Boundary terms in Complex Langevin simulations of full QCD

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- Intro to boundary-terms
- Results
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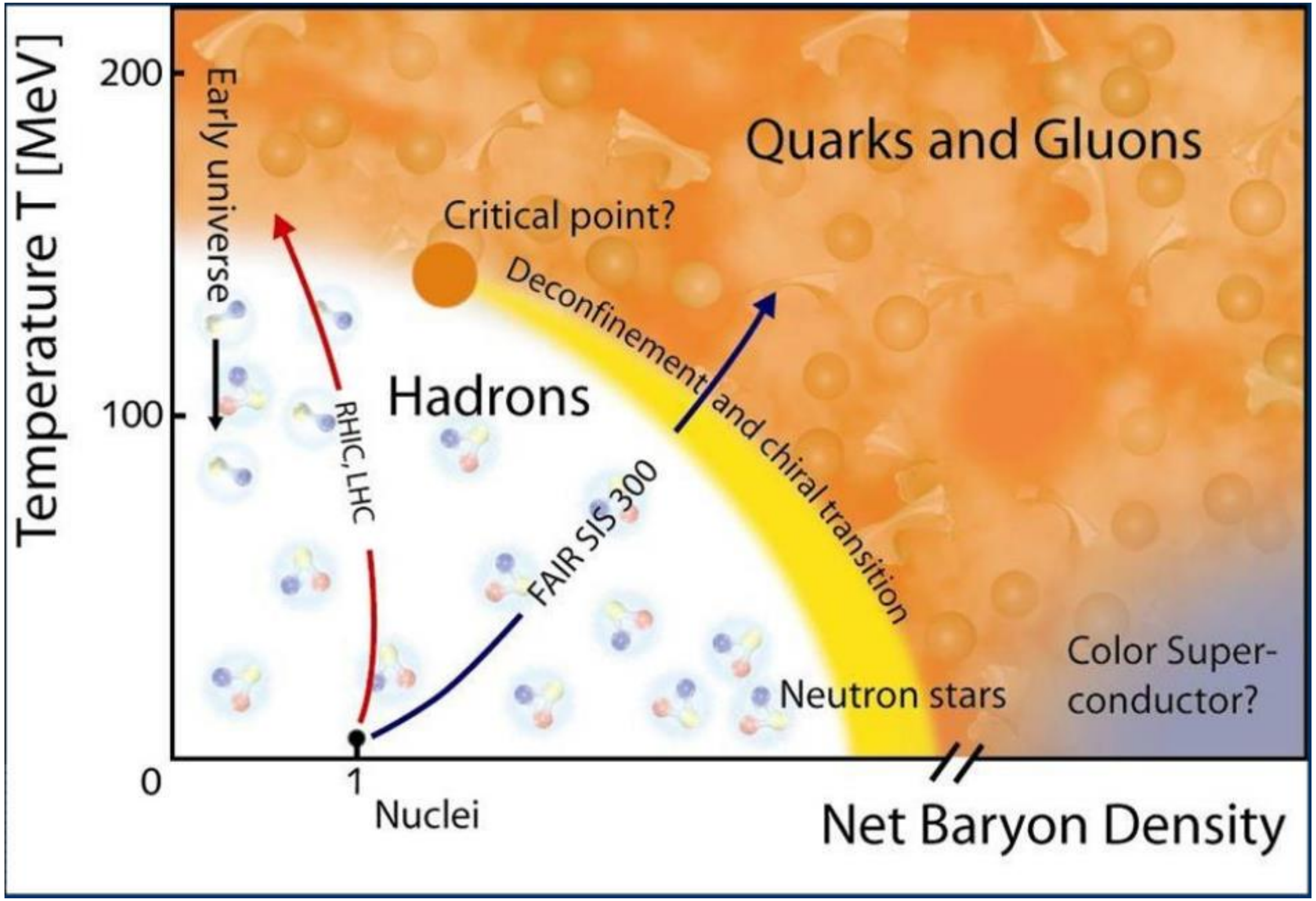
Motivation

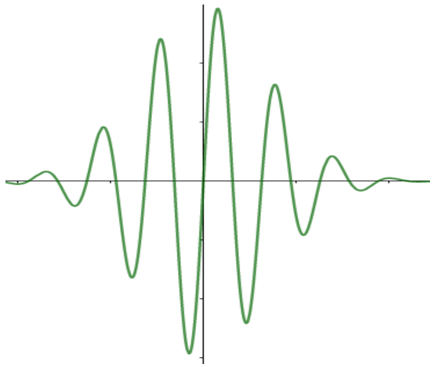
- Investigate compressibility of nuclear matter, and existence of critical point
- Sign-problem
- Difficult HMC calculations for large chemical potential
- Reweighting (Determinant costs $O(N_s^9)$)
- Taylor expansion (Limited convergence radius)



Complex Langevin

- Complex action => Sign problem
- Using stochastic equation
  instead of importance sampling.
- With the correct configuration space
- $\Re(d\phi) = \Re(K)dt + d\omega, \qquad \Im(d\phi) = \Im(K)dt$

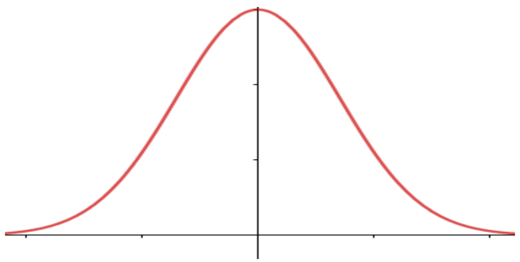
$$K = -\frac{d}{d\phi}S[\phi]$$



$$L_c = (\partial_z + K_z)\partial_z$$

$$\partial_t P(\phi, t) = L_c P(\phi, t)$$

$$\rho(\phi) \propto \exp(-S[\phi])$$



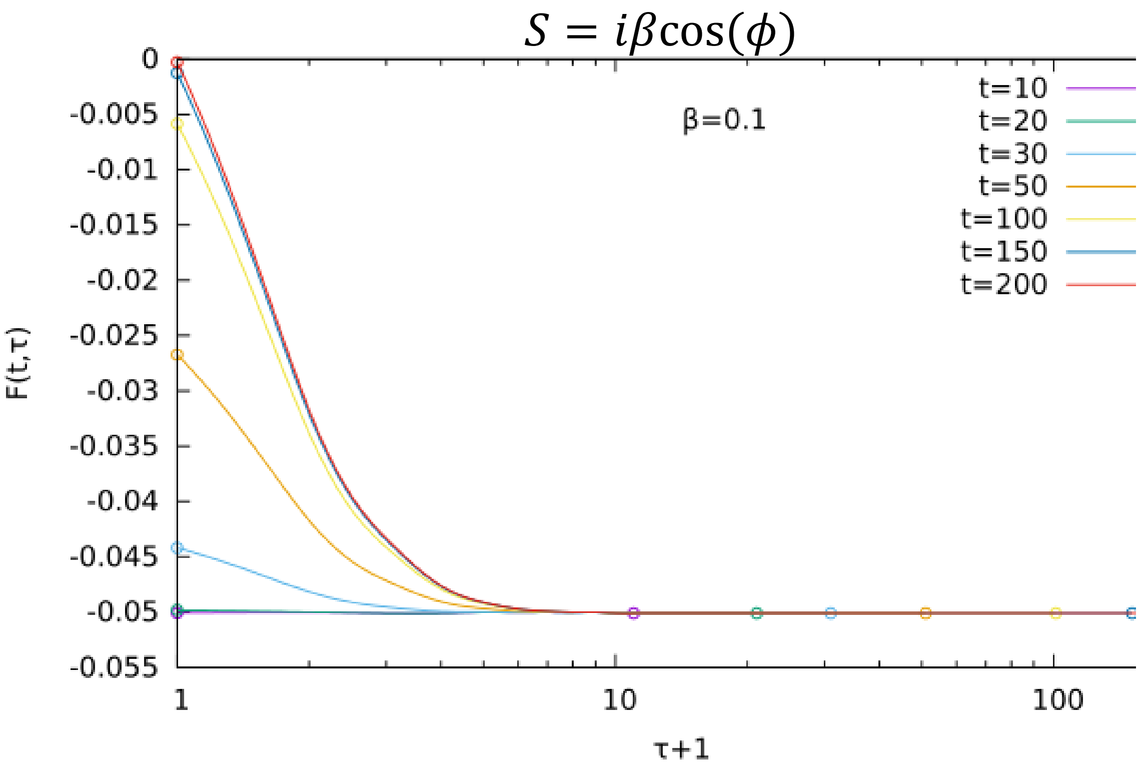
Boundary terms

- Interpolation function between $P(t)$ and $\rho(t)$

- $F_O(t,\tau) = \int P(x,y,t-\tau)\,\exp(\tau L_c)\,O(x+iy)\,dxdy$

$$F_O(t,0) = \langle O\rangle_{P(t)}, \qquad F_O(t,t) = \langle O\rangle_{\rho(t)}$$

- If $F_O(t,\tau)$ is constant in tau, then the observables are correct



Cut-off effect

- Big error at run-aways
- Limit the imaginary part, to “cut-off” run-aways

$$B_n(Y,t) = \partial_\tau^n F_O(t,\tau)|_{\tau=0}$$
$$= -\int_{|y|<Y} \partial_t^n P(x,y,t) O(x+iy) dxdy + \int_{|y|<Y} P(x,y,t) L_c^n O(x+iy) dxdy$$

- First integral vanishes as $t \to \infty$
- Second is easy to calculate on the lattice
- Higher order boundary terms

$$B_n(Y,t) = \int_{|y|<Y} P(x,y,t)\, L_c^n\, O(x+iy) dxdy$$

Toy model

- Action:

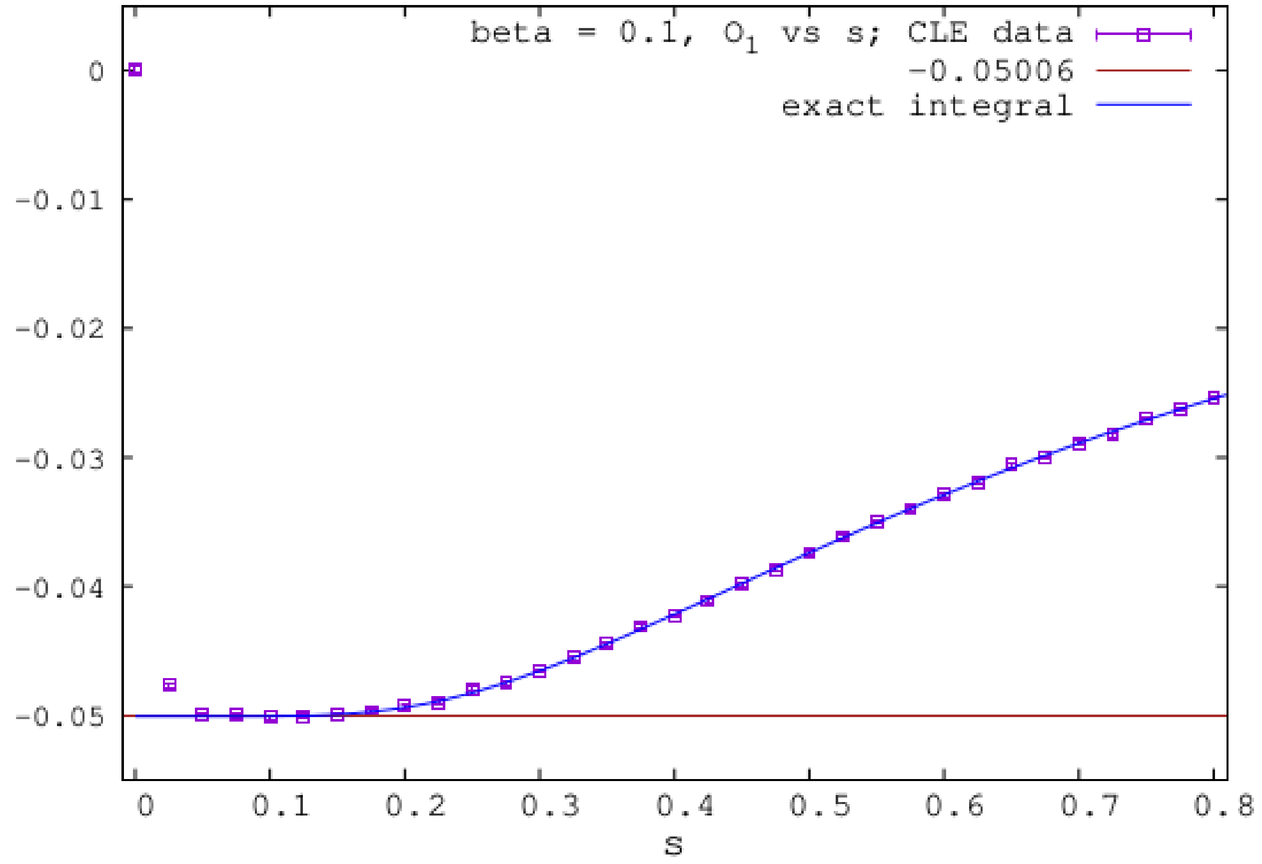
$$S[\phi] = i\beta\cos(\phi) + \frac{1}{2}s\phi^2$$

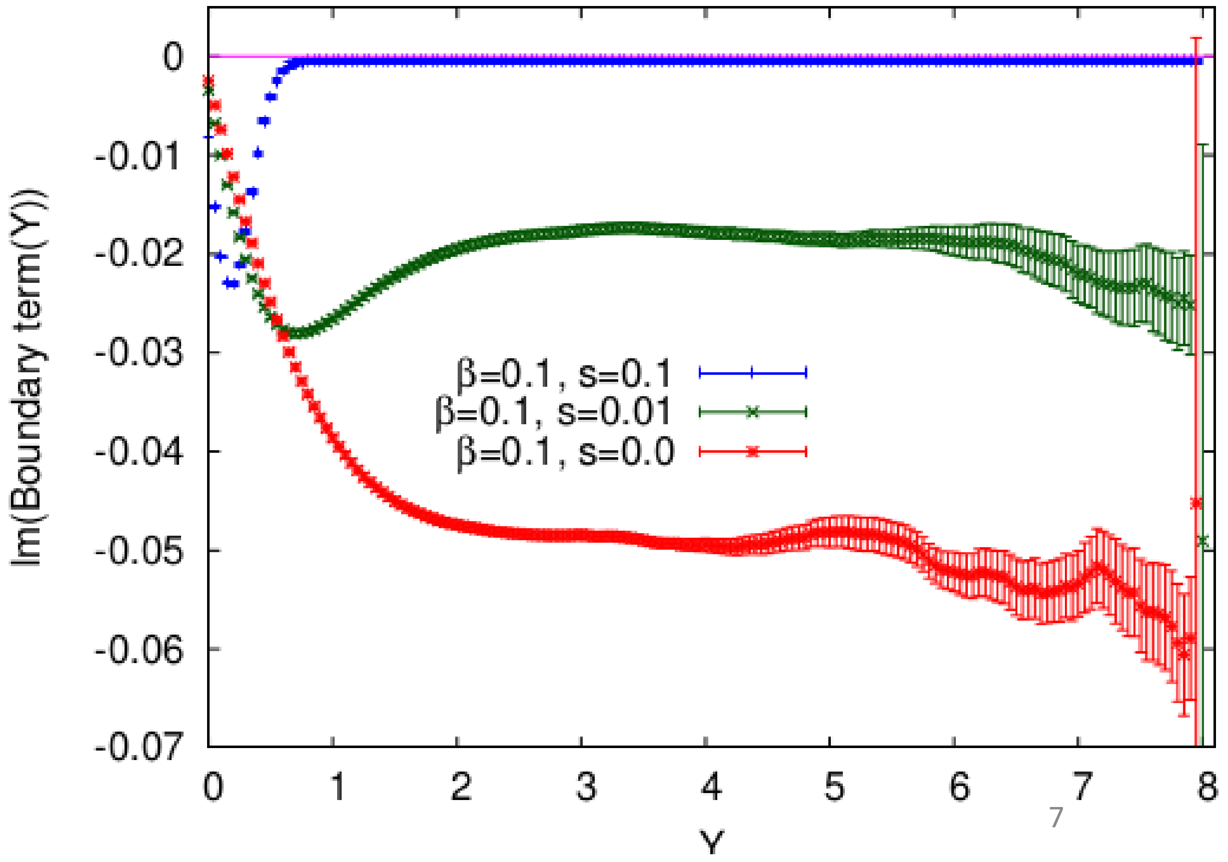
- Observable:

$$O[\phi] = \exp(i\phi)$$

- Bounary term:

$$L_c O[\phi] = i(i - S'[\phi]\exp(i\phi))$$





Updating the lattice using CLE

- Update:
$$U_\mu^{n+1}(x) = \exp\left[i\lambda_a\left(\epsilon K_{\mu a}(x) + \sqrt{\epsilon}\,\eta_{\mu a}(x)\right)\right] U_\mu^n(x)$$

- Using the left derivative
$$K_{\mu a}(x) = -D_{\mu a} S(x),$$
$$D_{\mu a} f(U) = \partial_\alpha f\left(\exp(i\alpha\lambda_a)\ U_\mu(x)\right)\Big|_{\alpha=0}$$

- If the drift is complex $\Rightarrow U \in SL(N)$
- Needs gauge cooling after each step

Reweighting

- Change the weights

$$\langle x\rangle_w = \frac{\sum w_i x_i}{\sum w_j} = \frac{\sum w_i x_i \frac{w_i'}{w_i'}}{\sum w_j \frac{w_j'}{w_j'}} = \frac{\sum w_i' x_i \frac{w_i}{w_i'}}{\sum w_j' \frac{w_j}{w_j'}} = \frac{\left\langle x \frac{w}{w'}\right\rangle_{w'}}{\left\langle \frac{w}{w'}\right\rangle_{w'}}$$

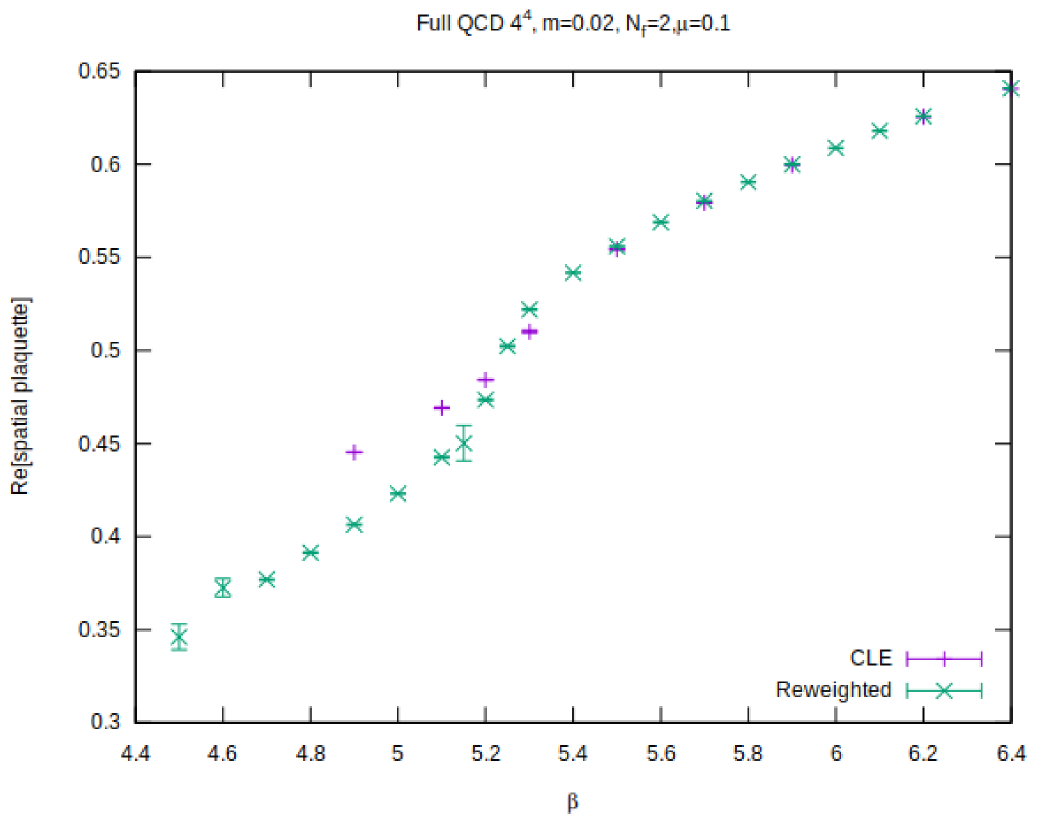
- Used in HMC, to simulate non-zero chemical potential

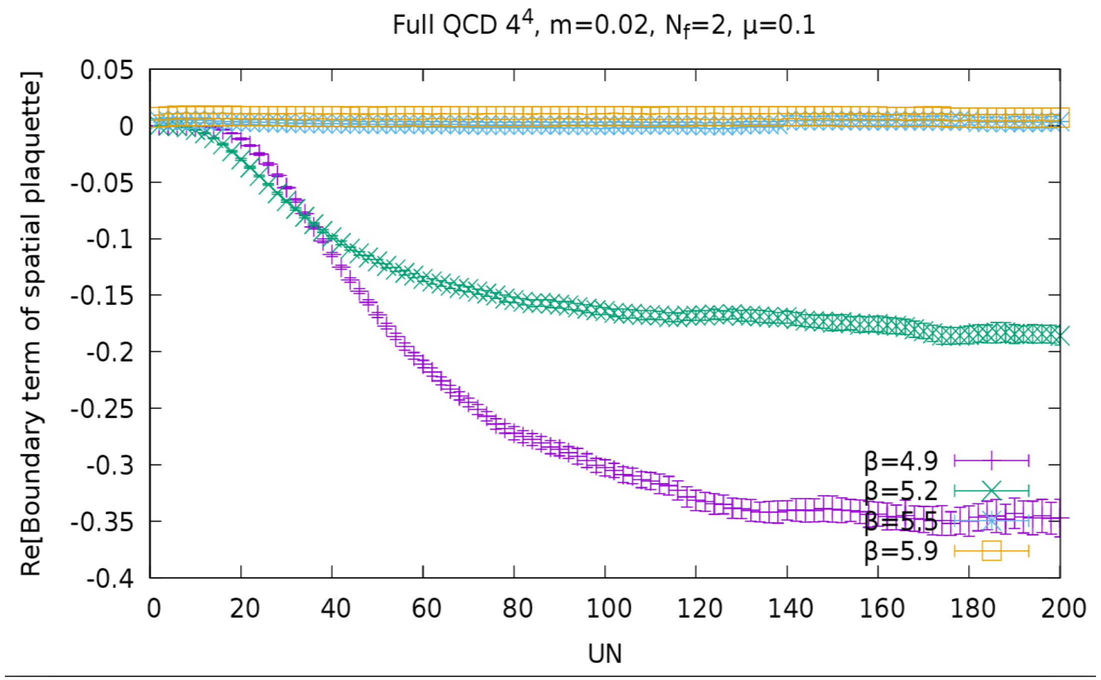
$$\left\langle \frac{w}{w'}\right\rangle = \left\langle \frac{\det M(\mu)}{\det M(\mu = 0)}\right\rangle = \exp\left(-\frac{V}{T}\Delta F(\mu, t)\right)$$

- Large $\mu \Rightarrow \left\langle \frac{w}{w'}\right\rangle$ goes towards zero

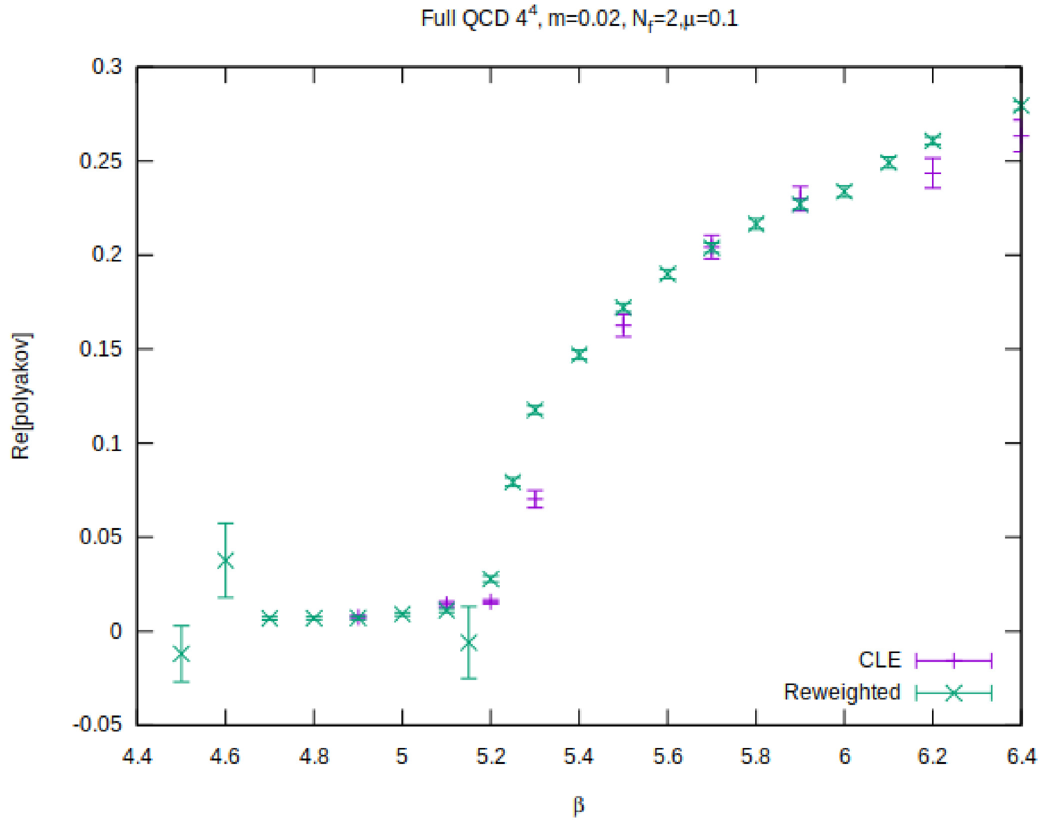
Results – Plaquettes

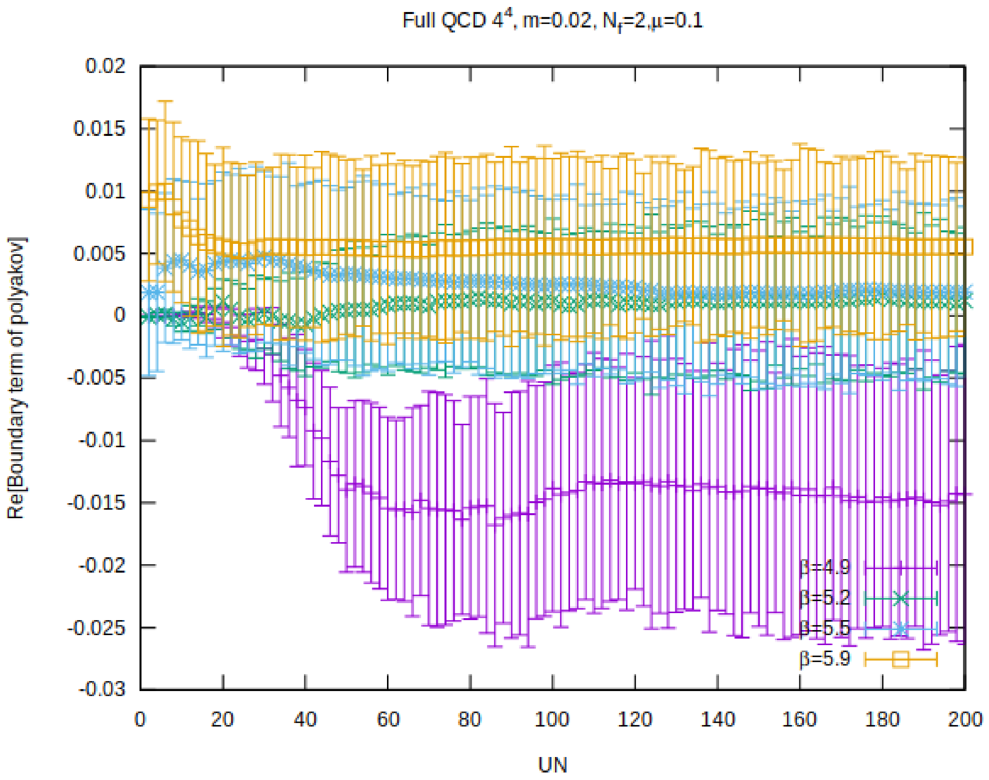
[Hansen, Sexty in preparation]





Results – Polyakov loops





Dynamical stabilization

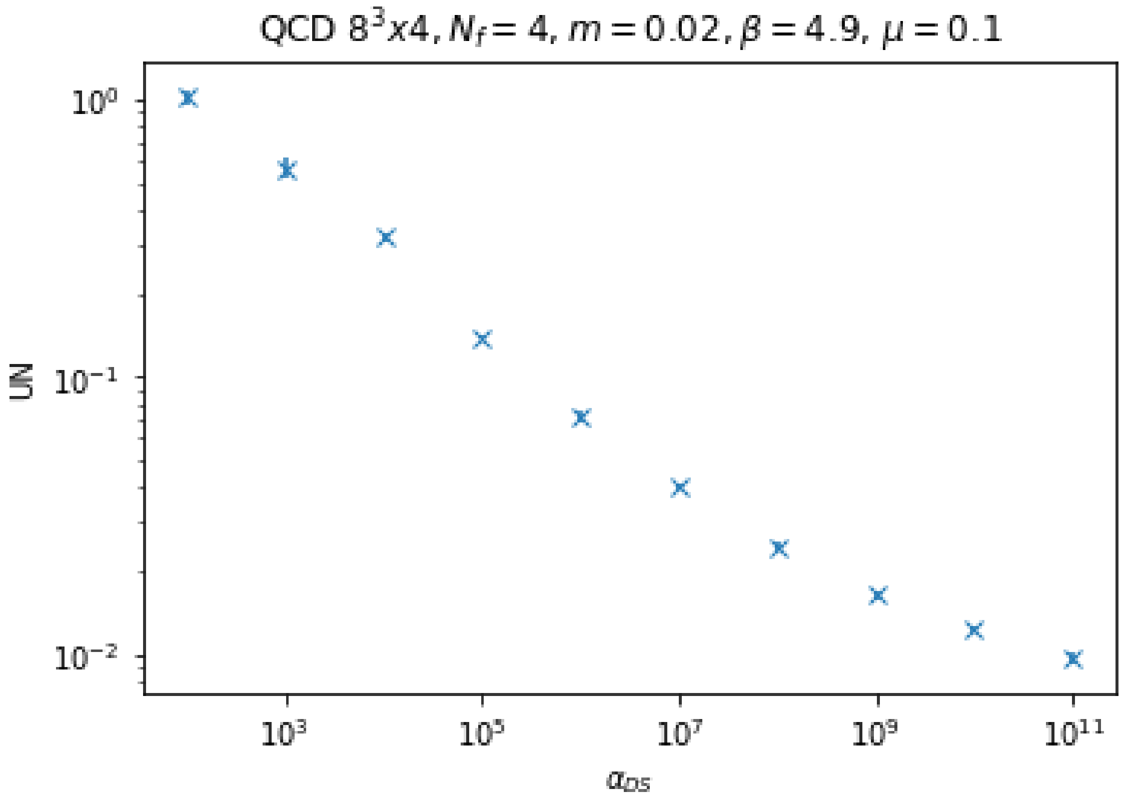
(Attanasio, Jäger, arxiv: 1808.04400)

- Introducing a Gauge invariant force, to the drift
- Designed to grow rapidly with the unitarity norm

$$K_{\mu a}(x) \to K_{\mu a}(x) + i\alpha_{DS} M_a(x)$$

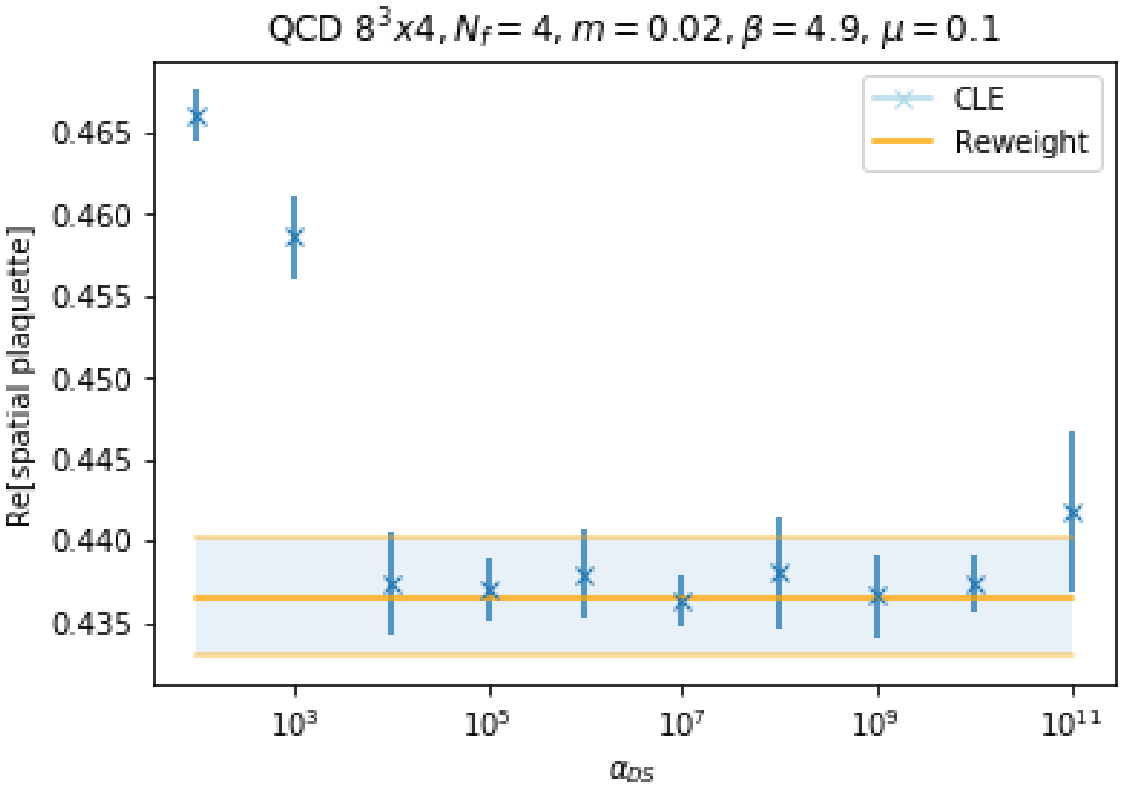
$$M_a(x) = i b_a \left(\sum_c b_c(x) b_c(x) \right)^3$$

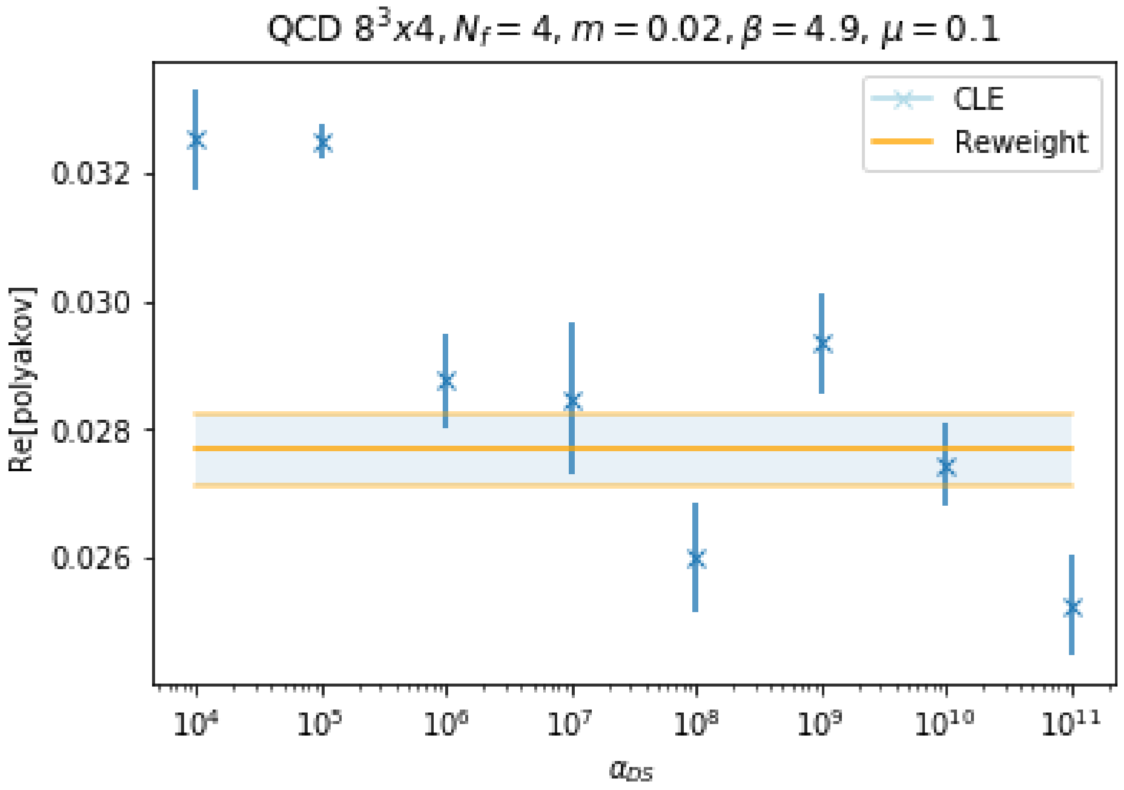
$$b_a(x) = Tr\left[\lambda_a \sum_\mu U_\mu(x) U_\mu^\dagger(x) \right]^3$$



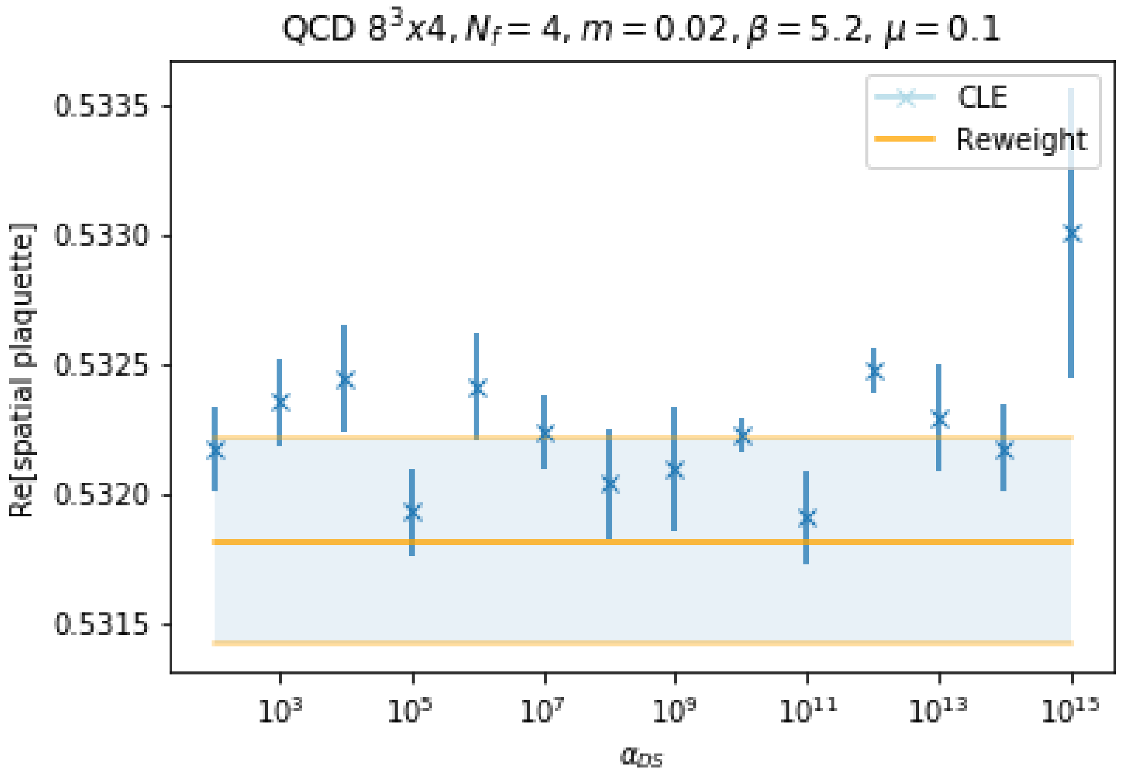
DynStab – Low temperature

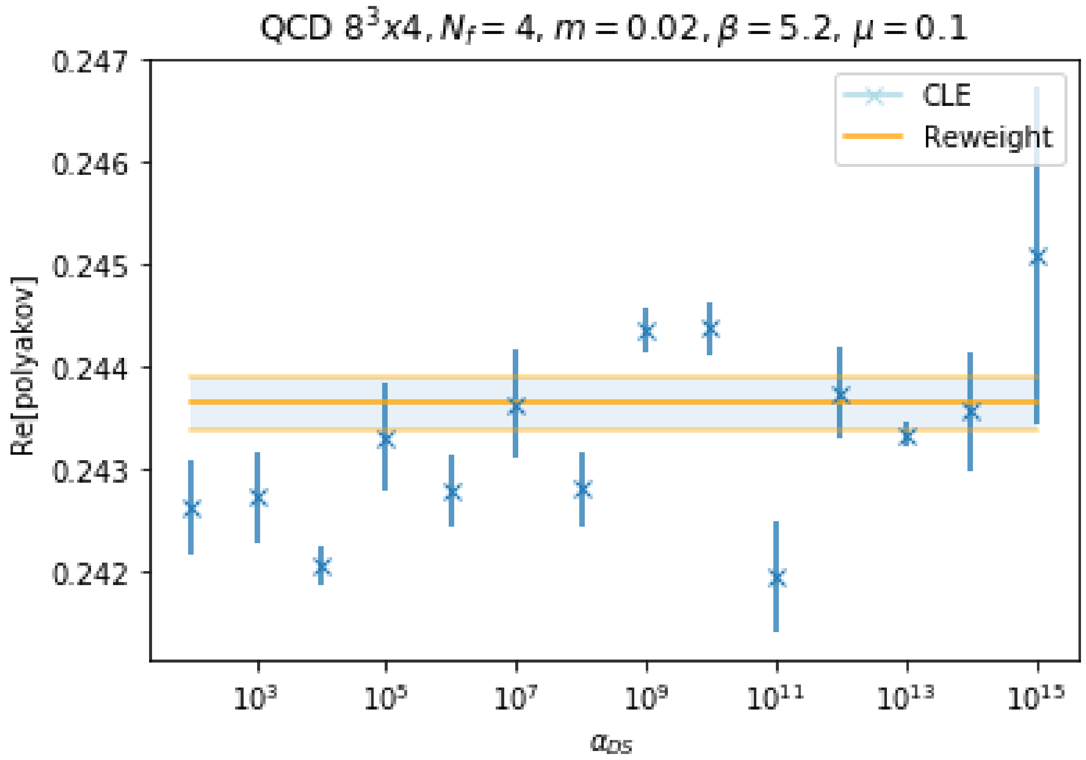
[Hansen, Sexty in preparation]



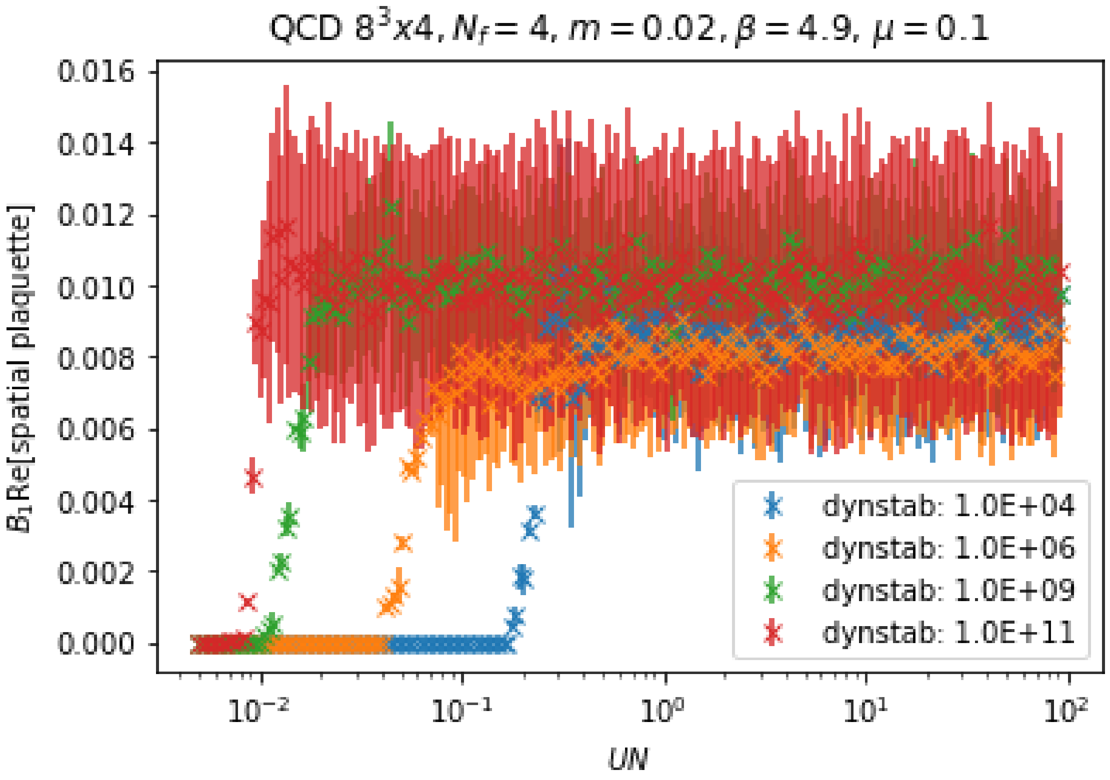


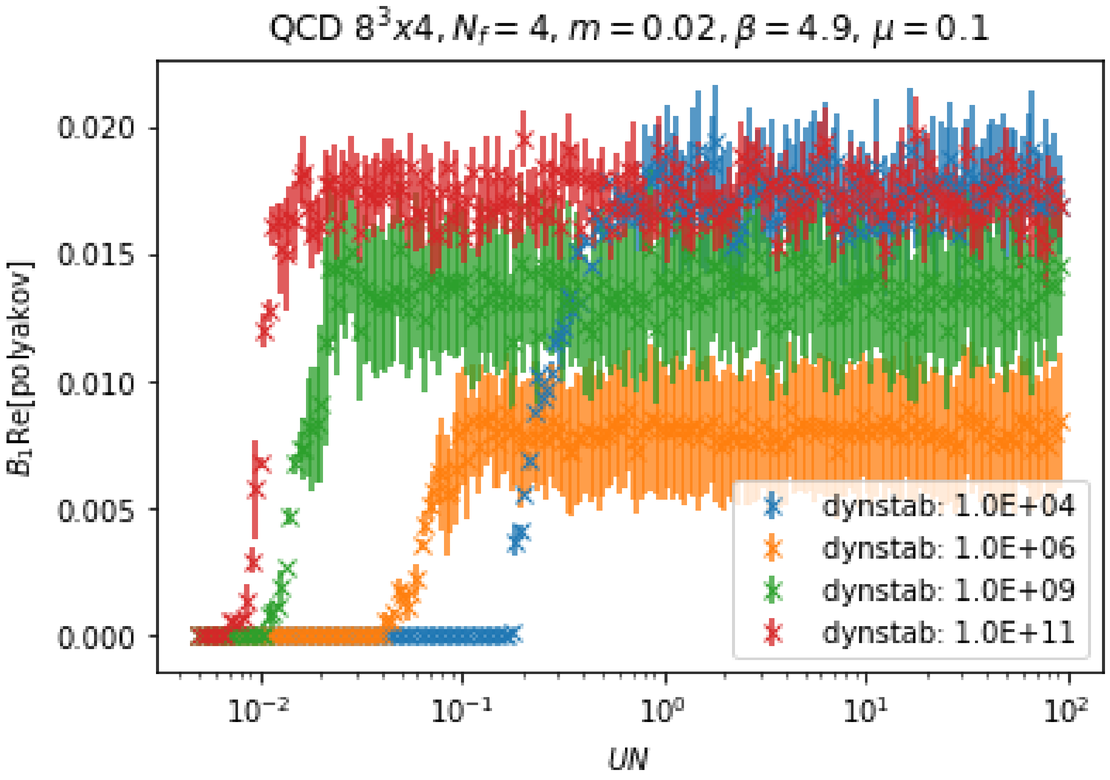
DynStab – High Temperature



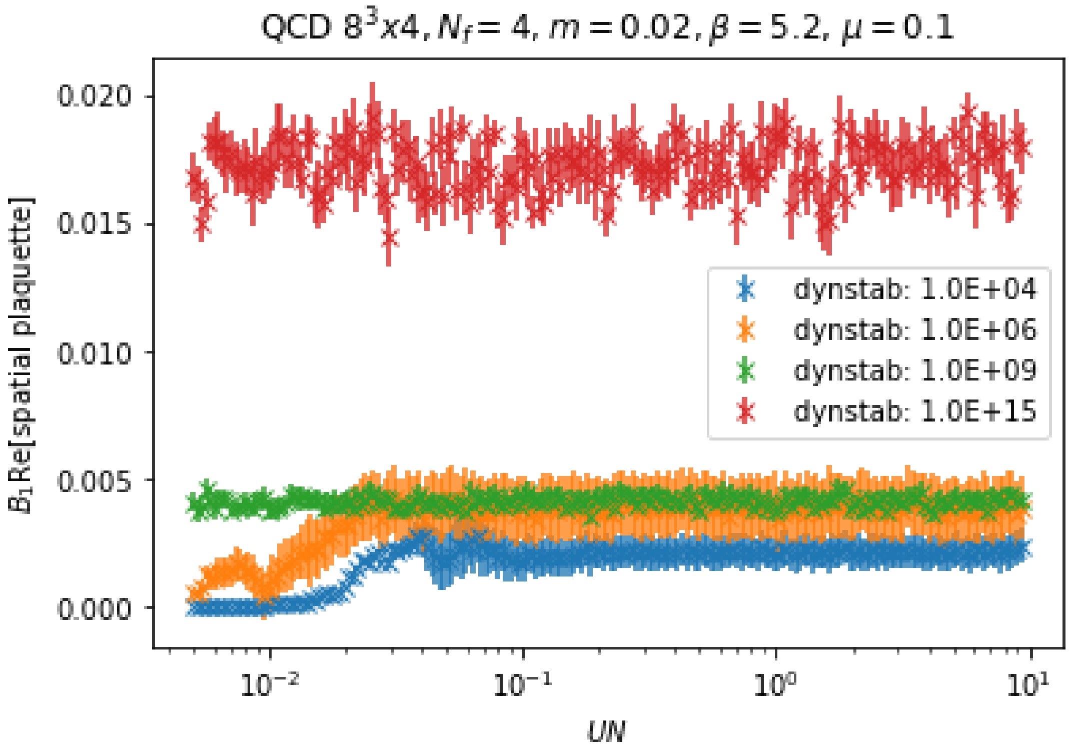


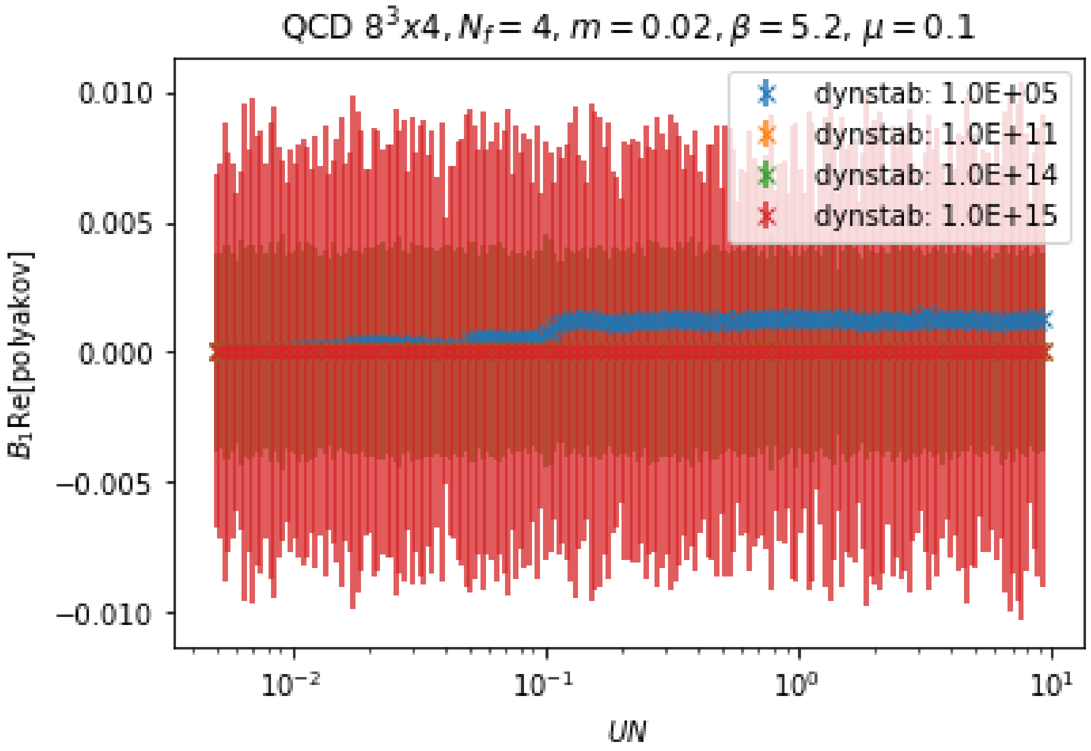
DynStab – Boundary terms low temp





DynStab – Boundary terms high temp





Conclusion

CL works very well for non-zero density (high temperature)

Low temperature deviations can be estimated using boundary terms

Dynamical Stabilization can slow the drifts from SU(3) to SL(3)

Dynamical Stabilization can help simulate low temperature

Calculating the boundary

- Long and difficult calculations
- $\Sigma = \frac{1}{|\Omega|} Tr(M^{-1})$
- $L_c\Sigma = \frac{2}{|\Omega|} \frac{N^2-1}{N} \left(Tr(M^{-1}) - mTr(M^{-2})\right)$

$$+ \frac{1}{|\Omega|} \sum_{j \in \Omega} 2Tr\left(M^{-1}\left(D_a^j M\right) M^{-1} \left(D_a^j M\right) M^{-1}\right)$$

$$+ \frac{1}{|\Omega|} \sum_{j \in \Omega} K_a^j D_a^j \, Tr(M^{-1})$$

$$\begin{aligned}
\frac{1}{2} \sum_{j \in \Omega} \mathrm{Tr}(M^{-1} D_i^j M M^{-1} D_i^j M M^{-1}) = \\
&- \mathrm{Tr}\left[\left(M^{-2}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
&- \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a+\mu,a+\mu}\right)\right] \\
&- \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a-\mu,a-\mu}\right)\right] \\
&- \mathrm{Tr}\left[\left(M^{-2}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \\
&+ \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\left(M^{-1}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
&+ \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\left(M_{a,a+\mu}\right)\left(M^{-1}_{a+\mu,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \\
&+ \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\left(M_{a,a-\mu}\right)\left(M^{-1}_{a-\mu,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
&+ \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\left(M^{-1}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right]
\end{aligned}$$