

Mean-field approximation of effective theories of lattice QCD

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Effective theories of lattice QCD

Underlying theory: 3+1 dimensional lattice QCD discretised à la Wilson

Derivation [[Langelage et al., 2011](#); [Fromm et al., 2012](#)]

1. Integrate out fermion fields
2. Combined strong coupling and hopping parameter expansion
3. Integrate out spatial link variables
4. Corrections to truncations of expansions can be derived systematically

Effective theories of lattice QCD

Example: spatial link integration over temporal plaquettes [Figure taken from: [Fromm et al., 2012](#)]



$$\sim \text{Tr} \left(\prod_{\tau=0}^{N_\tau-1} U_0(\mathbf{i}, \tau) \right) \text{Tr} \left(\prod_{\tau=0}^{N_\tau-1} U_0(\mathbf{j}, \tau) \right)^* := L_{\mathbf{i}} L_{\mathbf{j}}^*$$

Effective theories of lattice QCD

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Partition function given by [[Langelage et al., 2011](#); [Fromm et al., 2012](#)]

$$Z = \int [dU_{\mathbf{x}}] e^{-S_{\text{kin}}} \det Q_{\text{stat}} \prod_{\langle \mathbf{x}, \mathbf{y} \rangle} \left(1 + \lambda_1 (L_{\mathbf{x}} L_{\mathbf{y}}^* + \text{c.c.}) \right)$$

Effective theories of lattice QCD

Leading order effective action for dynamical quarks [\[Fromm et al., 2012\]](#)

$$S_{\text{kin}} = \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} 2h_2 W_{1111}^-(U_{\mathbf{x}}) W_{1111}^-(U_{\mathbf{y}})$$

$$W_{abcd}(U) = \text{tr} \left(\frac{(h_1 U)^a}{(1 + h_1 U)^b} \frac{(\bar{h}_1 U^\dagger)^c}{(1 + \bar{h}_1 U^\dagger)^d} \right)$$

$$W_{abcd}^-(U) = W_{ab00}(U) - W_{00cd}(U)$$

Mean-field approximations in general

Mean-field approximation in its simplest form:

1. Express field by mean-field and fluctuations $\phi \rightarrow \bar{\phi} + \delta\phi$
2. Expand the action around vanishing fluctuations
3. Neglect terms beyond leading order

Best approximation at the saddle point with the lowest free energy

Expectation values factorize self-consistently $\langle \phi_x \phi_y \rangle_{\text{mf}} = \langle \phi_x \rangle_{\text{mf}} \langle \phi_y \rangle_{\text{mf}}$

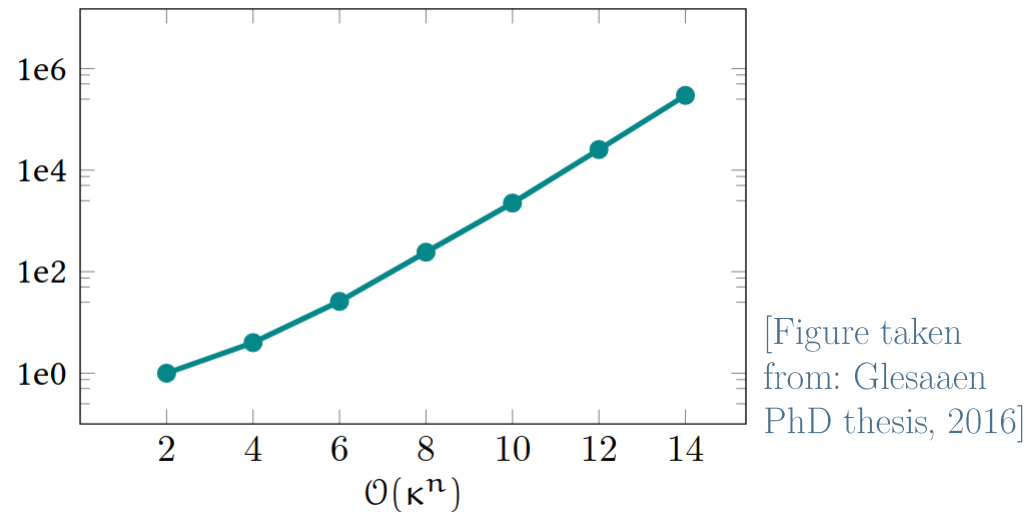
However: e.g. for Ising model with spins s_x we have $\langle s_x^2 \rangle_{\text{mf}} = 1 \neq \langle s_x \rangle_{\text{mf}}^2$

More appropriate: each power of the field receives its own mean-field

[See e.g. Zinn-Justin, 2002]

Mean-field approximations for the effective theories

Exponentially increasing number of interaction terms and linearly increasing interaction distance



Coordination number increases effectively by including corrections to the effective actions
Error introduced by mean-field is expected to decrease by improving the effective actions

Mean-field approximations for the effective theories

Mean-field has been applied before to Polyakov-loop effective theories

[See [Fukushima and Hidaka, 2011](#); [Greensite and Splittorff, 2012](#); [Rindlisbacher and de Forcrand, 2015](#)]

Effective actions contain all powers of the Polyakov loop

Introduce a mean-field for each monomial $L_{\mathbf{x}}^n L_{\mathbf{x}}^{*m}$

Thus: mean-field action is a function of infinitely many variational parameters

Mean-field approximations for the effective theories

Instead: choose an approximation for mean-field action in terms of less variables

1. should resemble original action more closely than a simple linearization in the fluctuations
2. should find self-consistency relations for the new variational parameters

Idea: resum a subset of fluctuations

Consequently: free energy density f_{mf} and self-consistency relations receive corrections

Application to the pure gauge effective theory

Consider the effective pure gauge action

$$-S_{\text{pg}}^{\text{eff}} = \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \log \left(1 + \lambda_1 (L_{\mathbf{x}} L_{\mathbf{y}}^* + L_{\mathbf{x}}^* L_{\mathbf{y}}) \right)$$

Substitute Polyakov loops by mean-fields and fluctuations

$$L_{\mathbf{x}} = \bar{L} + \delta L_{\mathbf{x}} \quad L_{\mathbf{x}}^* = \bar{L}^* + \delta L_{\mathbf{x}}^*$$

Neglect terms that are quadratic in the fluctuations $\mathcal{O}(\delta L_{\mathbf{x}} \delta L_{\mathbf{y}})$

Resummation: keep all local fluctuations $\sim \delta L_{\mathbf{x}}^n \delta L_{\mathbf{x}}^{*m}$

Application to the pure gauge effective theory

Unresummed mean-field action

$$-S_{\text{pg, mf}}^{\text{eff}} = \cdots + \sum_{\mathbf{x}} \frac{6\lambda_1 \bar{L}^*}{1 + 2\lambda_1 \bar{L} \bar{L}^*} L_{\mathbf{x}} + \frac{6\lambda_1 \bar{L}}{1 + 2\lambda_1 \bar{L} \bar{L}^*} L_{\mathbf{x}}^*$$

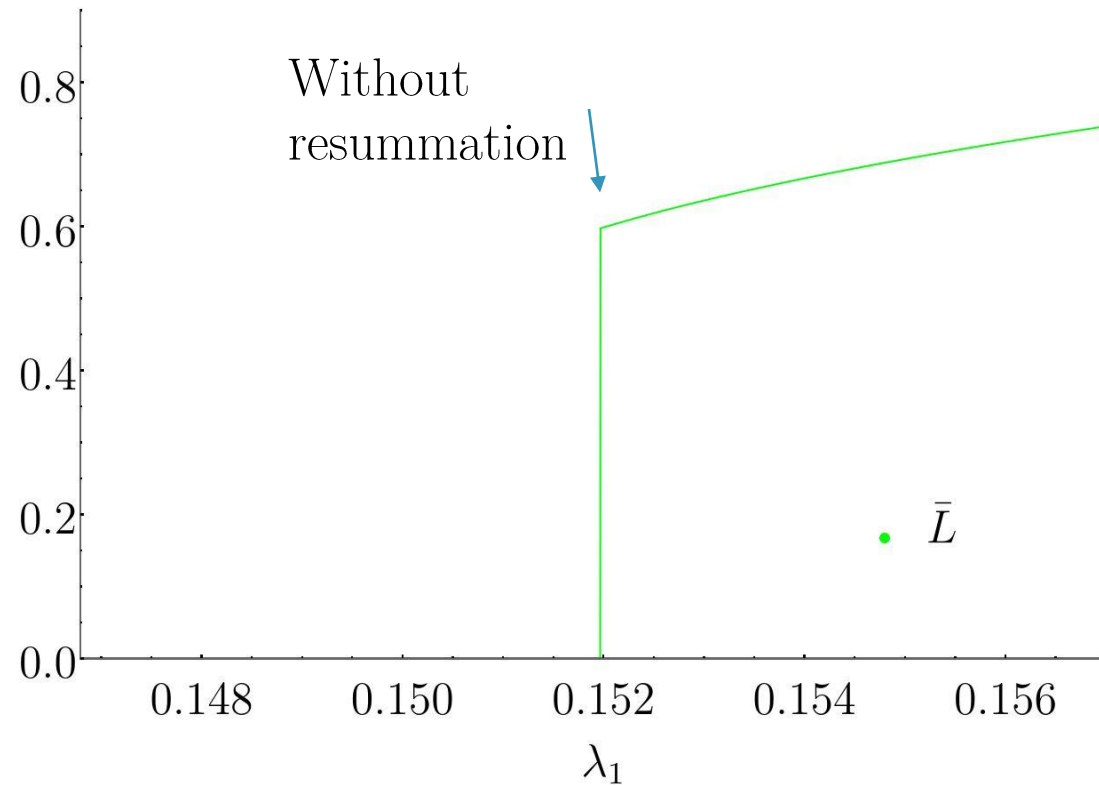
Resummed mean-field action contains a logarithm again

$$-S_{\text{pg, mf}}^{\text{eff}} = \cdots + 6 \sum_{\mathbf{x}} \log \left(1 + \lambda_1 (L_{\mathbf{x}} \bar{L}^* + L_{\mathbf{x}}^* \bar{L}) \right)$$

As we are at zero chemical potential: self-consistency relations for L and L^* are identical
Thus: tuning of only one mean-field necessary

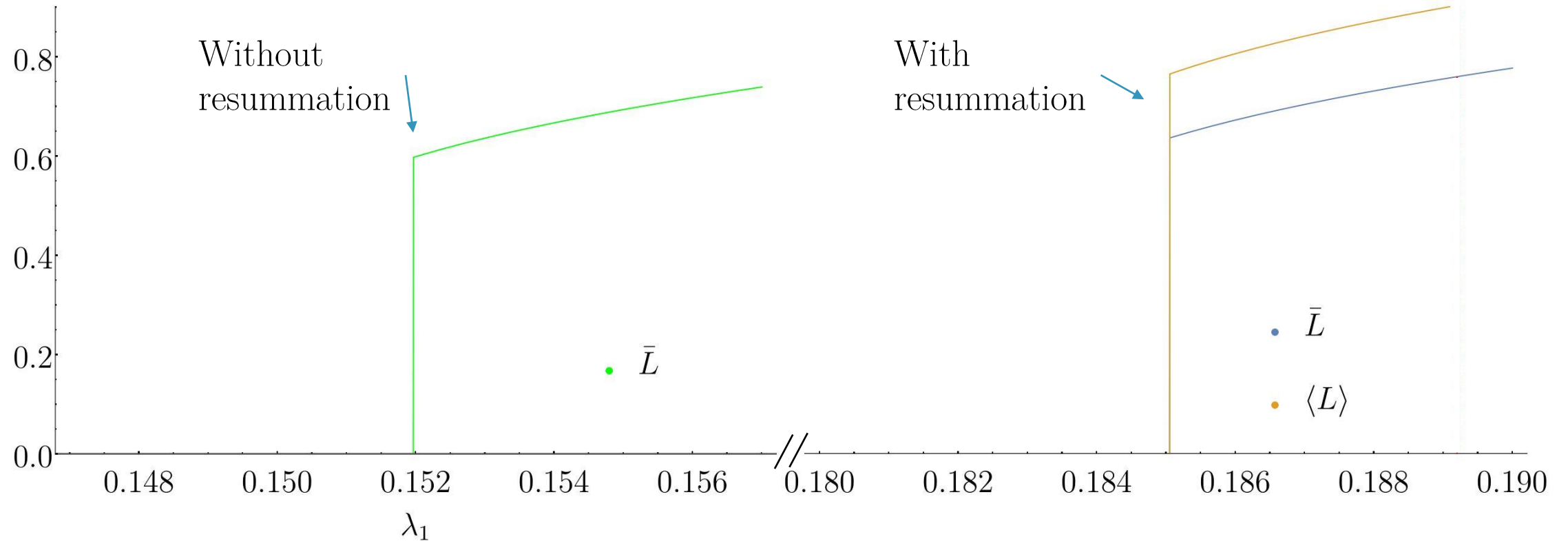
Application to the pure gauge effective theory

Probe for deconfinement transition by varying the effective pure gauge coupling



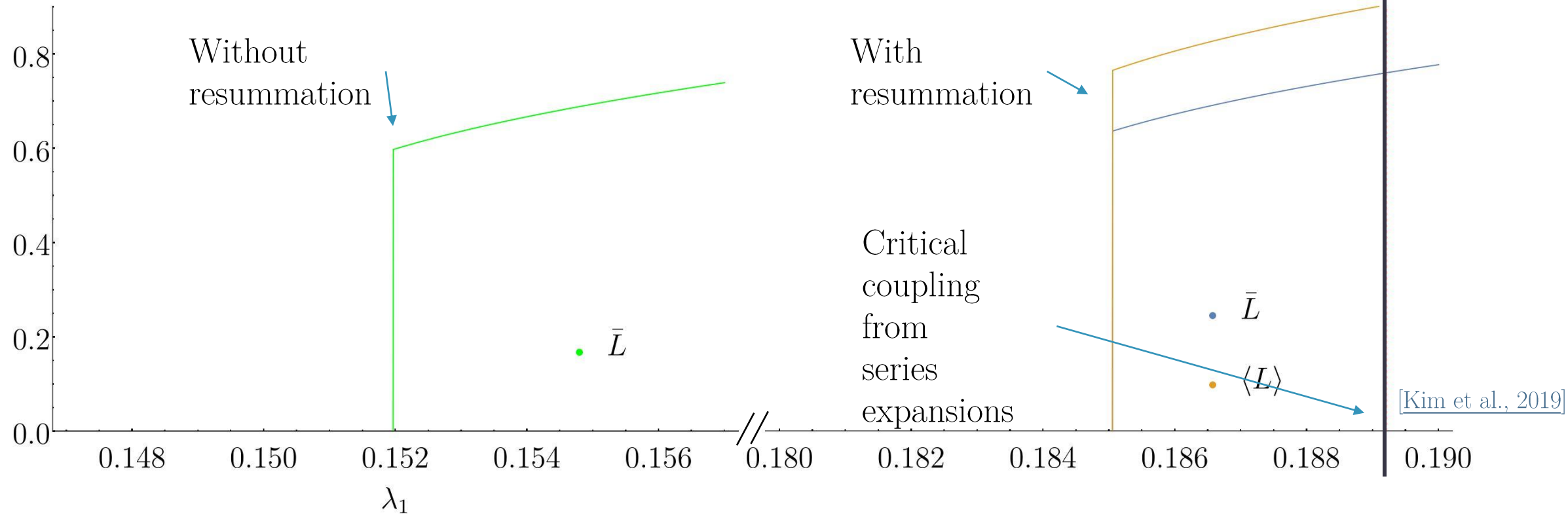
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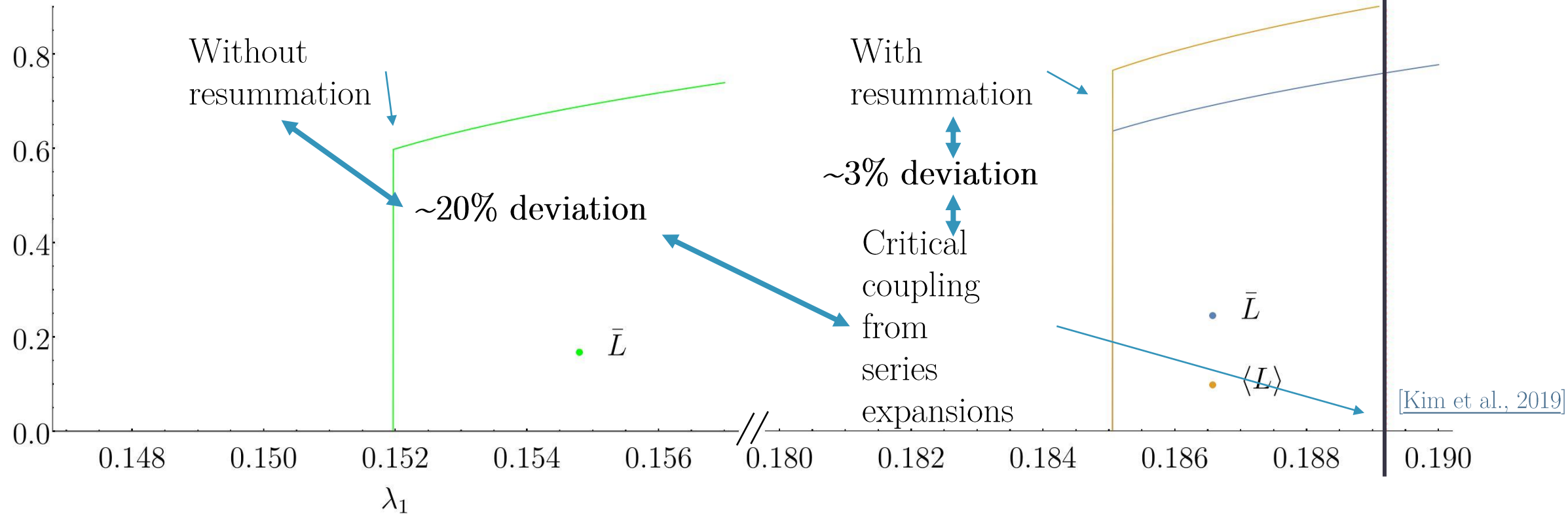
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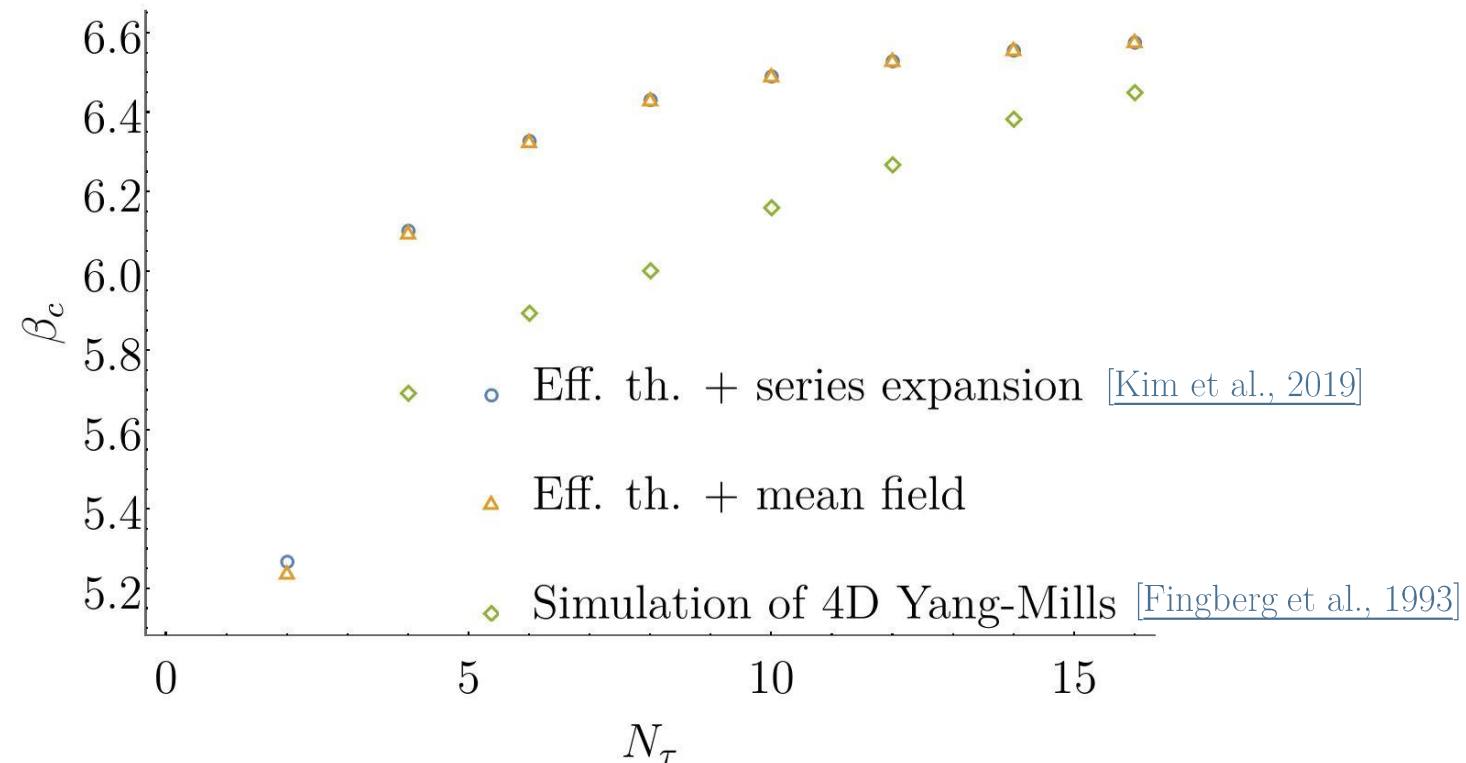
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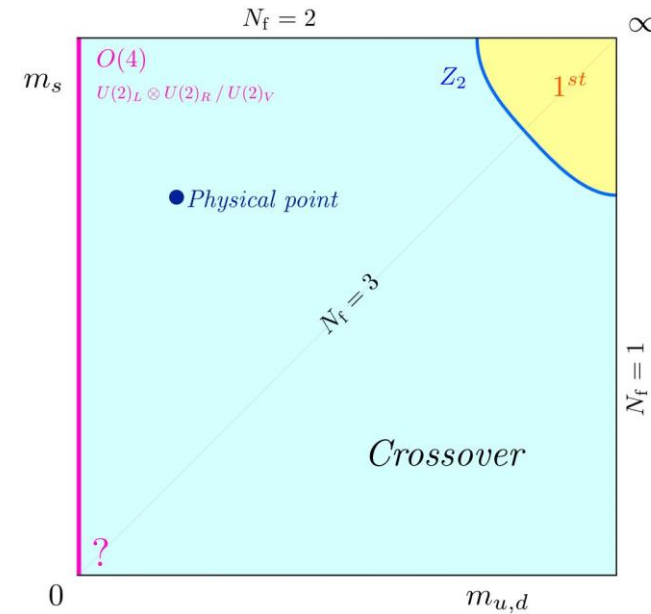
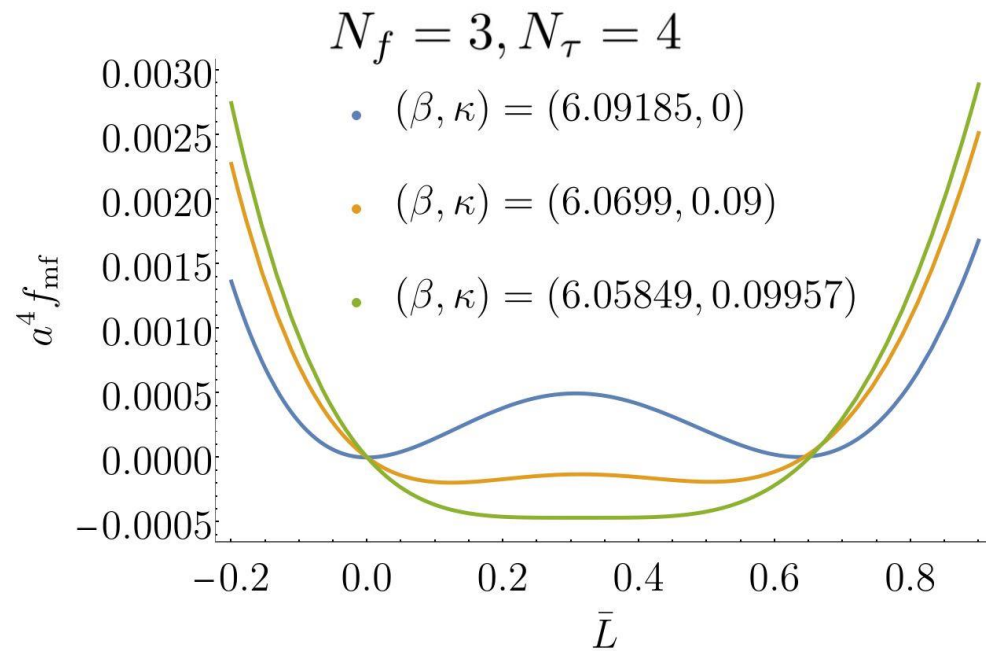
Application to the pure gauge effective theory

Critical gauge coupling at varying temporal extends compared to simulations of 4D Yang-Mills



Mean-field including dynamic quark effective actions

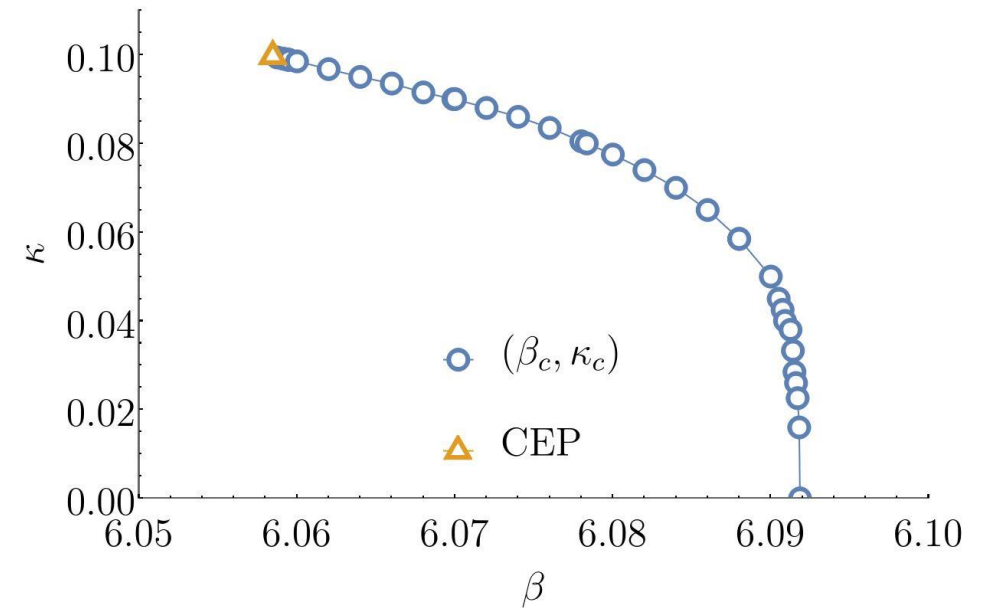
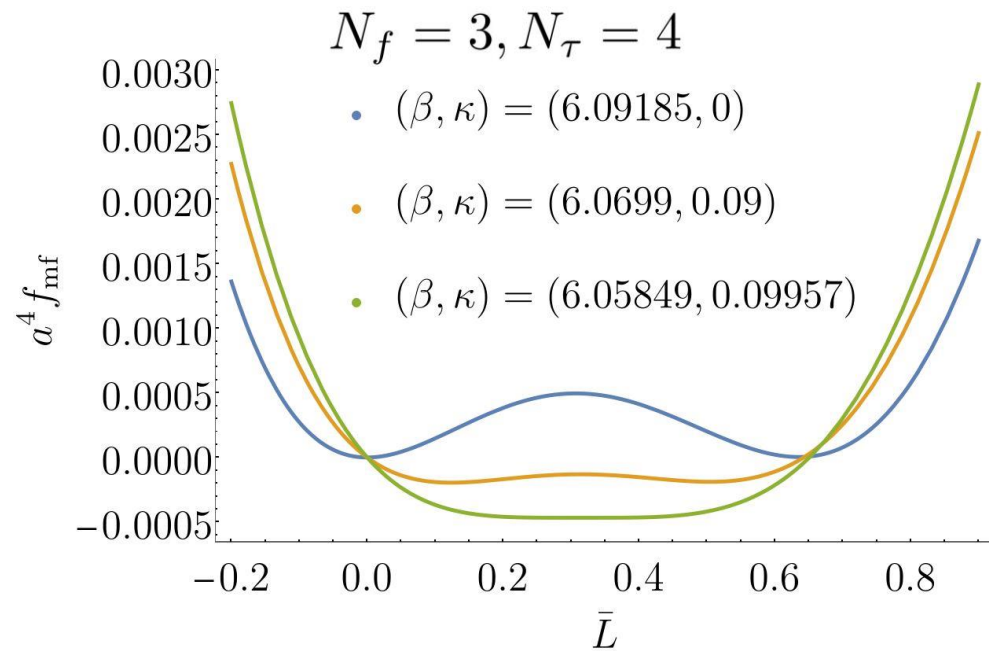
Demonstration: endpoint of 1st order deconfinement transition with $\mathcal{O}(\kappa^4)$ action



[Figure taken from: [Cuturi et al., 2021](#)]

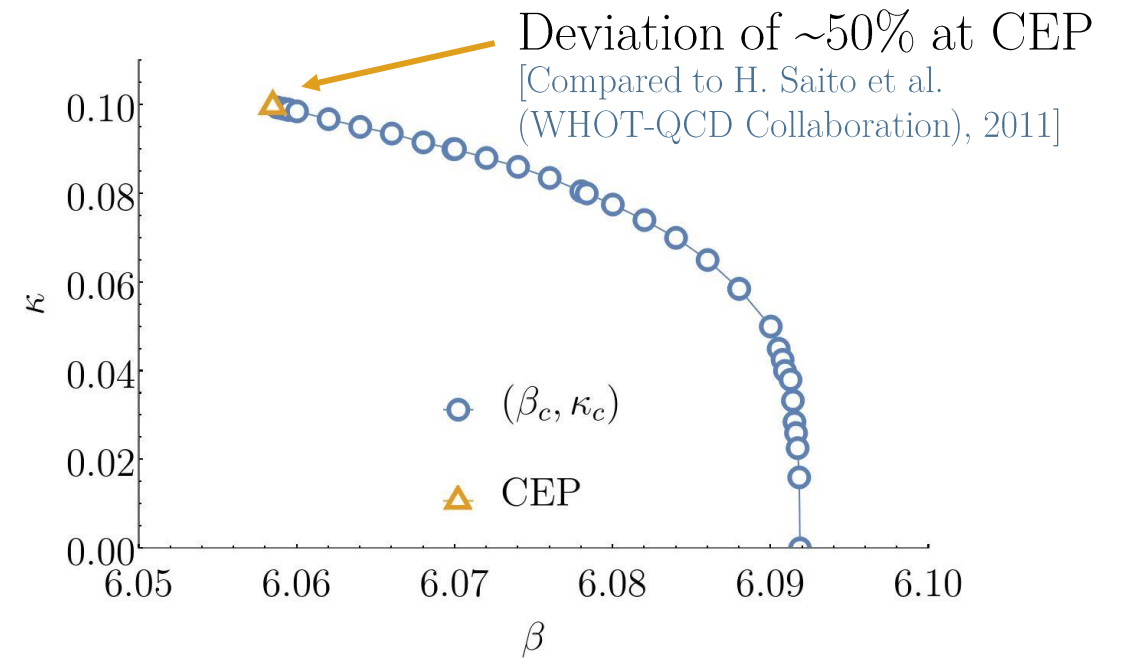
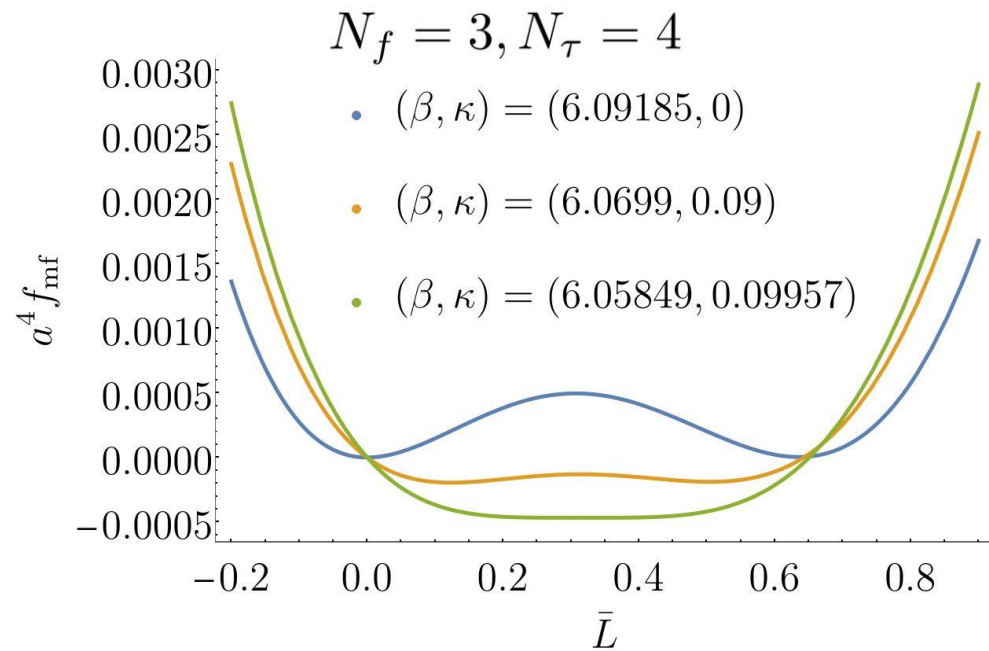
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Conclusions and Outlook

Found general formalism to obtain mean-field approximation of effective theories of lattice QCD
Uncertainty from effective theories of $\sim 10\%$ and from mean-field of $\sim 3 - 40\%$

Higher orders of the hopping parameter expansion should improve the effective actions and the mean-field approximation simultaneously

Application at non-zero chemical potential possible

Efficient way for evaluating lattice QCD at low T (large N_τ) and finite density

[see talk by Amine Chabane last monday]

Thank You!
