

The static energy of a quark-antiquark pair from Laplacian eigenmodes

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Motivation

- ▶ calculate the static potential energy with high resolution
 - ▶ matching the lattice QCD potential with the perturbative potential to determine $\Lambda_{\overline{MS}}$ in Fourier space, e.g., [Karbstein et al. \(2014\)](#)
 - ▶ observation of string breaking in QCD, e.g., [Bali et al. \(2008\)](#), [Bulava et al. \(2019\)](#)
- ⇒ we have to work with off-axis separated quarks
 - ▶ the spatial part of the Wilson loop has to go over stair-like paths through the lattice → not unique, computationally expensive
- ⇒ alternative operator which ensures gauge covariance of the quark-anti-quark $Q(\vec{x})\bar{Q}(\vec{y})$ trial state
 - ▶ required gauge transformation behavior:

$$U'_s(\vec{x}, \vec{y}) = G(\vec{x})U_s(\vec{x}, \vec{y})G^\dagger(\vec{y})$$



Laplacian Eigenmodes

- ▶ Consider the 3D covariant lattice Laplace operator:

$$\begin{aligned} \Delta V = & \frac{1}{a^2} [U_x^\dagger(x-a, y, z)V(x-a, y, z) - 2V(\vec{x}) + U_x(\vec{x})V(x+a, y, z) \\ & + U_y^\dagger(x, y-a, z)V(x, y-a, z) - 2V(\vec{x}) + U_y(\vec{x})V(x, y+a, z) \\ & + U_z^\dagger(x, y, z-a)V(x, y, z-a) - 2V(\vec{x}) + U_z(\vec{x})V(x, y, z+a)] \end{aligned}$$

- ▶ transformation behavior: $\Delta' = G(\vec{x})\Delta G^\dagger(\vec{y})$
- ▶ consider $V(\vec{x})$ an eigenvector: $\Delta V(\vec{x}) = \lambda V(\vec{x})$

$$\begin{aligned} \Delta' V'(\vec{x}) &= \lambda V'(\vec{x}) \\ G(\vec{x})\Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda V'(\vec{x}) \\ \Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda G^\dagger(\vec{x})V'(\vec{x}) \end{aligned}$$

- ▶ $V(\vec{x})$ and $G^\dagger(\vec{x})V'(\vec{x})$ are members of the same eigen-space



Static $Q\bar{Q}$ pair, Wilson loop alternative...

- ▶ idea taken from [Neitzel et al. \(2016\)](#) SU(2)
- ▶ SU(3): the eigenvalues are in general non-degenerate

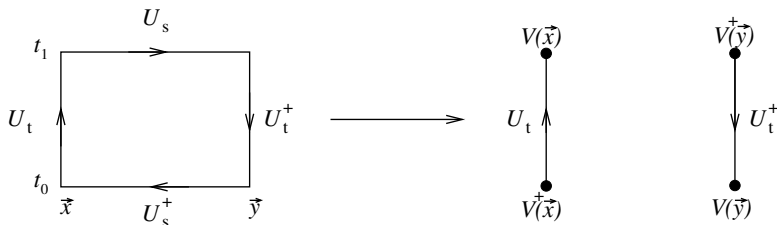
- ▶ spatial Wilson line: $U'_s(\vec{x}, \vec{y}) = G(\vec{x})U_s(\vec{x}, \vec{y})G^\dagger(\vec{y})$

$$V'(\vec{x})V^\dagger(\vec{y}) = G(\vec{x})V(\vec{x})V^\dagger(\vec{y})G^\dagger(\vec{y})$$

- ▶ Wilson loop of size $(R = |\vec{x} - \vec{y}|) \times (T = |t_1 - t_0|)$

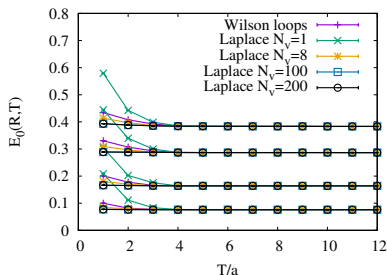
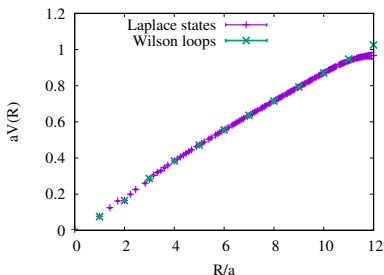
$$W(R, T) = U_t(\vec{x}; t_0, t_1)U_s(\vec{x}, \vec{y}; t_1)U_t^\dagger(\vec{y}; t_0, t_1)U_s^\dagger(\vec{x}, \vec{y}; t_0)$$

$$\rightarrow U_t(\vec{x}; t_0, t_1)V_j(\vec{x}, t_1)V_j^\dagger(\vec{y}, t_1)U_t^\dagger(\vec{y}; t_0, t_1)V_i(\vec{y}, t_0)V_i^\dagger(\vec{x}, t_0)$$



Simulations

- ▶ $24^3 \times 48$, $\beta = 5.3$, $N_f = 2$, $\kappa = 0.13270$, $a = 0.0658$ fm
- ▶ (on-axis) Wilson loops from 4646 measurements, 4D HYP
- ▶ Laplace states from 1161 measurements, 20 APE (0.5) for Lanczos, 1 4D HYP for temporal Wilson lines
- ▶ static potential $aV(R) = \lim_{T \rightarrow \infty} \log[W(R, T)/W(R, T + 1)]$
- ▶ at $R = 12a$ the force between $Q\bar{Q}$ must vanish due to symmetry
- ▶ more eigenvectors gives earlier plateau and increase precision



Gaussian profiles

$$W_{kl}(R, T) = \sum_{i,j}^{N_v} N_{kl}(\lambda_i, \lambda_j) \sum_{\vec{x}, t_0} \left\langle V_i^\dagger(\vec{x}, t_0) U_t(\vec{x}; t_0, t_1) V_j(\vec{x}, t_1) V_j^\dagger(\vec{y}, t_1) U_t^\dagger(\vec{y}; t_0, t_1) V_i(\vec{y}, t_0) \right\rangle$$

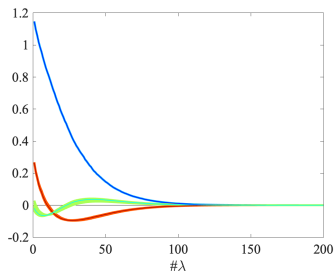
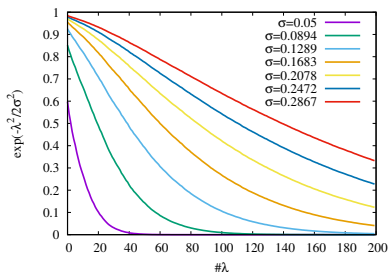
Gaussian profile functions:

$N_{kl}(\lambda_i, \lambda_j) = \exp(-\lambda_i^2/2\sigma_k^2) \exp(-\lambda_j^2/2\sigma_l^2)$ using 7 different σ values

⇒ prune W_{kl} using 3 most significant singular vectors u_i at t_0

...improves stability and keeps useful operators

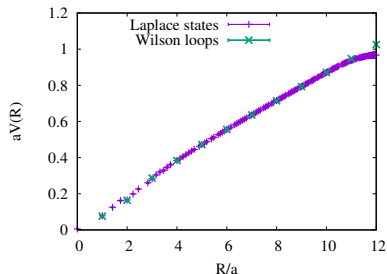
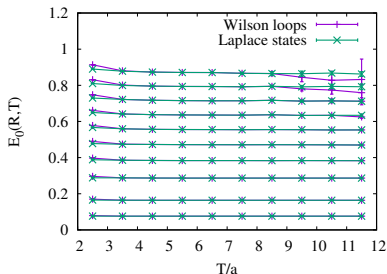
⇒ pruned ("optimal") profiles $u_{i,j} \exp(-\lambda^2/\sigma_j^2)$



Improved Results

Solve generalized eigenvalue problems for

- ▶ correlation matrix of Wilson loops with 3 spatial smearing levels
- ▶ Laplace trial states W_{kl} (or pruned version $\bar{W}_{ij} = u_i^\dagger W_{kl} u_j$)



- ⚡ on-axis Wilson loops for 3 spatial smearing levels (0,10,20HYP) are equally expensive as the calculation of 100 Laplacian eigenvectors and Laplace states with 3 Gaussian profiles including off-axis distances!



Fractional overlaps with the ground state

| R | $\#\lambda = 1$ | 8 | 64 | 200 | Gauss | Wloop | HYP2 |
|-----|-----------------|----------|----------|----------|----------|----------|----------|
| 1 | 0.773(3) | 0.945(1) | 0.970(1) | 0.982(1) | 0.993(1) | 0.921(1) | 0.983(1) |
| 2 | 0.747(4) | 0.929(2) | 0.964(1) | 0.987(1) | 0.989(1) | 0.891(1) | 0.978(1) |
| 3 | 0.723(4) | 0.878(2) | 0.984(2) | 0.986(1) | 0.988(1) | 0.867(1) | 0.972(2) |
| 4 | 0.726(5) | 0.874(3) | 0.921(2) | 0.984(2) | 0.986(2) | 0.841(2) | 0.965(3) |
| 5 | 0.637(6) | 0.871(4) | 0.979(3) | 0.982(3) | 0.983(3) | 0.813(2) | 0.956(5) |
| 6 | 0.629(6) | 0.869(4) | 0.978(4) | 0.980(3) | 0.981(3) | 0.793(3) | 0.948(6) |
| 7 | 0.619(7) | 0.869(5) | 0.977(4) | 0.979(4) | 0.987(4) | 0.772(3) | 0.934(7) |
| 8 | 0.598(8) | 0.862(6) | 0.972(5) | 0.970(4) | 0.964(4) | 0.745(4) | 0.953(8) |
| 9 | 0.572(8) | 0.857(6) | 0.960(5) | 0.934(4) | 0.963(3) | 0.708(4) | 0.947(9) |
| 10 | 0.540(9) | 0.840(7) | 0.965(5) | 0.931(5) | 0.95(1) | 0.671(5) | 0.94(1) |
| 11 | 0.426(9) | 0.807(7) | 0.943(6) | 0.93(1) | 0.956(9) | 0.649(4) | 0.93(1) |
| 12 | 0.33(7) | 0.79(2) | 0.94(1) | 0.92(1) | 0.95(1) | 0.64(2) | 0.92(1) |

t -average over mass-plateau region of

$$\frac{W(R, t)}{W(R, t_0)} \frac{\cosh\left(\left(\frac{T}{2} - t_0\right) m_0\right)}{\cosh\left(\left(\frac{T}{2} - t\right) m_0\right)}$$



Conclusions & Outlook

- ✓ alternative operator for a static quark-anti-quark pair based on Laplacian eigenmodes
- ✓ improved version (several eigenvectors weighted with Gaussian profiles) gives earlier plateau and better signal
- ✓ much higher resolution of the potential energy as off-axis distances basically come "for free"
- ✓ implementation of static-light (charm) correlator using "perambulators" $V(t_1)D^{-1}V(t_2)$ from distillation framework (quark field smearing via projection $\psi \rightarrow VV^\dagger\psi$), see [Knechtli et. al \(2022\)](#), \Rightarrow talk by J.-A. Urrea-Niño on hadron spectroscopy
- 🔧 putting together building blocks for observation of string breaking in QCD (mixing matrix of static and light quark propagators)
- 🔧 also working on hybrid static potentials (hybrid meson masses), instead of "gluonic handles" (excitations) use derivatives of V



Acknowledgements

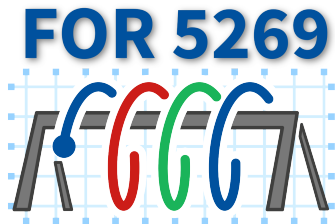
THANK YOU

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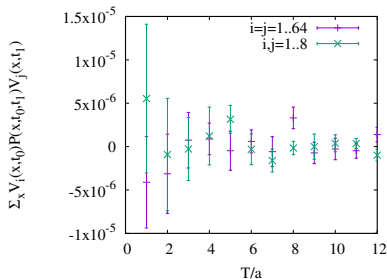
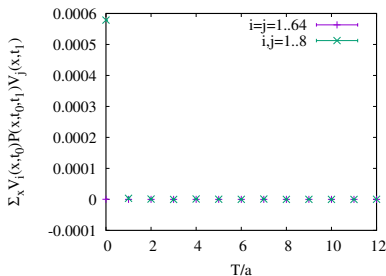
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Extra: Static quark line, SVD, GEVP

$$Q(T) = \sum_{i,j} \sum_{\vec{x}, t_0}^{N_v} \langle V_i^\dagger(\vec{x}, t_0) U_t(\vec{x}; t_0, t_1) V_j(\vec{x}, t_1) \rangle$$



- ▶ SVD: $W = UDV^\dagger$; U, V unitary, columns are orthonormal bases
- ▶ GEVP: $W(t)v_k = \rho_k W(t_0)v_k$, ρ_k give effective energies
- ▶ u_i or v_k can be used to get 'optimal' profiles for energy states

Bonus: Static-light (charm) meson

$$C(t) = - \sum_{t_0, i, j} \left\langle \text{tr}_d \{ [V_i^\dagger D^{-1} V_j](t_0 + t, t_0) P_+ \} \right. \\ \left. \sum_{\vec{x}} V_j^\dagger(\vec{x}, t_0) U_t(\vec{x}; t_0, t_0 + t) V_i(\vec{x}, t_0 + t) \right\rangle$$

- ▶ with light (charm) perambulators $V^\dagger D^{-1} V$
- ▶ HYP smearing of temporal links in U_t removes free energy

