# Anisotropy from the Wilson flow in QED 2+1

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# Lattice action

We consider a $U(1)$ gauge theory in $2+1$ dimensions and regularize it on a asymmetric lattice (Degrand and DeTar (2006)):

$$S = \frac{\beta}{\xi}\sum_x \sum_\nu \left[1 - \frac{1}{N_c}\mathrm{ReTr}(P_{0\nu}(x))\right] + \beta\xi \sum_x \sum_{j<i}\left[1 - \frac{1}{N_c}\mathrm{ReTr}(P_{ij}(x))\right]$$

where $P_{\mu\nu}$ is the standard Wilson Plaquette (Gattringer and Lang (2009)):

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x) \quad .$$

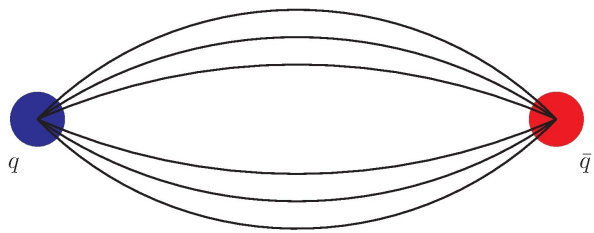
At the boundary we impose periodic boundary conditions. $\beta = \frac{2N_c}{g^2}$ and $\xi$ is the *bare anisotropy*.

Due to its group structure, this theory is often referred as (here *anisotropic*) QED $2+1$ or $\mathrm{QED}_3$.

It is relevant for some condensed matter systems (see e.g. Kosiński et al. (2012)).

It resembles QCD (in $3+1$ dimensions), showing:

- dynamical mass generation (Maris and Lee (2003))
- confinement (DeGrand (2019)).



- It is studied with Monte Carlo techniques :

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi e^{-\sum_x \mathcal{L}[\phi](g_i^L)}$$

**software**: https://github.com/urbach/su2

- There exist Hamiltonian formulations (Clemente et al. (2022)) suited to run on NISQ (Noisy Intermediate-Scale Quantum) devices in the near future.

$$U(t) = e^{-iH(g_i^H)t}$$

We want to combine Lagrangian and Hamiltonian results

$\rightarrow$ required non-perturbative matching of the couplings $g_i$.

# Hamiltonian limit

*A little background…*

Lattice correlators are defined as:

$$C_T(t) = \langle O_1(t) O_2(0) \rangle_T = \frac{\mathrm{Tr}[e^{-H(T-t)} O_1 e^{-Ht} O_2]}{\mathrm{Tr}[e^{-HT}]} = \frac{\sum_n \langle n | e^{-H(T-t)} O_1 e^{-Ht} O_2 | n \rangle}{\sum_n \langle n | e^{-HT} | n \rangle}$$

- We insert $N$ identity operators for each timeslice $t_k = \epsilon k$ (aka $a_t k$):

$$1 = \sum_m \frac{1}{2E_m} | m \rangle \langle m |$$

- After some algebra we find:

$$C_T(t) = \frac{\int \mathcal{D}\phi\, e^{-S_T[\phi]} O_1(t) O_2(0)}{\int \mathcal{D}\phi\, e^{-S[\phi]}}$$

where $S_T$ is the discretized action (but with different lattice spacings in the time and space directions!).

- Now one says that in the limit $a_t, a_s \to 0$ spacetime symmetry is recovered.
  $\implies$ we **define** the path integral with $a_t = a_s$.

- The limit $a_t \to 0\,,\ a_s \neq 0$ is the limit to the lattice Hamiltonian.

# Non perturbative matching

**Steps**:

- Calculation of $n$ observables $O_i$ in the two formalisms
- Lagrangian $\rightarrow$ Monte Carlo
- Hamiltonian $\rightarrow$ tensor network, quantum simulation, …
- In the action we introduce an anisotropy $\xi$ $\Longrightarrow$ different temporal and spatial lattice spacings:

$$a_t \neq a_s$$

- Find the limit $a_t \rightarrow 0$ of the $O_i$
- Find the couplings $g_i^H$ that match this limit of the Lagrangian

# Motivation

- Small $\beta$ region: harder determination of $\xi_R$ from the static potential
- Wilson flow renormalizes fields and the determination of $\xi_R$ from the Wilson flow is known to work in QCD (Borsányi et al. (2012))

**Main idea**

- Compute observables at fixed $a_s$:
  - $$\tau_0^2 E(\tau_0) = c$$
  - vary $\xi$ and find the $\beta$ s.t. $a_s$ is constant
  - Find $\xi_R = \xi(\tau_0)$ for each ensemble
  - Go to $\xi_R \to 0$
- Extrapolate to $a_t \to 0$ through $\xi_R = a_t / a_s$

# Matching with the Hamiltonian

Here we consider the pure gauge theory and want to match the 1st eigenvalue (mass gap). This is the lightest state of the theory (Loan and Ying (2006)): **glueball** $0^{--}$

Numerical studies in $U(1)$ (Athenodorou and Teper (2019)) and $SU(N)$ (Teper (1998)) theories find that the glueball $0^{--}$ is the lightest state.

**Interpretation**

In the continuum limit we expect a theory of free *screened* photons of mass $m_D = m^{gs}_{0^{--}}$ (Athenodorou and Teper (2019)):

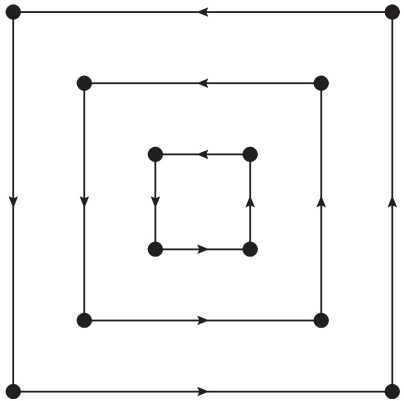
$$m^{ex1}_{0^{--}} = \frac{3}{2} m^{gs}_{0^{++}} = 3 m^{gs}_{0^{--}}$$

# Glueball correlators

$$\phi_i(x) = \mathrm{Tr}\left[\prod_{k\in\mathcal{L}_i} U_{\mu_k}(x_k + \hat{e}_k)\right] \quad , \; x_1 = x_n = x$$

We use spatial square loops of size length $r$:

$$U_{ij}^{(r)}(x) = \prod_{k=0}^{r} U_i(x + k\hat{e}_1)\prod_{k=0}^{r} U_j(x + r\hat{e}_1 +$$



# $PC$ combination

Combination of $\phi_i$ with its transform under parity $\phi^{(P)}$ (Gattringer and Lang (2009)) and taking their real/imaginary part:

$$\phi_i^{PC} = \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} \left[ \phi_i \pm \phi_i^{(P)} \right] \quad ,$$

- Finally, we project to $\vec{p} = \vec{0}$:

$$\varphi_i^{PC}(t, \vec{0}) = \frac{1}{V} \sum_{\vec{x}} \phi_i^{PC}(t, \vec{x}) = \frac{1}{L^2} \frac{2}{(d-1)(d-2)} \sum_{\vec{x}} \sum_{j<i} \mathrm{Re}\ \mathrm{Tr}[U_{ij}(t, \vec{x})]$$

# Glueball correlation functions

The glueball correlators are build averaging over all possible timeslices:

$$C^{PC}_{r_i r_j}(t) = \frac{1}{T}\sum_{\tau=1}^{T} \langle \varphi^{PC}_{r_i}(t+\tau)\varphi^{PC}_{r_j}(\tau)\rangle$$

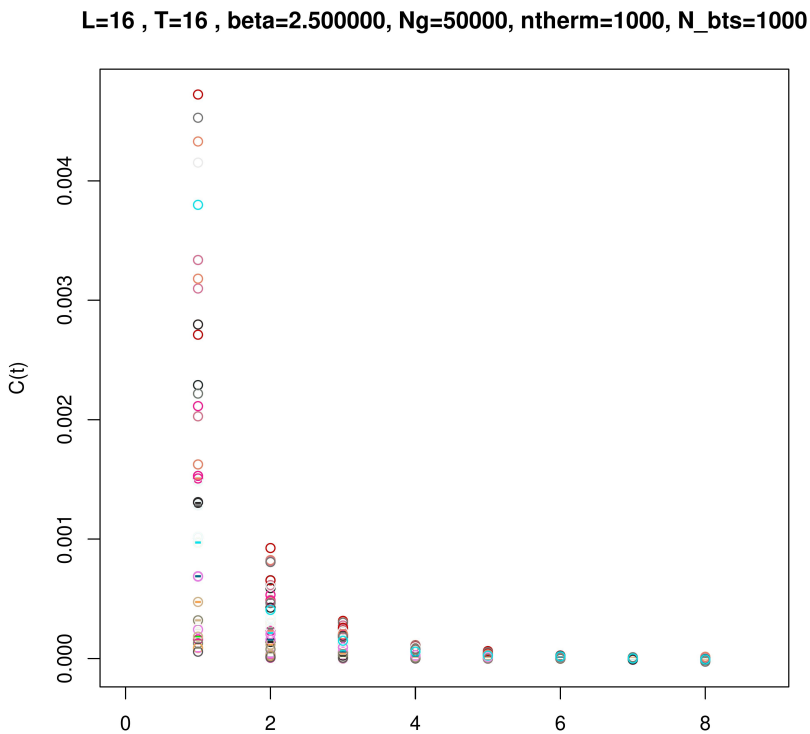
The large-time behavior is:

$$C^{PC}_{r_i r_j}(t) \to v_{r_i r_j} + A_{r_i r_j}(e^{-E_0^{PC} t} \pm e^{-E_0^{PC}(T-t)})$$

where $v_{r_i r_j} = \langle \varphi^{PC}_{r_i}(0)\varphi^{PC}_{j}(0)\rangle$.

The VEV is subtracted exactly as $C_{\text{sub}}(t) = C(t) - C(t+1)$. Combining the $C_{r_i r_j}$ we can apply the GEVP.

(work in progress...)



# Renormalized anisotropy

Lagrangian and Hamiltonian formalism are equivalent at zero temporal lattice spacing: $a_t \to 0$.

We write the action as:

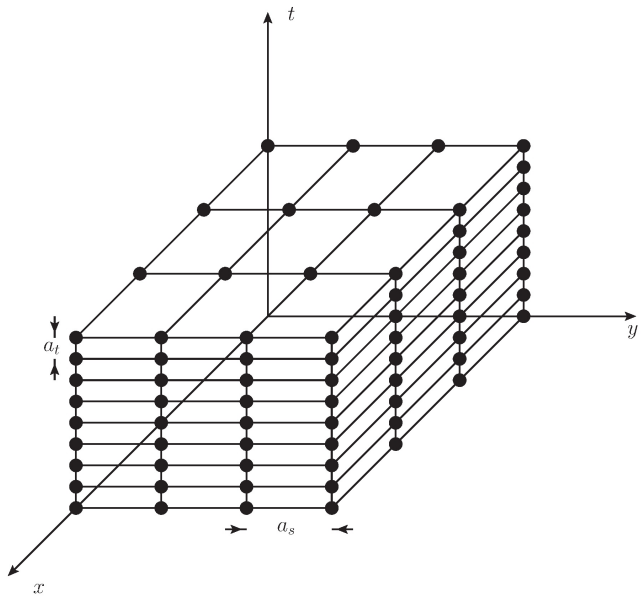
$$S = \beta_t S_t + \beta_s S_s$$

- $\beta_t = \beta / \xi$ temporal coupling (electric field)

- $\beta_s = \beta \xi$ spatial coupling (magnetic field)

$\beta_t \neq \beta_s \;\to$ anisotropy of the lattice: we have a temporal and spatial lattice spacings, $a_t$ and $a_s$.

$\xi \neq 1$



**How to go to $a_t \to 0$ ?**

Need to compute the renormalized anisotropy:

$$\xi_R = \frac{a_t}{a_s} \neq \xi$$

and extrapolate along $a_s = \text{const.}$

**How to compute $\xi_R$?**

- Static potential $V(r)$ between $q\bar{q}$ pairs at distance $r$.
- Wilson flow: $\xi(t)$ fixed at some $t_0/a_s^2$ along the flow.

# Static potential

In $2+1\,d$ the pair $e^+e^-$ at distance $r$ has potential (Clemente et al. (2022)):
$V(r) = V_0 + \alpha \log(r) + \sigma r$

## Anisotropy

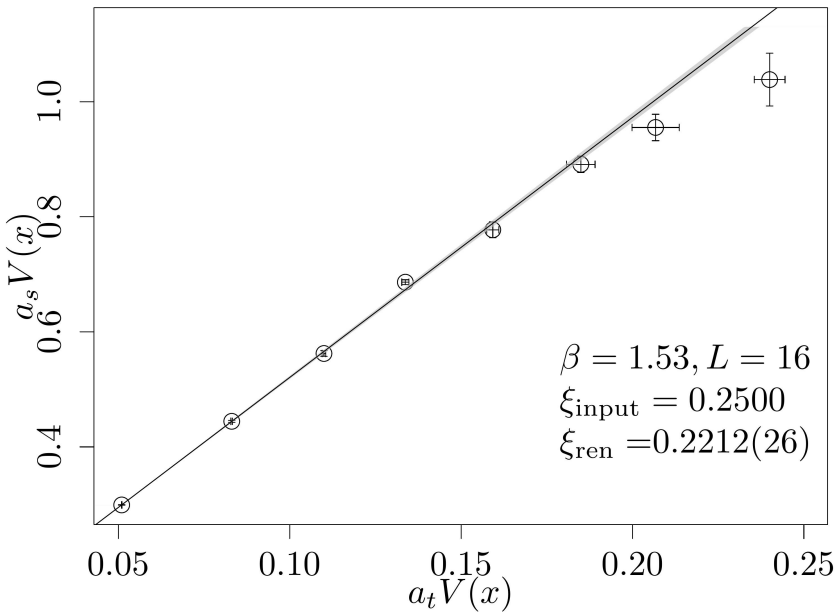
$$a_t V(a_s x) = \xi_R a_s V(a_s x)$$

$V$ is extracted from the $t \gg 1$ behavior of Wilson loop expectation values ($r = |x|$):

$$\frac{W_{ts}(x, t+1)}{W_{ts}(x, t)} \xrightarrow{t \to \infty} e^{-a_t V(r)}$$

$$\frac{W_{ss}(x, y+1)}{W_{ss}(x, y)} \xrightarrow{y \to \infty} e^{-a_s V(r)}$$

# $\xi_R$ from the static potential



# Wilson flow

Wilson flow evolution (Lüscher (2010)):

$$\dot{V}_\tau(x,\mu) = -\frac{1}{\beta}\{\nabla_\mu(x) S_G[V_\tau]\}\, V_\tau(x,\mu), \qquad V_0(x,\mu) = U(x,\mu)$$

where

$$\nabla_\mu(x) f(U) = -iT^a \frac{d}{d\omega} f\left(e^{i\omega T^a} U\right)\Big|_{\omega=0}$$

**Main properties**:

- For any $t > 0$ the fields are renormalized.

$\rightarrow$ we can extract $\xi_R$ from the flow evolution of $E_{ts}$ and $E_{ss}$

- Perturbative calculation:

$$\langle E \rangle \propto t^{-D/2}$$

$\rightarrow$ choose $\tau_0$ in the non perturbative region of the flow.

# $\xi(t)$

The energy density of the system is:

$$\mathcal{E} \propto \sum_{i \neq j} F_{ij}^2 + 2 \sum_i F_{0i}^2 = (d-1)(d-2)E_{ss} + 2(d-1)E_{ts}$$

In the continuum limit $E_{ss} = E_{ts} = \bar{E}$

On the lattice we compute:

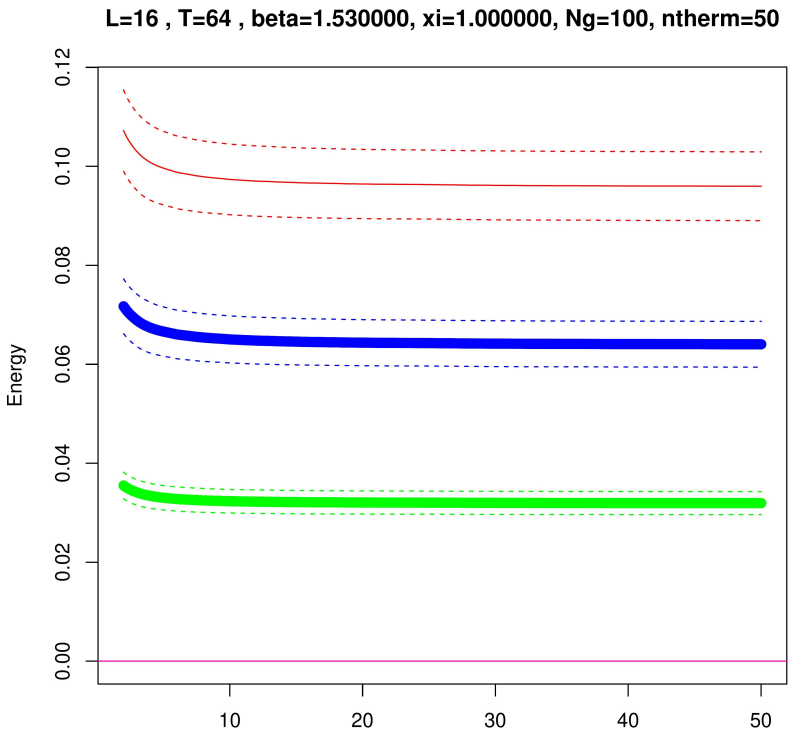
$$E_{ss}^{LAT} = a_s^4 (d-1)(d-2) E_{ss}\,, \quad E_{ts}^{LAT} = 2 a_t^2 a_s^2 (d-1) E_{ts}$$

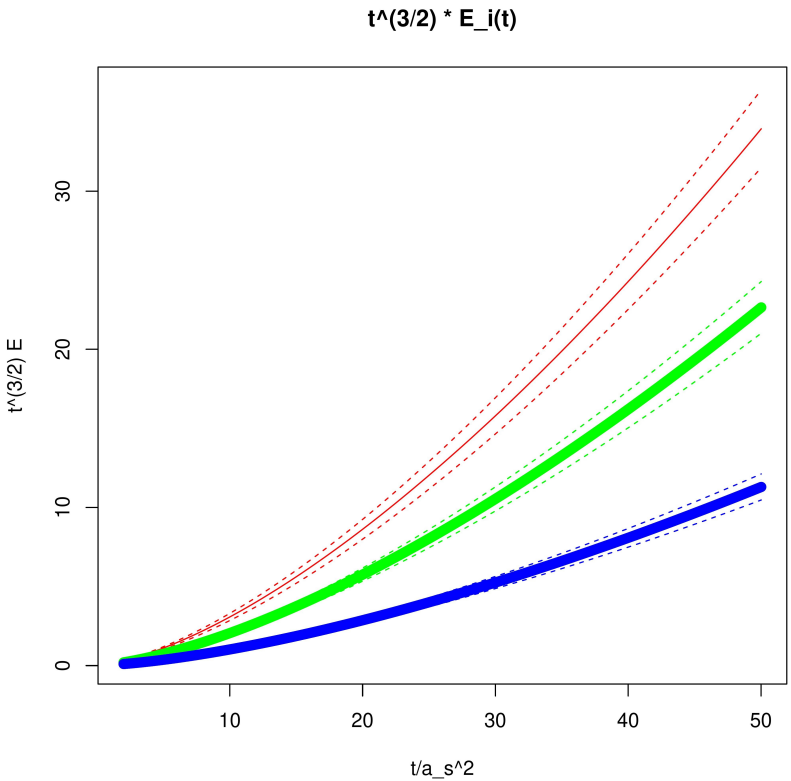
Renormalized anisotropy:

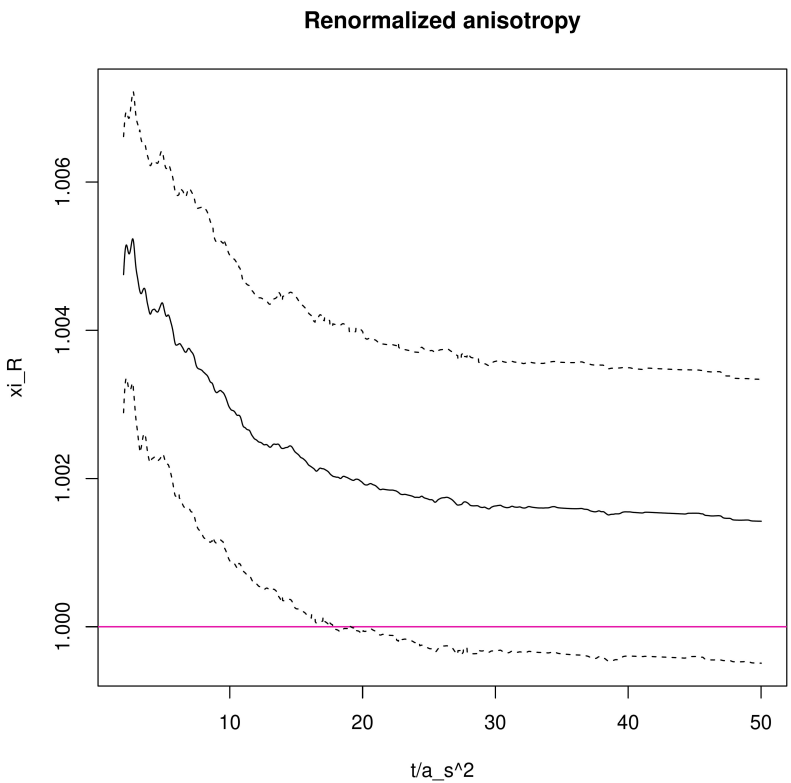
$$\xi^{LAT\,2}_{R} = \frac{d-2}{2} \frac{\langle E_{ts}^{LAT} \rangle}{\langle E_{ss}^{LAT} \rangle} = \frac{d-2}{2} \frac{a_t^2 a_s^2 \langle E_{ts} \rangle}{a_s^4 \langle E_{ss} \rangle}$$

In the limit $a_s \to 0$ we have: $\xi_R^{LAT} \to \frac{a_t}{a_s}$.

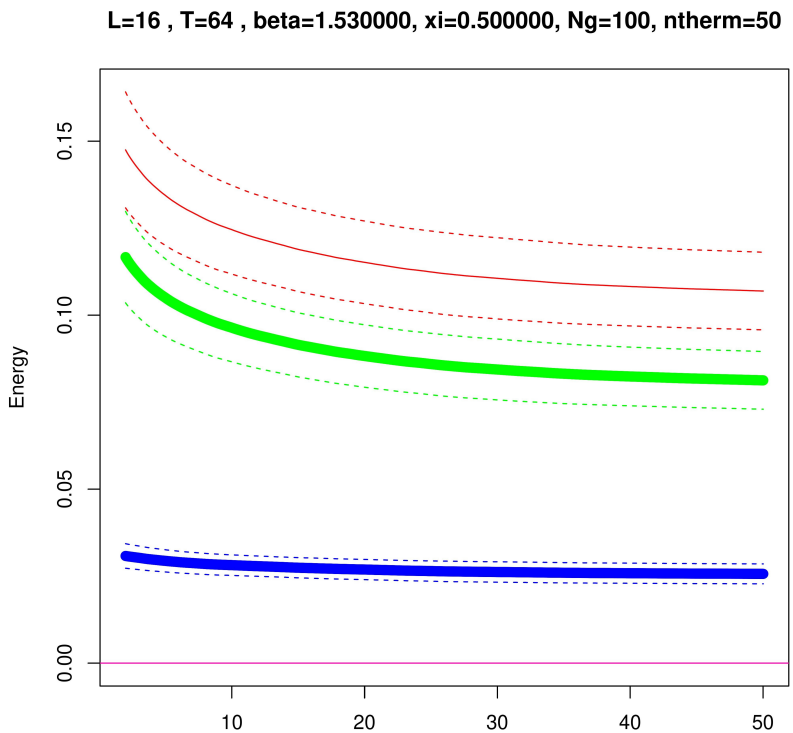
$\xi = 1.0$

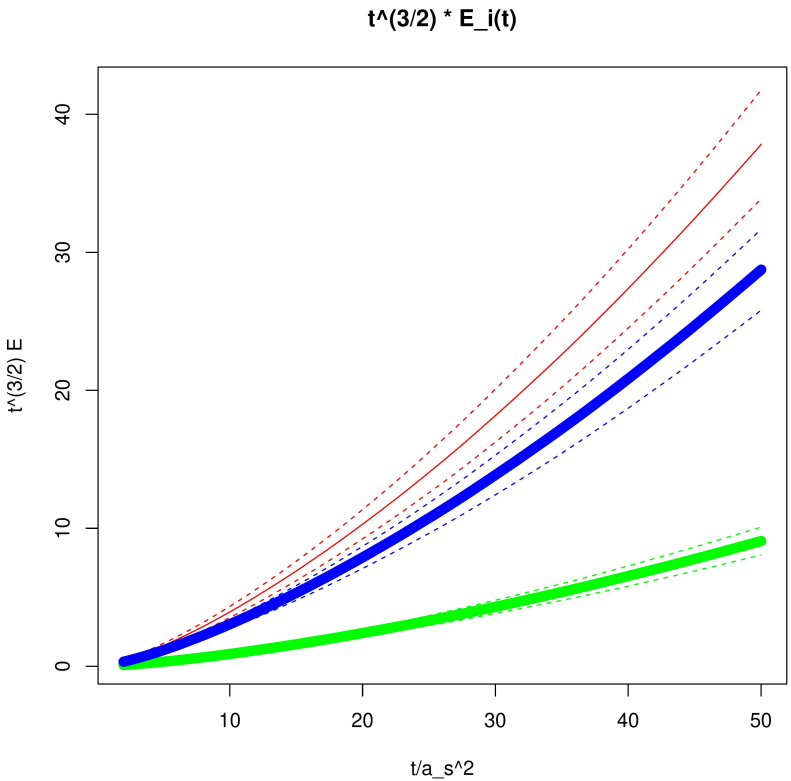

L=16 , T=64 , beta=1.530000, xi=1.000000, Ng=100, ntherm=50
Energy
0.00
0.02
0.04
0.06
0.08
0.10
0.12
10
20
30
40
50

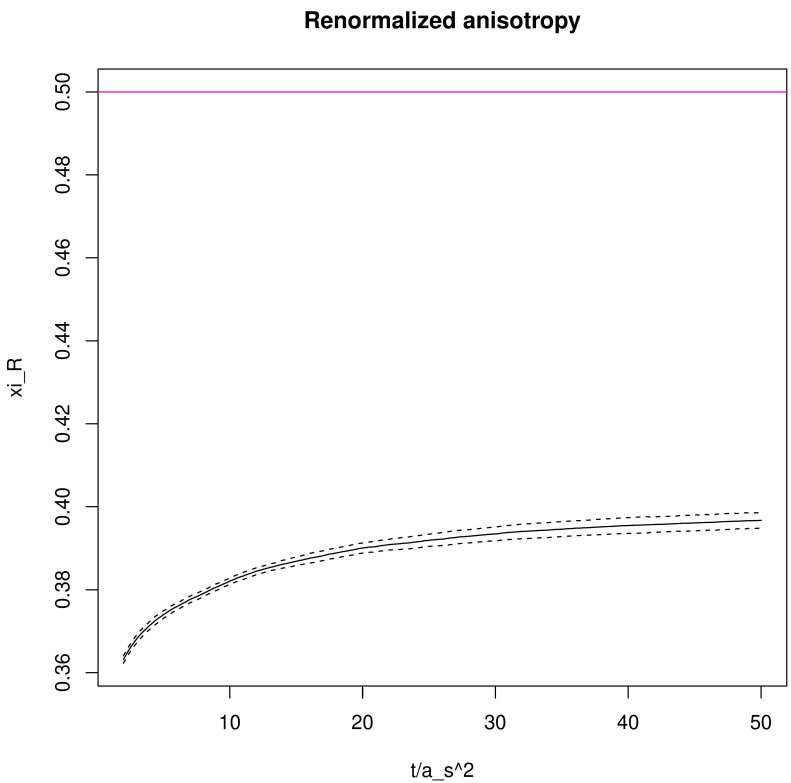

t^(3/2) * E_i(t)
t^(3/2) E
0
10
20
30
10
20
30
40
50
t/a_s^2


Renormalized anisotropy
xi_R
1.000
1.002
1.004
1.006
10
20
30
40
50
t/a_s^2

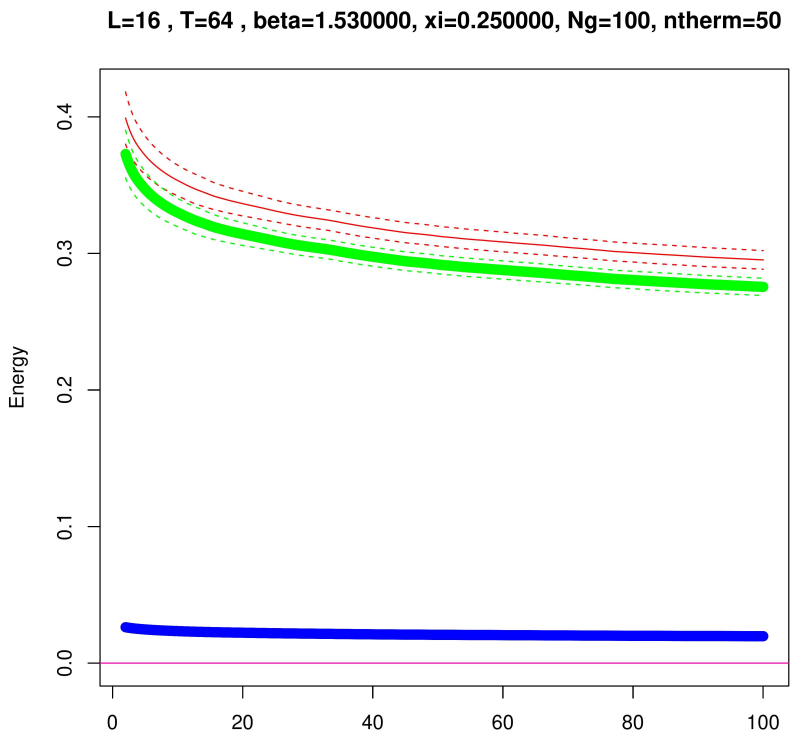
$\xi = 0.5$

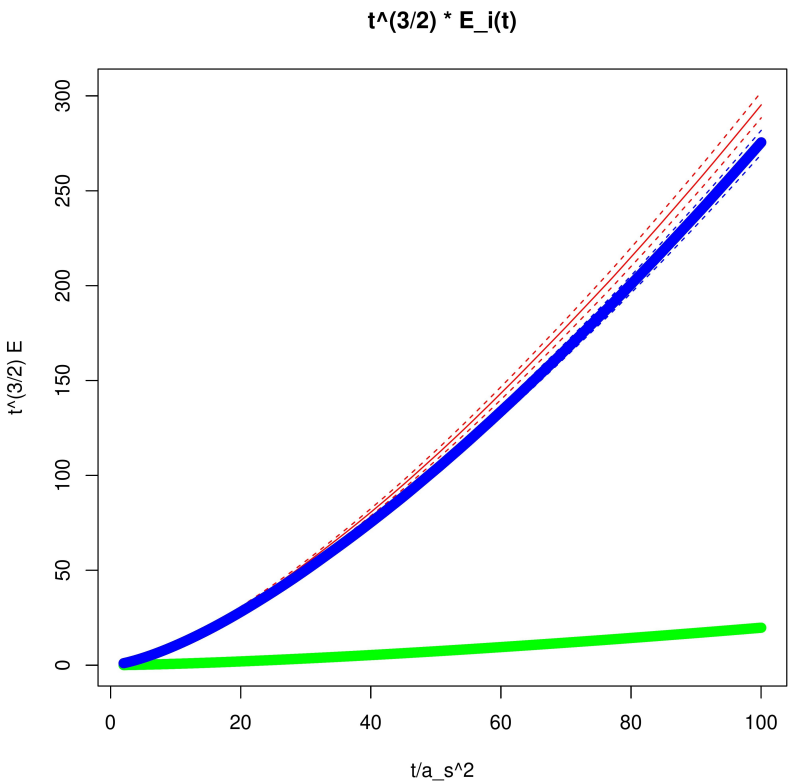

L=16 , T=64 , beta=1.530000, xi=0.500000, Ng=100, ntherm=50
Energy
0.00
0.05
0.10
0.15
10
20
30
40
50

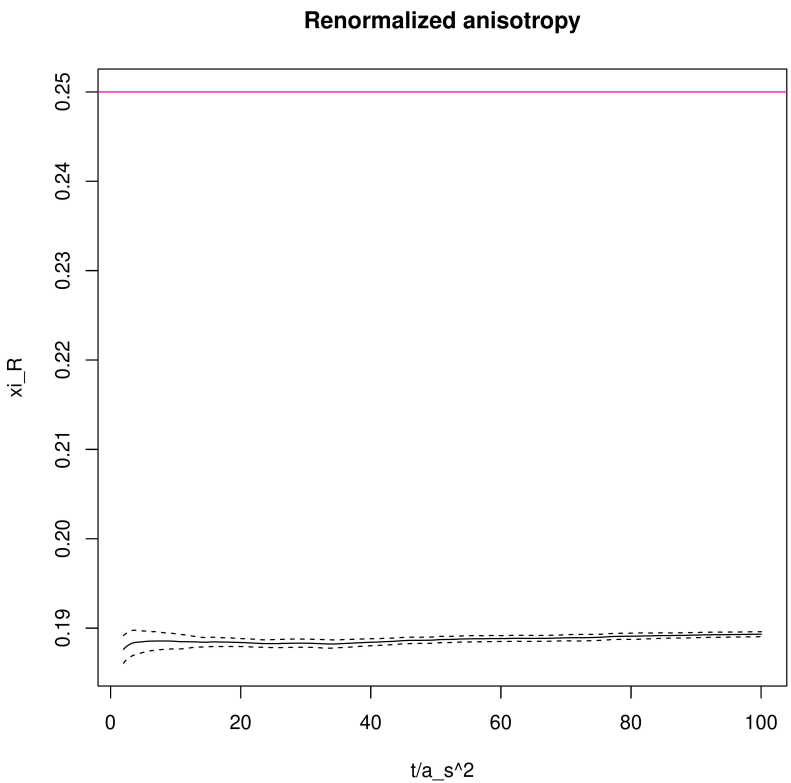

t^(3/2) * E_i(t)
t^(3/2) E
0
10
20
30
40
10
20
30
40
50
t/a_s^2


Renormalized anisotropy
xi_R
0.36
0.38
0.40
0.42
0.44
0.46
0.48
0.50
10
20
30
40
50
t/a_s^2

$\xi = 0.25$


L=16 , T=64 , beta=1.530000, xi=0.250000, Ng=100, ntherm=50
Energy
0.0
0.1
0.2
0.3
0.4
0
20
40
60
80
100


t^(3/2) * E_i(t)
t^(3/2) E
0
50
100
150
200
250
300
0
20
40
60
80
100
t/a_s^2


Renormalized anisotropy
xi_R
0.19
0.20
0.21
0.22
0.23
0.24
0.25
0
20
40
60
80
100
t/a_s^2

# Future prospects

- Complete determination of the glueball spectrum
- Determine $\xi_R$ from the Wilson flow at small $\beta$
  - Expected better precision with respect to static potential
  - $V(a_s) - V(a_s\sqrt{2})$ available for the matching at small volume
- Inclusion of staggered fermions (*in progress*):

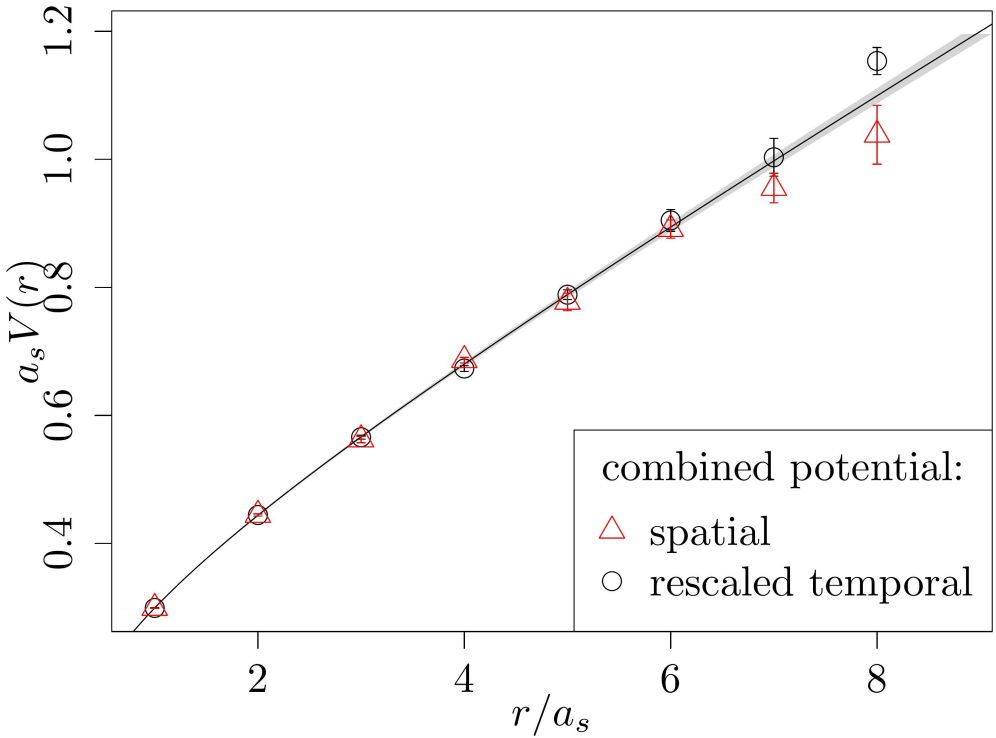
$$\mathcal{L}_F = \sum_f \sum_{x,y} \bar{\psi}(x) D(x|y) \psi(y)$$

# Thank you for the attention !

# Backup

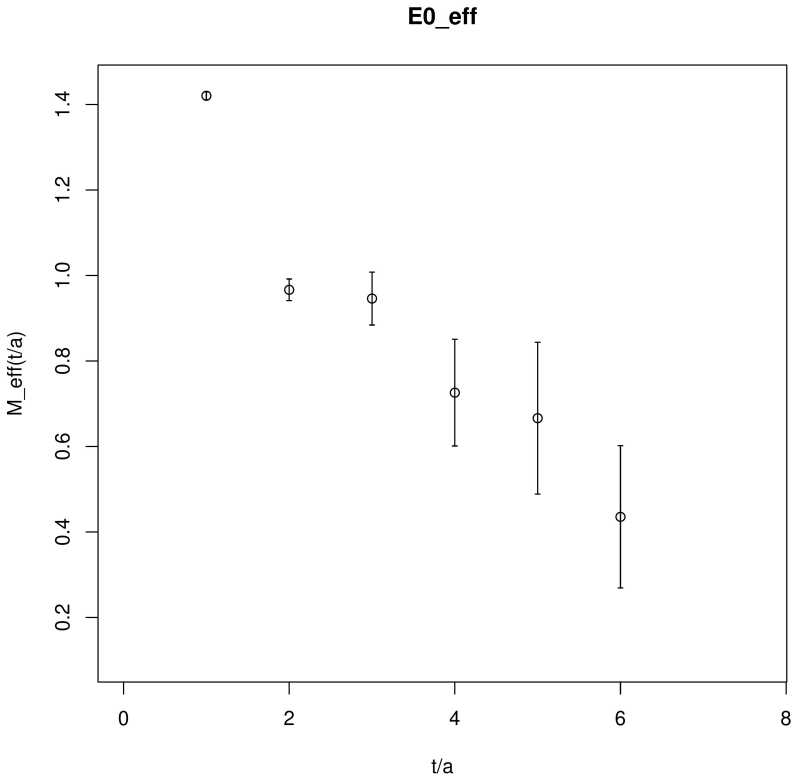
Backup slides following

# $V_{ts}$ and $V_{ss}$ after rescaling with $\xi_R$



# Preliminary results

(work in progress...)


E0_eff
M_eff(t/a)
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0
2
4
6
8
t/a

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