# Determination of Lee-Yang edge singularities in QCD by rational approximations

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Bonn, 08/08/2022

- One of our main goals is the study of the critical points in the QCD phase diagram
- We can think of at least three critical regions:

  (1) The Roberge-Weiss transition region

  (2) The chiral transition region

  (3) The region around the critical endpoint of QCD

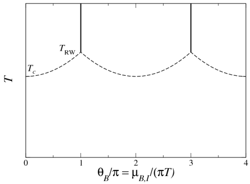



Fig from Phys.Rev. D 93, 74504 (2016)

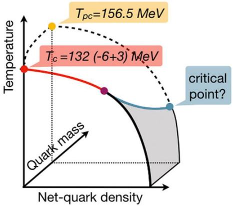



Fig taken from C. Schmidt

In this work we study the singularities of QCD in the complex $\mu$-plane.

- There is a deep connection between phase transitions and the singularities in the complex plane.

Consider for instance the Ising model

(see also previous talk by F. Di Renzo).

The free energy has a branch cut on the imaginary axis for the symmetry breaking field $h$. The branch point is known as Yang-Lee edge singularity.

[Yang, Lee, 1952]

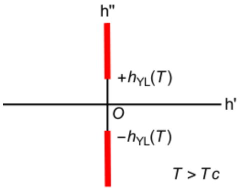



Fig from New J. Phys. 19, 083009 (2017)

- Why are YLE singuarities useful?

(1) If for $T \to T_c$ the YLE singularities end up on real axis this signals the presence of a physical phase transition

(2) At $T \neq T_c$ the presence of a YLE singularities determines a finite radius of convergence

- How do we get the information that we need from lattice data?

  Because of the sign problem we cannot directly explore regions at real (or even complex) chemical potentials.

- In our approach:

  (1) We perform simulations at imaginary $\mu$

  (2) We calculate the Taylor coefficients of a given observable

  (3) We merge the results using multi-point Padé approximants

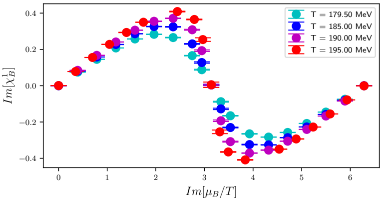
- This is a combination of the Taylor expansion and analytic continuation methods.

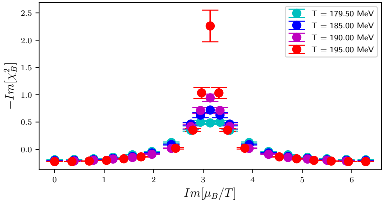
  We can extract information about the singularities of the observable by studying the uncanceled poles of the rational approximants and we can analytically continue the rational approximants to real $\mu$.

# RW transition region
## Numerical set-up

Let's first consider the high-temperature regime. The numerical set-up that we adopted is the following.

- We simulate highly-improved staggered quarks (HISQ) in $N_f = 2 + 1$ QCD using $36^3 \times 6$ lattices
- Simulations performed at $T = 195.0, 190.0, 185.0$ and $179.5$ $MeV$
- We have measured the cumulants $\chi^1_{B,Q} \equiv \frac{1}{Z}\frac{\partial Z}{\partial \mu_{B,Q}}$ and their first derivative w.r.t $\hat{\mu}_B$ at $O(10)$ imaginary chemical potentials

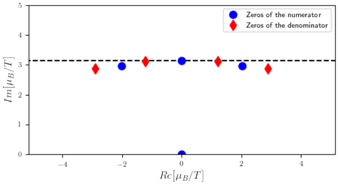


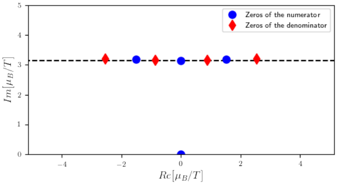
# RW transition region

## Singularity structure from Padé analysis (I)

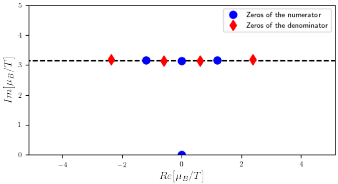
$T = 179.5\ MeV$



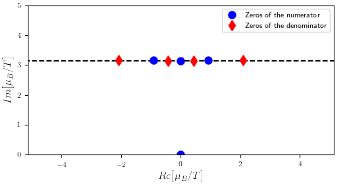

$T = 185.0\ MeV$




$T = 190.0\ MeV$




$T = 195.0\ MeV$

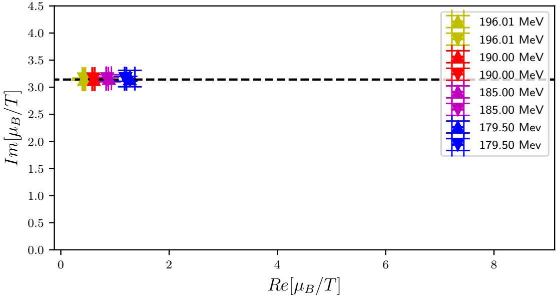



$\Rightarrow$ signature of branch cut singularities pinching the imag axis

# RW transition region

## Singularity structure from Padé analysis (II)

Summary:




- For all 4 temperatures we find singularities lying on $\hat{\mu}_B^I = \pi$. We find consistent results from the Padé approximants for $\chi_B^1$ and $\chi_Q^1$.
- Are these the YLE singularities associated to the RW critical point?

# RW transition region
## Scaling analysis (I)

Theoretical expectations:

- Magnetic EoS: $M(h,t) = h^{\frac{1}{\delta}} f_G(z) + M_{reg}(h,t)$, where
  - $t = t_0^{-1} \frac{T_{RW}-T}{T_{RW}}$ and $h = h_0^{-1} \frac{\hat{\mu}_B - i\pi}{i\pi}$ are scaling fields
  - $f_G$ is a scaling function depending only on the scaling variable $z = t/h^{\frac{1}{\beta\delta}}$
  - $\beta$ and $\delta$ are the critical exponents [$3d, Z(2)$ universality class]
- We can solve $t/h^{\frac{1}{\beta\delta}} = z_c \equiv |z_c| e^{i\frac{\pi}{2\beta\delta}}$

$$\rightarrow \hat{\mu}^R_{YLE} = \pm\pi\left(\frac{z_0}{|z_c|}\right)^{\beta\delta}\left(\frac{T_{RW}-T}{T_{RW}}\right)^{\beta\delta} \quad , \quad \hat{\mu}^I_{YLE} = \pi$$

Scaling analysis:

- We can fit the real part of the singularities using the ansatz:

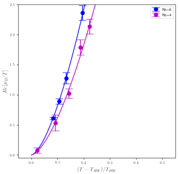
$$\hat{\mu}^R_{YLE} = a\left(\frac{T_{RW}-T}{T_{RW}}\right)^{\beta\delta} + b$$

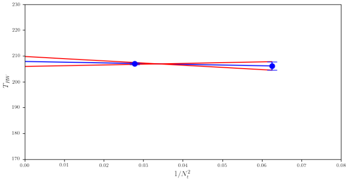
$\rightarrow$ estimate non-universal parameters $T_{RW}$, $z_0$ from fit parameters

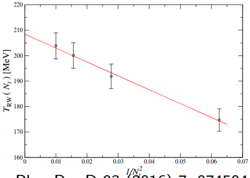
# RW transition region
## Scaling analysis (II)

- By fitting our data using $\beta\delta = 1.5635$ we obtain the estimate $T_{RW} = 207.08(35)\ MeV$.

  A rudimentary continuum extrapolation using our previous data for $N_\tau = 4$ yields $T_{RW}^{cont} = 207.1(2.4)\ MeV$.



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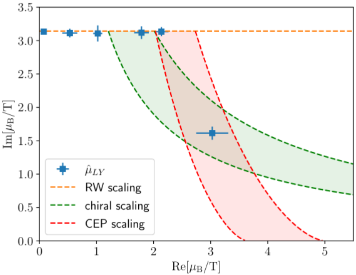
- Using $|z_c| = 2.452$ from [Connely et al., 2020] we also find $z_0 = 10.2$-$10.5$

# Chiral transition and CEP regions

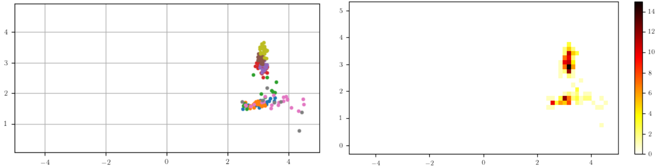
## Scaling analysis



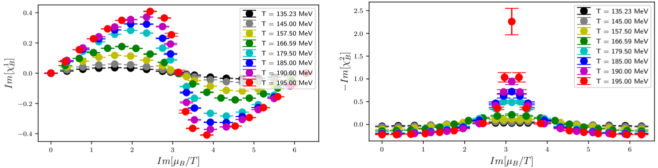

- Chiral transition region
  - $t = t_0^{-1}\left[\frac{T-T_c}{T_c} + k_2^B\left(\frac{\mu_B}{2}\right)^2\right]$ ,
    $h = h_0^{-1}\frac{m_l}{m_s^{phys}}$ [Mukherjee and Skokov, 2021]
  - $\hat{\mu}_{YLE} = \left[\frac{1}{k_2^B}\frac{z_c}{z_0}\left(\frac{m_l}{m_s^{phys}}\right)^{\frac{1}{\beta\delta}} - \frac{T-T_c}{T_c}\right]^{\frac{1}{2}}$
  - non-universal parameters $T_c, k_2^B, z_0$ from [HotQCD data], $|z_c|$ from [Connely]
- CEP region
  - linear ansatz [Basar, 2021]

    $$t = \alpha_t(T - T_{CEP}) + \beta_t(\mu_B - \mu_{CEP}) \quad ,$$
    $$h = \alpha_h(T - T_{CEP}) + \beta_h(\mu_B - \mu_{CEP})$$
  - $\mu_{YLE} \sim \mu_{CEP} - c_1(T - T_{CEP}) + ic_2|z_c|^{-\beta\delta}(T - T_{CEP})^{\beta\delta}$
  - parameters unknown: to visualize one can guessestimate
    $\mu_{CEP} = 500 - 630\ MeV,\ T_{CEP} = T_{pc}(1 - k_2^B(\frac{\mu_{CEP}}{T_{pc}})^2)$

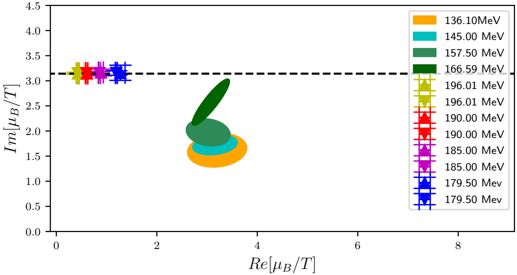
- Previously at 145 $MeV$ we had found a candidate for a chiral singularity by restricting the fit interval from $[0, 2\pi]$ to $[0, \pi]$



- We performed and additional set of simulations in the low-temperature regime ($T = 135.23, 157.50, 166.59$ $MeV$)



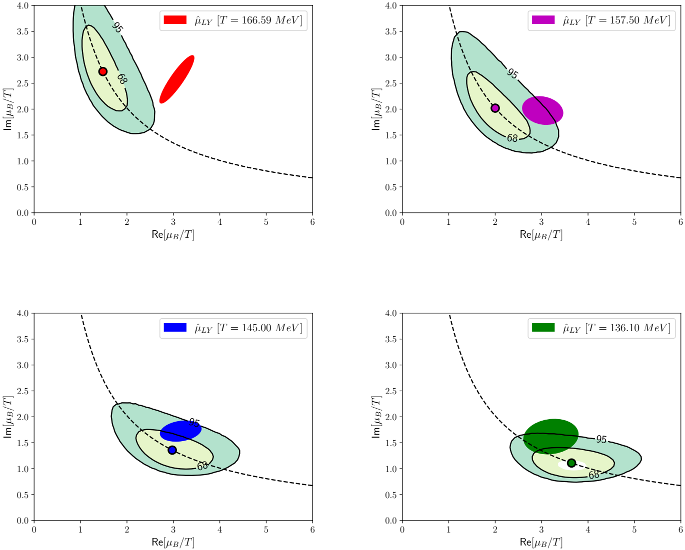
Summary:




- We observe a singularity moving towards the real axis as T is decreased
- We can infer a radius of convergence $\mathcal{R} = |\hat{\mu}_B - 0| \approx 3 \div 4$ for $T = 136 \div 166\ MeV$
- Are these the YLE singularities associated to the chiral transition?

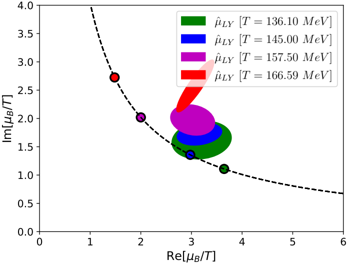
# Chiral transition and CEP regions

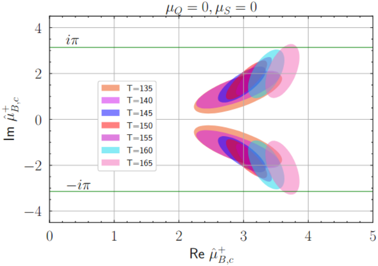
## Singularity structure from Padé analysis (II)



# Chiral transition and CEP regions

## Singularity structure from Padé analysis (III)

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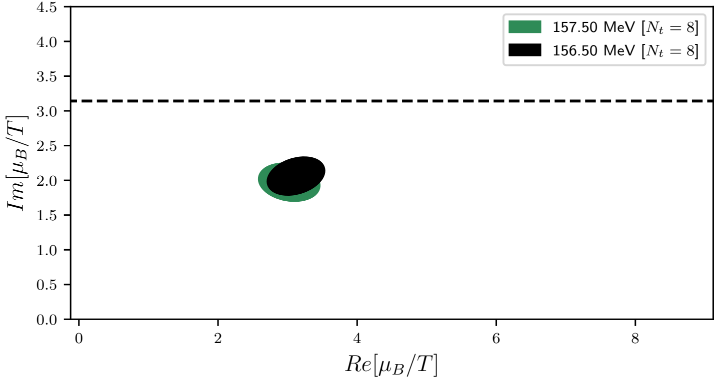
- Individually the results are in agreement with theoretical expectations within errors, but looking at the big picture the trajectory has a steeper slope than expected...

# Chiral transition and CEP regions

## $N_\tau = 8$ data

- We also have data from $N_\tau = 8$ lattices for one temperature. We find a singularity consistent with the singularity that we had found for a very similar temperature on the $N_\tau = 6$ lattice.



# Conclusions

- We have used the multi-point Padé method to study the singularities of QCD in the complex plane.
- We have identified the YLE singularities associated to the Roberge-Weiss critical point.

  The singularities follow the expected scaling behaviour for a transition belonging to the $3d, Z(2)$ universality class.

  We also have a preliminary continuum extrapolation for $T_{RW}$.

  Future prospects: for a proper extrapolation we would need $N_\tau = 8$ data.
- We have found singularities that *might* be identified as the YLE singularities associated to the chiral transition. These imply a radius of convergence $\mathcal{R} \approx 3 \div 4$ for $T = 136 \div 166\ MeV$.

  Future prospects: simulations with smaller $m_l/m_s$ ratio are currently ongoing.

Thank you for listening!

# Backup slide