Static Energy in $(2+1+1)$-Flavor Lattice QCD: Scale Setting and Charm Effects

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Based on [N. Brambilla, et al., arXiv/2206.03156[hep-lat]]

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Motivation

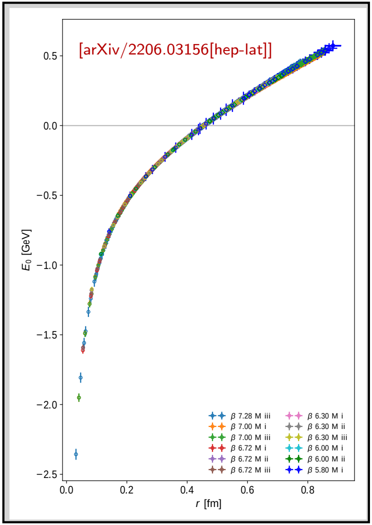
- Static energy $E_0(r)$ studied on the lattice since earliest times (**confinement**) [1], defined as ground state of Wilson loop.
- High accuracy, numerically cheap observable on the lattice.
- Of major importance for **scale setting** in the past – still now?
- Wilson loops and related correlators contribute in most **effective field theories** for heavy quarks and quarkonia.
- **No scheme change** required between lattice and continuum.
- To date, extraordinarily **detailed lattice studies** in pure gauge theory ($N_f = 0$) or $(2+1)$-flavor QCD [2]) $\Lambda_{\overline{MS}}^{N_f=0;3}$.
- So far, dearth of results in $N_f = (2+1+1)$-flavor lattice QCD.

[1] [K. Wilson, PRD 10 (1974) 2445]

[2] [A. Bazavov, et al., PRD 100 (2019) 11, 114511]

Summary

- We calculate the static energy in $(2+1+1)$-**flavor QCD** covering r 0:03 0:9 fm.
- We **self-consistently convert** the distance r=a and the static energy aE_0 **to physical units** using scales obtained from E_0.
- We resolve dependence of the scales and the effective string tension on **sea quark masses**.
- We study in detail **effects of the charm sea** for r ? 1=m_c.



Lattice setup

- $(2+1+1)$-flavor QCD **ensembles with HISQ** [3] and 1-loop Symanzik gauge action generated by MILC collaboration [4].
- **Three light quark masses** ($m_l{=}m_s \quad 1{=}27;\ 1{=}10;\ 1{=}5$), physical strange and charm sea.
- **Six lattice spacings** via f_{p4s} scale ($a_{f_{p4s}} \quad 0{:}03 \quad 0{:}15\,\mathrm{fm}$).
- We compute the **Wilson line correlator in Coulomb gauge**, with or without one step of HYP smearing [5].

$$W(r;\ ;a) = \prod_{u=0}^{=a\ \ 1} U_4(r;ua;a)\,;$$

$$C(r;\ ;a) = \left\langle \frac{1}{N^3} \sum_{x} \sum_{y=R(r)} \frac{1}{N_c N_r} \operatorname{tr}\left[W^{y}(x+y;\ ;a)\, W(x;\ ;a)\right] \right\rangle;$$

[3] [E. Follana, et al., PRD 75 (2007) 054502]

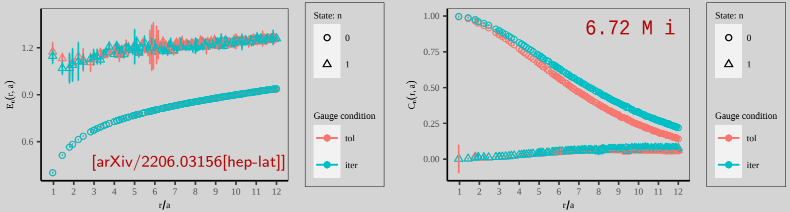
[4] [A. Bazavov, et al., PRD 98 (2018) 7, 074512]

[5] [A. Hasenfratz, F. Knechtli, PRD 64 (2001) 034504]

Spectrum of Wilson line correlators

- **Improved action**) spectral representation holds at =a 2.
- **Physical $T_{\mathbf{max}}$ and $r_{\mathbf{max}}$** ' 0:6 0:9 **fm** implies much fewer time slices on coarse lattices (factor five variation in a).
- On coarse lattices **limitation in number of states**, N_{st}.

$$C\,(r;\ ;a) = \mathrm{e}^{\ E_0(r;a)}\left(C_0\,(r;a) + \sum_{n=1}^{N_{\mathrm{st}}\ 1} C_n\,(r;a)\prod_{m=1}^{n}\mathrm{e}^{\ \Delta_m(r;a)}\right) + :::;$$



Energies $aE_{0;1}$ and overlap factors $C_{0;1}$ in different gauges.

Fits

- Up to N_r 500 $jrj{=}a$ values after averaging, cover factor 24.
- Up to $T_{\mathrm{max}}{=}a$ N_r 10^4 correlated data) automation!
- **Fit range** [i.e. $_{\mathrm{min};N_{\mathrm{st}}}(r{=}a)$] **varies with N_{st} and jrj**

$jrj + 0{:}2$ fm	$_{\mathrm{min};1}$	$0{:}3$ fm		for $N_{\mathrm{st}} = 1$	)	prior values;
$\frac{2}{3}jrj + 0{:}1$ fm	$_{\mathrm{min};2}$	$_{\mathrm{min};1}$	$2a$	for $N_{\mathrm{st}} = 2$	)	final choice;
$\frac{1}{3}jrj$	$_{\mathrm{min};3}$	$_{\mathrm{min};2}$	$2a$	for $N_{\mathrm{st}} = 3$	)	cross-check

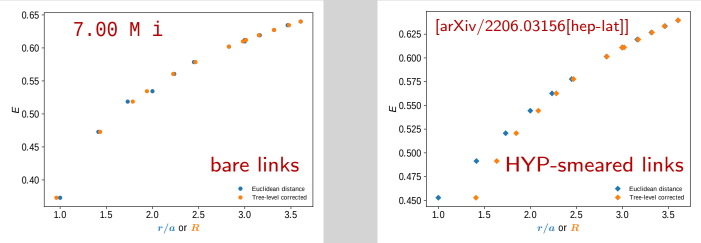
- We use Bayesian fits with **loose, linear priors** (10% width).
- Priors for excited state overlap factors with 100% width: no unacceptable examination of data, robust for prior variation.
- Priors for $\Delta_1 = E_1$ E_0 are from SU(3) pure gauge theory [6].

[6] [K. Juge, et al., PRL90, 161601 (2003)]

Discretization artifacts

- $E_0\,(r\,;a)$ is available only at discrete distances, and depends on the direction of r) not smooth in $|r| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
- Distance r_I defined via tree-level [7] gluon propagator $D\ \ (k)$

$$E_0^{\text{tree}}(r;a) = \quad C_F g_0^2 \int \frac{\mathrm{d}^3 k}{(2\ \ )^3} \mathrm{e}^{ik\ r} D_{44}(k) \qquad \frac{C_F g_0^2}{4}\,\frac{1}{r_I}:$$



Tree-level-correction (aka: -improvement) in use for $|r|/a \quad 6$.

[7] Extension to one loop ongoing: [G. v. Hippel, et al., in preparation: TUM-EFT 171/22]

Determination of scales

- The scales are defined in terms of **the force $F(r)$** $\quad \dfrac{\mathbf{d}E_0(r)}{\mathbf{d}r}$

$$r_i^2 F(r_i) = c_i; \quad i = 0; 1; 2$$

with $c_0 = 1{:}65$ [8], $c_1 = 1$ [9], and $c_2 = 1{=}2$ [10].

- The r_i correspond to **different regimes**. $(2+1)$-flavor QCD:

$$r_0 \quad 0{:}475 \text{ fm}; \quad r_1 \quad 0{:}3106 \text{ fm}; \quad r_2 \quad 0{:}145 \text{ fm};$$

- A **Cornell Ansatz** encodes the main features of $E_0(r)$ locally

$$E(R; a) = \quad \frac{A}{R} + B + \Sigma R;$$

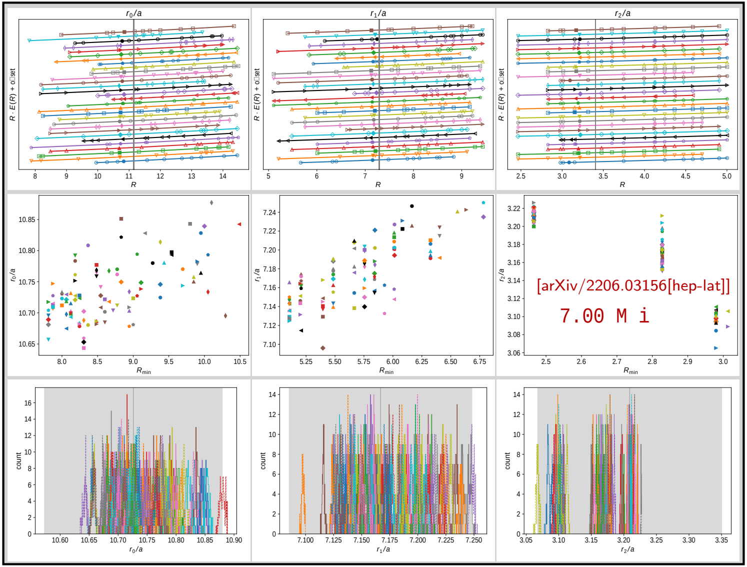
- $r_i{=}a = \sqrt{(c_i \quad A){=}\Sigma}$; identify Σ at large r with string tension
- **Consistent definition** of $F(r; a)$ across various a is difficult

[8] [R. Sommer, NPB 411 (1994) 839]

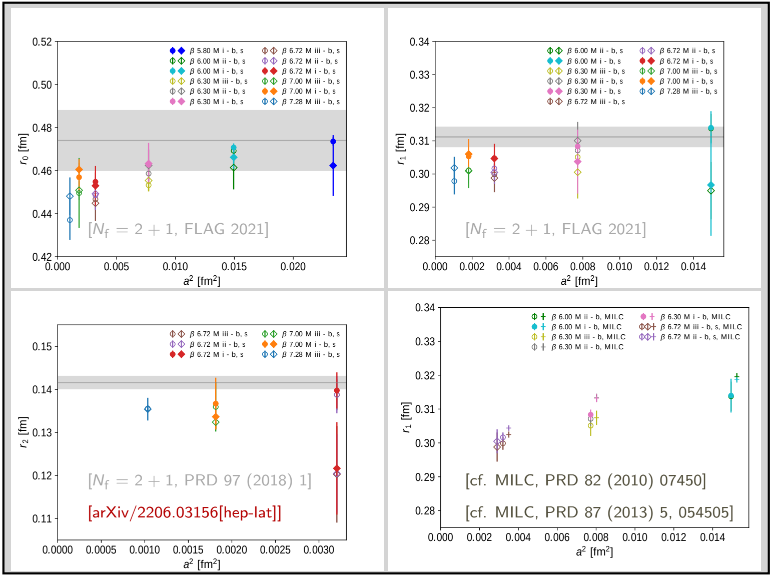
[9] [C. Bernard, et al., PRD 62 (2000) 034503]

[10] [A. Bazavov, et al., PRD 97 (2018) 1, 014510]

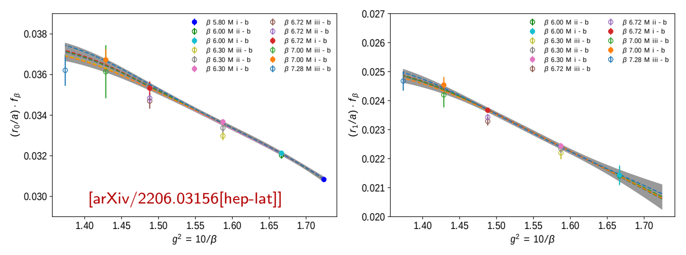
Defining the force through asymmetric, random picking



Lattice spacing, N_f, and quark mass dependence



Smoothing (via Allton-Ansatz)



$$\frac{a}{r_i} = \frac{C_0 f \;\; + C_2 g_0^2 f^3 + C_4 g_0^4 f^3}{1 + D_2 g_0^2 f^2}; \quad f \;\; = \text{two-loop} \quad \text{fct.} \;\; (N_f = 4)$$

$$C_0 = C_{00} + C_{01l} \frac{am_l}{f} + C_{01s} \frac{am_s}{f} + C_{01} \frac{am_{tot}}{f} + C_{02} \frac{(am_{tot})^2}{f};$$

$$C_2 = C_{20} + C_{21} \frac{am_{tot}}{f}; \quad am_{tot} = 2am_l + am_s + am_c;$$

Continuum extrapolation

- Using smoothed results we extrapolate to the continuum limit: r_0/r_1, r_1/r_2; $a_{f_{p4s}} r_0/a$, $a_{f_{p4s}} r_1/a$; $\sqrt{\;/a^2}\, r_0/a$
- Lattice spacing dependence via $x = (a/r_i)^2;\ i = 0,1$
- m_l dependence via $y = m_l/m_s$ (sea or tuned [4] m_s), where we use PDG values [11] in the continuum GMOR

$$m_l/m_s = 1/(2M_K^2/M^2 \quad 1) \quad \Rightarrow \quad \begin{cases} m_l/m_s|_{M^2=M^2} & = 0.04128; \\ m_l/m_s|_{M^2=M^2{}_0} & = 0.03851. \end{cases}$$

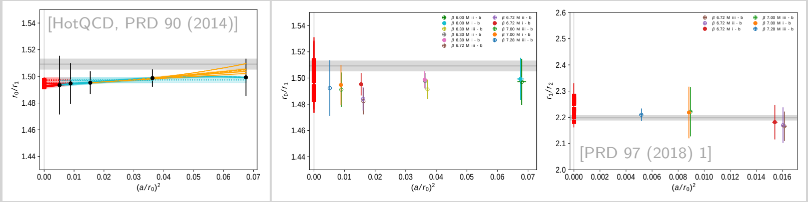
- Logs in the Symanzik expansion via $\ = 1$ or $_b = g_0^2/(4\ u_0^2)$

$$= {}_0 + {}^2[{}_1 x + {}_2 x y^{1,2}] + {}_3 x^2 + {}_4 y \qquad (l,q;l,qm;mc);$$

[4] [A. Bazavov, et al., PRD 98 (2018) 7, 074512]

[11] We use: $2M_K^2 = M_K^2 + M_{K^0}^2$ and $M^2 = M^2$ or $M^2{}_0$

The ratios r_0=r_1 and r_1=r_2

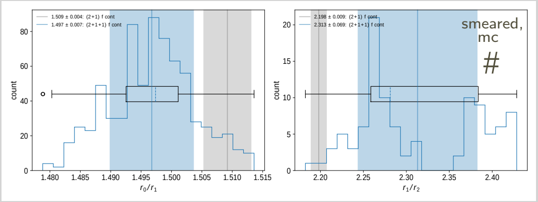


r_0=r_1 = 1:4968   0:0069;

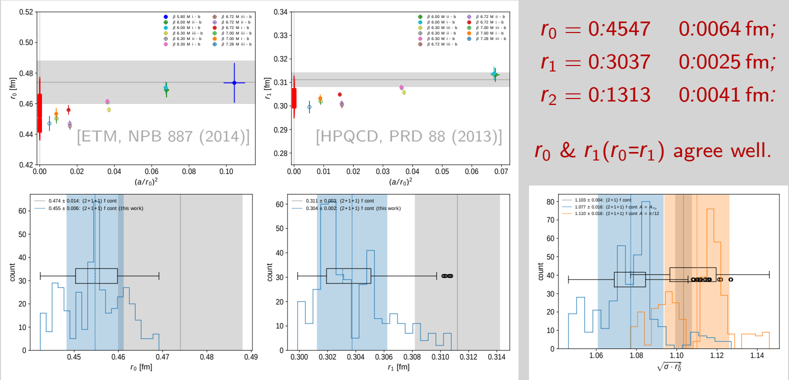
r_1=r_2 = 2:313   0:069:

AIC favors weighted average:

central r_0=r_1, lowest r_1=r_2.



The scales r_0 and r_1 and the string tension



$r_0 = 0{:}4547 \quad 0{:}0064$ fm;
$r_1 = 0{:}3037 \quad 0{:}0025$ fm;
$r_2 = 0{:}1313 \quad 0{:}0041$ fm:

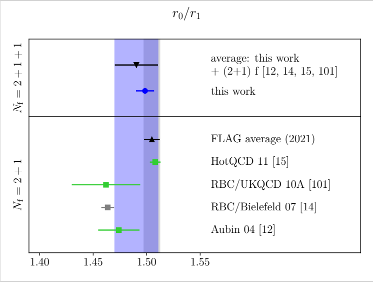
r_0 & $r_1(r_0{=}r_1)$ agree well.

The string tension is consistent with $(2+1)$-flavor QCD [12]:

$$\sqrt{r_0^2} = 1{:}077 \quad 0{:}016 \quad (A = A_{r_0}); \qquad \sqrt{r_0^2} = 1{:}110 \quad 0{:}016 \quad (A = \;\; {=}12):$$

[12] [RBC/Bielefeld, PRD 77 (2008) 014511]

Comparison to published results



$r_0 = r_1$ averaged between $N_f = 2+1$ & $N_f = 2+1+1$:

$r_0 = r_1 = 1{:}490(20)$; $\ ^2 = \mathrm{dof}\quad 14{:}800 = 3$

cf. $r_0 = r_1 = 1{:}5049(74)$; $\ ^2 = \mathrm{dof}\quad 4{:}741 = 2$

r_0 or r_1 averaged within $(2+1+1)$-flavor QCD:

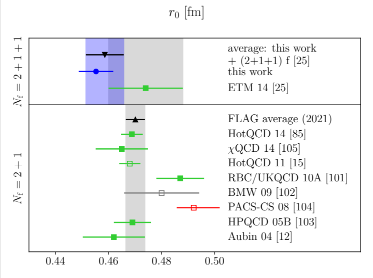
$r_0 = 0{:}4586(71)$ fm; $\ ^2 = \mathrm{dof}\quad 1{:}472 = 1$;

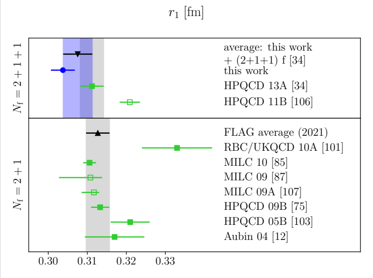
cf. $r_0 = 0{:}4701(36)$ fm; $\ ^2 = \mathrm{dof}\quad 3{:}790 = 4$

$r_1 = 0{:}3076(37)$ fm; $\ ^2 = \mathrm{dof}\quad 3{:}021 = 1$;

cf. $r_1 = 0{:}3127(30)$ fm; $\ ^2 = \mathrm{dof}\quad 7{:}281 = 4$

[(2+1)-flavor QCD excluding RBC/Bielefeld result.]





Static energy in perturbation theory with massless sea

- The static energy is defined as

$$E_0(r) = \lim_{t\,!\,1} \frac{\mathrm{i}}{t} \ln \left\langle \mathrm{tr}\, P \exp \left[ig \oint_{r \quad t} \mathrm{d}z \quad A \quad (z) \right] \right\rangle ;$$

and–for massless sea quarks–perturbatively known at $\mathrm{N}^3\mathrm{LL}$

$$E_0(r) = \Lambda \quad \frac{C_\mathrm{F} \quad _\mathrm{s}}{r} \left(1 + \# \quad _\mathrm{s} + \# \quad ^2_\mathrm{s} + \# \quad ^3_\mathrm{s} \ln \quad _\mathrm{s} + \# \quad ^3_\mathrm{s} + \# \quad ^4_\mathrm{s} \ln^2 \quad _\mathrm{s} + \# \quad ^4_\mathrm{s} \ln \quad _\mathrm{s} + : : : \right) ;$$

- Integrating the force eliminates the leading renormalon

$$E_0(r) = \int_r^r \mathrm{d}r^0 \, F(r^0) + \mathrm{const}:$$

- $E_0(r)$ depends on N_f via its coefficients $\#$ and via $\quad _\mathrm{s}$
- $_\mathrm{s}$ depends on a scale $\quad 1{=}r$ (we use $\quad = 1{=}r$)

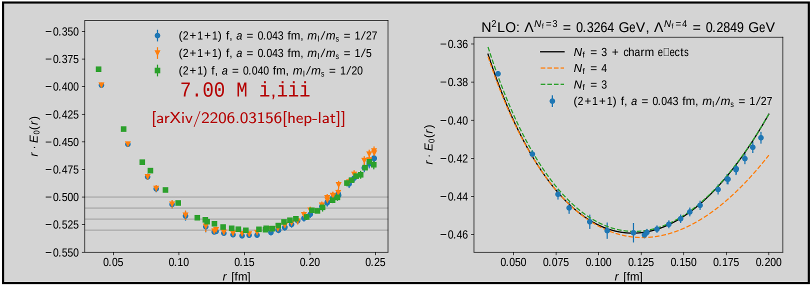
Massive charm sea

- Massive charm quark decouples for large r:

$$rm_c \quad 1 \quad) \quad \text{charm is effectively infinitely heavy} \quad) \quad N_f = 3$$
$$rm_c \quad 1 \quad) \quad \text{charm is effectively massless} \quad) \quad N_f = 4$$

- For $rm_c \quad 1 \quad)$ massive correction $V_m^{(N_f)}(r)$ known at N^{2}LO

$$E_{0;m}^{(N_f)}(r) = \int_{r}^{r} \mathrm{d}r^{0} \, F^{(N_f)}(r^{0}) + V_m^{(N_f)}(r) + \text{const};$$



Conclusions

- We have computed the static energy $E_0(r)$ in $(2+1+1)$-flavor QCD over a wide range of lattice spacings and quark masses.
- First simultaneous determination of both the scales r_0, r_1 and their ratio r_0=r_1, and the ratio r_1=r_2 and the string tension .
- r_1=a on coarse lattices is slightly lower than earlier results [13]; procedural differences in defining r_1=a lead to larger errors.
- r_0=r_1 and $r_0{}^{p\,-}$ are consistent with $(2+1)$-flavor QCD results.
- r_1=r_2 differs significantly from $(2+1)$-flavor QCD results.
- We study the charm quark contribution across its decoupling.
- $E_0(r)$ at r . 0:2 fm is inconsistent with $(2+1)$-flavor QCD, but well reproduced by perturbation theory at two-loop order.

[13] [MILC, PRD 82 (2010) 07450]; [MILC, PRD 87 (2013) 5, 054505]

Data sets

MILC $(2+1+1)$-flavor gauge ensembles [1] used in this study.

our naming	N^3	N		$a_{f_{p4s}}$ (fm)	u_0	am_l	am_s	am_c	m_l=m_s	$(am_s)_{\rm tuned}$	M (MeV)	#conf.
`5.80 M i`	32^3	48	5.80	0.15294	0.85535	0.00235	0.0647	0.831	phys	0.06852	131	1041
`6.00 M ii`	32^3	64	6.00	0.12224	0.86372	0.00507	0.0507	0.628	1=10	0.05296	217	1000
`6.00 M i`	48^3	64				0.00184			phys		132	709
`6.30 M iii`	32^3	96	6.30	0.08786	0.874164	0.0074	0.037	0.44	1=5	0.03627	316	1008
`6.30 M ii`	48^3	96				0.00363	0.0363	0.43	1=10		221	1031
`6.30 M i`	64^3	96				0.0012		0.432	phys		129	1074
`6.72 M iii`	48^3	144	6.72	0.05662	0.885773	0.0048	0.024	0.286	1=5	0.02176	329	1017
`6.72 M ii`	64^3	144				0.0024			1=10		234	1103
`6.72 M i`	96^3	192				0.0008	0.022	0.26	phys		135	1268
`7.00 M iii`	64^3	192	7.00	0.0426	0.892186	0.00316	0.0158	0.188	1=5	0.01564	315	1165
`7.00 M i`	144^3	288				0.000569	0.01555	0.1827	phys		134	478
`7.28 M iii`	96^3	288	7.28	0.03216	0.89779	0.00223	0.01115	0.1316	1=5	0.01129	309	821

Time and distance intervals in the full data set.

$a_{f_{p4s}}$ (fm)		$T_{\rm min}$=a	$T_{\rm max}$=a	$T_{\rm max}$ (fm)	$r_{\rm max}$=a	$r_{\rm max}$ (fm)
0.15294	5.8	1	9	1.35	6	0.92
0.12224	6.0	1	6	0.73	6	0.73
0.08786	6.3	1	8	0.70	8	0.70
0.05662	6.72	1	10	0.57	12	0.68
0.0426	7.0	1	20	0.85	20	0.85
0.03216	7.28	1	28	0.91	24	0.78

[1] [A. Bazavov, et al., PRD 98 (2018)]

Fits

- For each N_{st}, $_{\mathrm{min};N_{\mathrm{st}}}(r=a)$ is varied by a) fits are robust.
- 100 Jackknife pseudoensembles to propagate statistical errors.
- Randomized thinning of range to invert correlation matrix [14].
- Or full range, but restricted to the diagonal) consistent.
- We use Bayesian fits with loose, linear priors (10% width).

[14] Modi ed eigenvalue smoothing a la [C. Michael, A. McKerrell, PRD 51 (1995) 3745].

A consistent definition of the force on the lattice?

Wanted: **independent determinations** of the three scales!

- Small R: discretization artifacts $0{:}1$ 1% stat. errors
- The slope determining the $1{=}R$ coefficient A decreases with R
- Stat. errors of $E(R;a)$ data increases with R or a
- Density of $E(R;a)$ data increases with R or a
- Correlations between many data at similar R) many small eigenvalues) cannot invert correlation matrix

Can we possibly **trust the statistical errors** we obtain for $r_i{=}a$?

A consistent definition of the force on the lattice?

Asymmetric, randomized thinning in intervals (expected r_i=a x%)

- $x = 30$ (or 35 for the finest lattice)
- Out of $10 = \sqrt{N_J}$ pick 3(7) below (above) expected R
- Extend inverval if necessary by demanding at least six(fourteen) below(above) to permit randomized picking
- If not possible relax to at least five(eleven) below(above)
- Pick $N_P = 100$ (or $N_P = 200$ for the finest lattice) R values