fits are robust.
- 100 Jackknife pseudoensembles to propagate statistical errors.
- Randomized thinning of τ range to invert correlation matrix ¹⁴.
- Or full τ range, but restricted to the diagonal \Rightarrow consistent.
- We use Bayesian fits with loose, linear priors ($\geq 10\%$ width).

¹⁴Modified eigenvalue smoothing à la [C. Michael, A. McKerrell, PRD 51 (1995) 3745].



A consistent definition of the force on the lattice?

Wanted: **independent determinations** of the three scales!

- Small R : discretization artifacts $\sim 0.1 - 1\% \gg$ stat. errors
- The slope determining the $1/R$ coefficient A decreases with R
- Stat. errors of $E(R, a)$ data increases with R or a
- Density of $E(R, a)$ data increases with R or a
- Correlations between many data at similar $R \Rightarrow$ many small eigenvalues \Rightarrow cannot invert correlation matrix

Can we possibly **trust the statistical errors** we obtain for r_i/a ?



A consistent definition of the force on the lattice?

Asymmetric, randomized thinning in intervals (expected $r_i/a \pm x\%$)

- $x = 30$ (or 35 for the finest lattice)
- Out of $10 = \sqrt{N_J}$ pick 3(7) below (above) expected R
- Extend interval if necessary by demanding at least six(fourteen) below (above) to permit randomized picking
- If not possible relax to at least five(eleven) below (above)
- Pick $N_P = 100$ (or $N_P = 200$ for the finest lattice) R values

