Equation of state and Taylor expansions at nonzero isospin chemical potential

Bastian Brandt

Universität Bielefeld

Francesca Cuteri & Gergely Endrődi

UNIVERSITÄT BIELEFELD
Faculty of Physics

CRC-TR 211
Strong-interaction matter under extreme conditions

11.08.2022

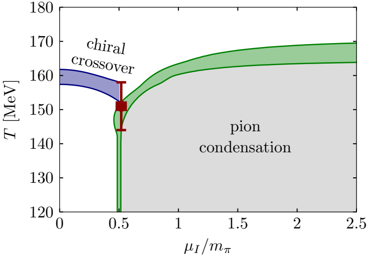
QCD at non-zero chemical potential

Convenient chemical potential basis for simulations: ("isospin" basis)

$$\mu_u = \mu_L + \mu_I \qquad \mu_d = \mu_L - \mu_I \qquad \mu_s$$

$\boxed{\mu_I \neq 0, \quad \mu_L = \mu_s = 0}$ pure isospin chemical pot. – no sign problem

Continuum extrapolated phase diagram:



[Brandt, Endrődi, Schmalzbauer '18]

- improved actions (staggered)
- physical pion masses
- $T \neq 0$: $N_t = (6,)\,8,\ 10,\ 12$
- $T = 0$: $a = 0.22,\ 0.15$ [0.1] fm

Contents

I The equation of state

Equation of state: $T = 0$

$$p(T=0,\mu_I=0)=0 \qquad n_I=\frac{\partial p}{\partial \mu_I} \qquad \epsilon=-p+n_I\mu_I$$

$$\Rightarrow \quad p(0,\mu_I)=\int_0^{\mu_I} d\mu\, n_I(0,\mu_I) \qquad I(0,\mu_I)=-4p+n_I(0,\mu_I)\mu_I$$

can be obtained from $n_I(0,\mu_I)$

here: use all spline interpolations which provide “good” description of n_I

Equation of state: $T = 0$

$$p(T=0,\mu_I=0)=0 \qquad n_I=\frac{\partial p}{\partial \mu_I} \qquad \epsilon=-p+n_I\mu_I$$

$$\Rightarrow \quad p(0,\mu_I)=\int_0^{\mu_I} d\mu\, n_I(0,\mu_I) \qquad I(0,\mu_I)=-4p+n_I(0,\mu_I)\mu_I$$

can be obtained from $n_I(0,\mu_I)$

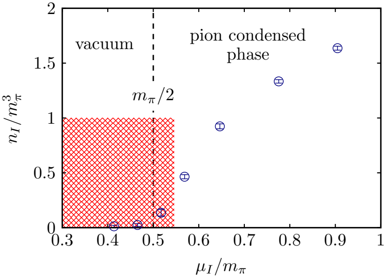
here: use all spline interpolations which provide “good” description of n_I

- $T=0$ never exactly fulfilled

  → use large N_t so that $T\approx 0$

- correct for $T\neq 0$ effects with χPT

  particularly relevant for $\mu_I \approx m_\pi/2$



Equation of state: $T = 0$

$$p(T=0,\mu_I=0)=0 \qquad n_I=\frac{\partial p}{\partial \mu_I} \qquad \epsilon=-p+n_I\mu_I$$

$$\Rightarrow \quad p(0,\mu_I)=\int_0^{\mu_I} d\mu\, n_I(0,\mu_I) \qquad I(0,\mu_I)=-4p+n_I(0,\mu_I)\mu_I$$

can be obtained from $n_I(0,\mu_I)$

here: use all spline interpolations which provide "good" description of n_I

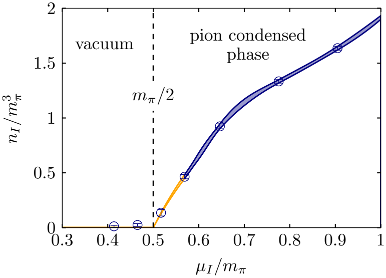
- $T=0$ never exactly fulfilled

  ➡ use large N_t so that $T\approx 0$
- correct for $T\neq 0$ effects with χPT

  particularly relevant for $\mu_I \approx m_\pi/2$

- initial $a \approx 0.29$ fm simulations

[Brandt *et al* '18]



Equation of state: $T = 0$

$$p(T=0,\mu_I=0)=0 \qquad n_I = \frac{\partial p}{\partial \mu_I} \qquad \epsilon = -p + n_I\mu_I$$

$$\Rightarrow \quad p(0,\mu_I) = \int_0^{\mu_I} d\mu\, n_I(0,\mu_I) \qquad I(0,\mu_I) = -4p + n_I(0,\mu_I)\mu_I$$

can be obtained from $n_I(0,\mu_I)$

here: use all spline interpolations which provide "good" description of n_I

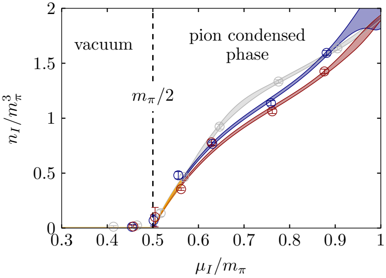
- $T = 0$ never exactly fulfilled

  ⟶ use large N_t so that $T \approx 0$
- correct for $T \neq 0$ effects with χPT

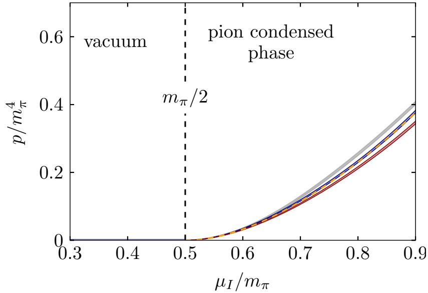
  particularly relevant for $\mu_I \approx m_\pi/2$

- initial $a \approx 0.29$ fm simulations

  [Brandt *et al* '18]
- $a \approx 0.22$, 0.15 fm – towards continuum



Equation of state: $T = 0$

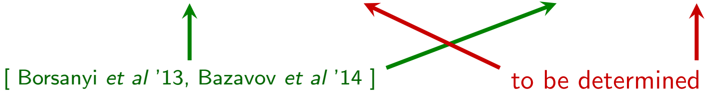


- in good agreement with NLO χPT [Adhikari, Andersen '19]
- currently: analyse $a \approx 0.1$ fm lattices

→ continuum limit

Equation of state: $T \neq 0$

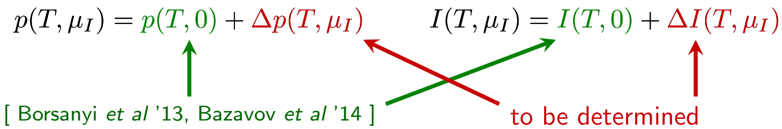
$$p(T,\mu_I) = p(T,0) + \Delta p(T,\mu_I) \qquad I(T,\mu_I) = I(T,0) + \Delta I(T,\mu_I)$$



[Borsanyi *et al* '13, Bazavov *et al* '14]

to be determined

Equation of state: $T \neq 0$



$$p(T,\mu_I) = p(T,0) + \Delta p(T,\mu_I) \qquad I(T,\mu_I) = I(T,0) + \Delta I(T,\mu_I)$$

[Borsanyi *et al* '13, Bazavov *et al* '14]

to be determined

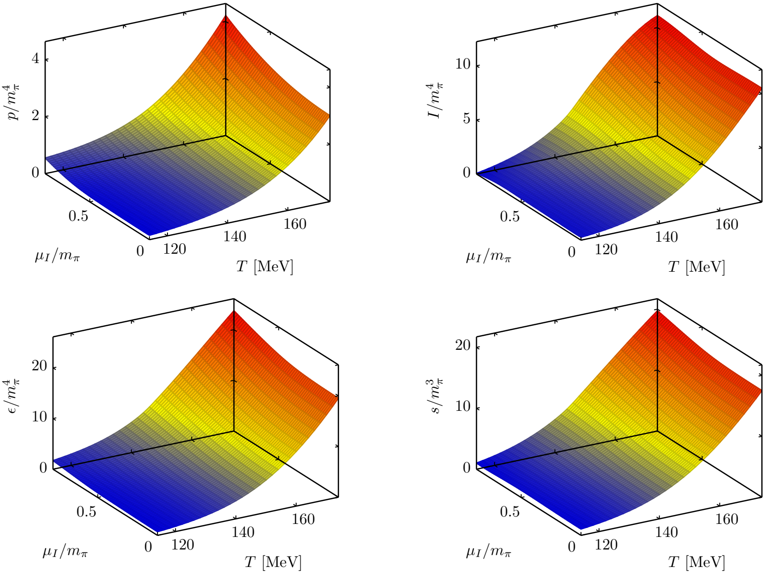
- computation of Δp: $\Delta p(T,\mu_I) = \int_0^{\mu_I} d\mu \, n_I(T,\mu)$
- computation of ΔI:

Starting point: $$\frac{\Delta I(T,\mu_I)}{T^4} = T\frac{\partial}{\partial T}\left(\frac{\Delta p(T,\mu_I)}{T^4}\right) + \frac{\mu_I n_I(T,\mu_I)}{T^4}$$

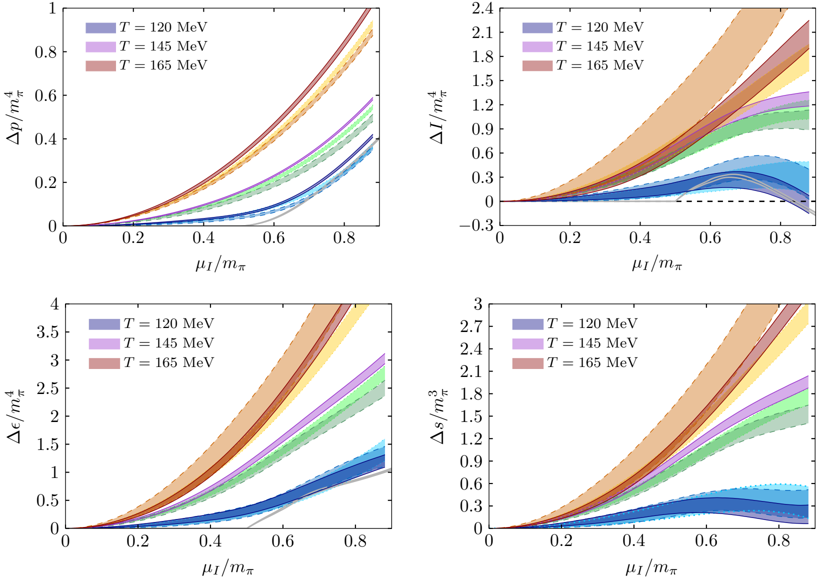
$$\longrightarrow \Delta I(T,\mu_I) = \int_0^{\mu_I} d\mu_I' \left(T\frac{\partial}{\partial T} - 4\right) n_I(T,\mu_I') + \mu_I n_I(T,\mu_I)$$

again: use all spline interpolations which provide “good” description of n_I

Computing the EoS at $T \neq 0$ – $N_t = 8$



Computing the EoS at $T \neq 0$



II The speed of sound

Computing the speed of sound

Other interesting quantity: speed of sound c_s

definition at $\mu \neq 0$: $c_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|$ @ $\left[\frac{s}{n_I} = \text{const}; \quad \frac{n_I}{n_L} = \text{const}; \quad \ldots\right]$

Computing the speed of sound

Other interesting quantity: speed of sound c_s

definition at $\mu_I \neq 0$: $c_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|$ @ $\left[\frac{s}{n_I} = \text{const}\right]$

→ compute directional derivative of p and ϵ:

$$c_s^2 = \frac{\partial p}{\partial \xi}\left(\frac{\partial \epsilon}{\partial \xi}\right)^{-1} \qquad \text{with} \qquad \frac{\partial}{\partial \xi}\left(\frac{s}{n_I}\right) = 0$$

Computing the speed of sound

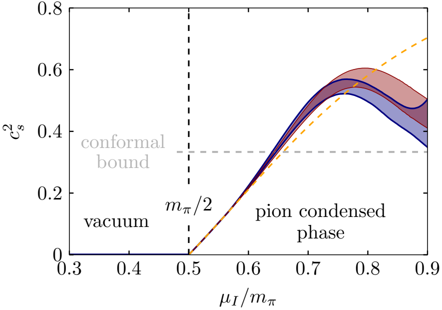
Other interesting quantity: speed of sound c_s

definition at $\mu_I \neq 0$: $c_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|$ @ $\left[\frac{s}{n_I} = \text{const}\right]$

→ compute directional derivative of p and ϵ:

$$c_s^2 = \frac{\partial p}{\partial \xi}\left(\frac{\partial \epsilon}{\partial \xi}\right)^{-1} \qquad \text{with} \qquad \frac{\partial}{\partial \xi}\left(\frac{s}{n_I}\right) = 0$$

- $T = 0$: $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \mu_I}$



Computing the speed of sound

Other interesting quantity: speed of sound c_s

definition at $\mu_I \neq 0$: $c_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|$ @ $\left[\frac{s}{n_I} = \text{const}\right]$

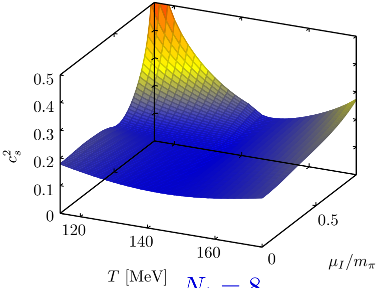
→ compute directional derivative of p and ϵ:

$$c_s^2 = \frac{\partial p}{\partial \xi}\left(\frac{\partial \epsilon}{\partial \xi}\right)^{-1} \qquad \text{with} \qquad \frac{\partial}{\partial \xi}\left(\frac{s}{n_I}\right) = 0$$

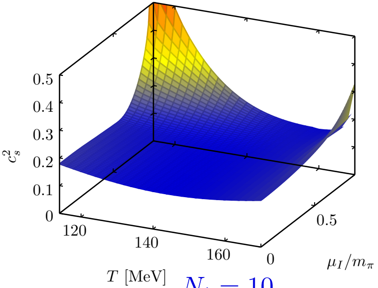
- $T = 0$: $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \mu_I}$
- $T \neq 0$: $\frac{\partial}{\partial \xi}$ mixed T & μ_I dervivative – compute analytically via
  - the spline interpolation for $n_I(T, \mu_I)$
  - analytic interpolations for $p(T,0)$ and $I(T,0)$ from [Borsanyi *et al* '13]

$$\frac{I(T,0)}{T^4} = e^{-h_1/t - h_2/t^2}\left(h_0 + \frac{f_0\left[\tanh(f_1\,t + f_2) + 1\right]}{1 + g_1 t + g_2 t^2}\right)$$

Speed of sound at $T \neq 0$



$N_t = 8$



$N_t = 10$

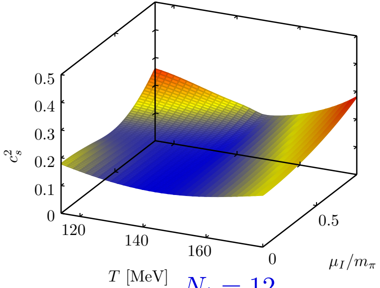
- $T \ll T_c$ & $\mu_I \gg m_\pi/2$:

  $c_s^2 > 1/3$ – conformal limit

  [Cherman, Cohen, Nellore '09]

  towards continuum:

  pushed to smaller T



$N_t = 12$

III Extension to $\mu_L \neq 0$ via Taylor expansion

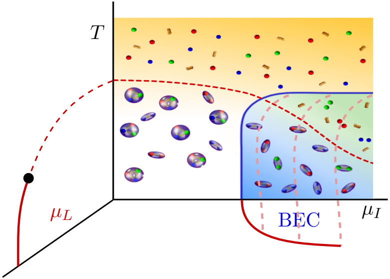
Extension to baryon chemical potential

Eventually: $\mu_I,\, \mu_L,\, \mu_s$ parameter space

Extension to $\mu_L,\, \mu_s \neq 0$:

Taylor expansion around $\mu_I \neq 0$

here: focus on μ_L direction



Extension to baryon chemical potential

Eventually: $\mu_I,\, \mu_L,\, \mu_s$ parameter space

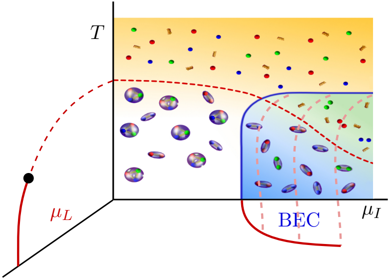
Extension to $\mu_L,\, \mu_s \neq 0$:

Taylor expansion around $\mu_I \neq 0$

here: focus on μ_L direction

$$\frac{p(T,\mu)}{T^4} = \sum_{m=0}^{\infty} \frac{\chi_m^L(T,\mu_I)}{m!} \left(\frac{\mu_L}{T}\right)^m$$

$$\chi_m^L(T,\mu_I) = \frac{\partial^m [p(T,\mu)/T^4]}{\left(\partial \mu_L/T\right)^m}\Bigg|_{\mu_L=0}$$



Extension to baryon chemical potential

Eventually: μ_I, μ_L, μ_s parameter space

Extension to $\mu_L, \mu_s \neq 0$:

Taylor expansion around $\mu_I \neq 0$

here: focus on μ_L direction

$$\frac{p(T,\mu)}{T^4} = \sum_{m=0}^{\infty} \frac{\chi_m^L(T,\mu_I)}{m!} \left(\frac{\mu_L}{T}\right)^m$$

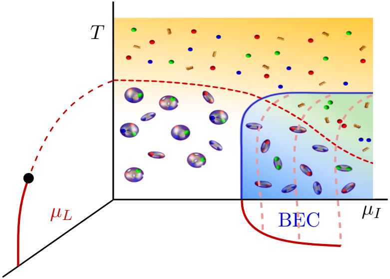
$$\chi_m^L(T,\mu_I) = \left.\frac{\partial^m [p(T,\mu)/T^4]}{(\partial \mu_L/T)^m}\right|_{\mu_L=0}$$

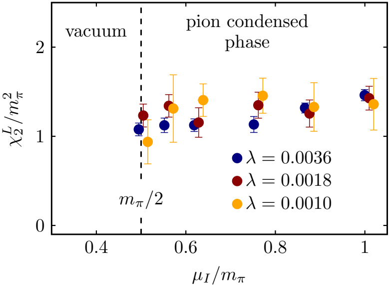
- first results for $\chi_2^L(T,\mu_I)$:

  $T \approx 0$ (30 MeV)
  for $a \approx 0.15$ fm

  at first glance: looks good

  but what about this λ?

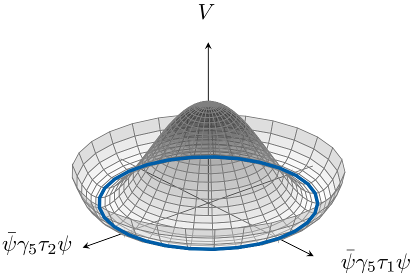




The regulator and its removal

Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode



The regulator and its removal

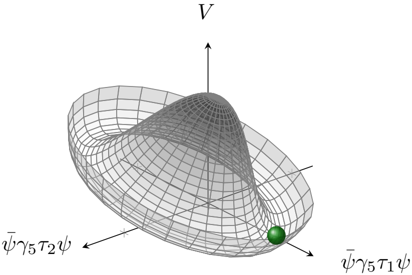
Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode

- for simulations:

  need to introduce a regulator

  – λ (pion source) [Kogut, Sinclair '02]



The regulator and its removal

Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode

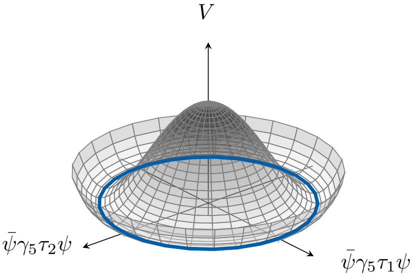
- for simulations:

  need to introduce a regulator

  – λ (pion source) [Kogut, Sinclair '02]
- physical results: extrapolate $\lambda \to 0$

  reliable extrapolations:

  main task for analysis



The regulator and its removal

Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode

- for simulations:

  need to introduce a regulator

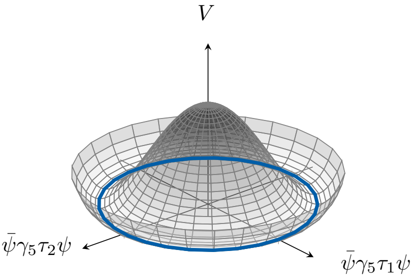
  – λ (pion source) [Kogut, Sinclair '02]
- physical results: extrapolate $\lambda \to 0$

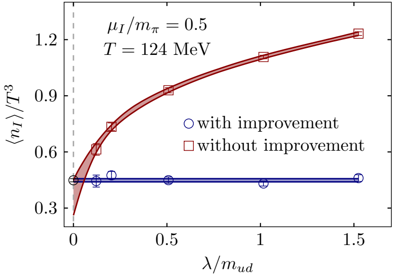
  reliable extrapolations:

  main task for analysis
- facilitated by improvement program:

  [Brandt, Endrődi, Schmalzbauer '18]
  - leading order reweighting
  - valence quark improvement
    - condensates/densities:

      works well for $T \neq 0$





The regulator and its removal

Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode

- for simulations:

  need to introduce a regulator

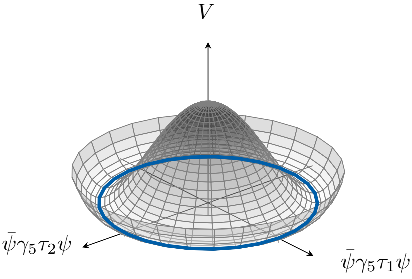
  – λ (pion source) [Kogut, Sinclair '02]
- physical results: extrapolate $\lambda \to 0$

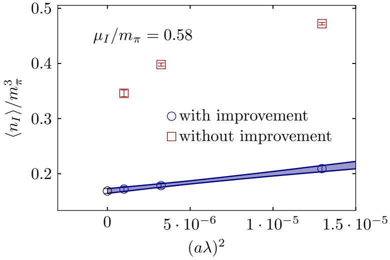
  reliable extrapolations:

  main task for analysis
- facilitated by improvement program:

  [Brandt, Endrődi, Schmalzbauer '18]
  - leading order reweighting
  - valence quark improvement
    - condensates/densities:

      works well for $T \neq 0$ & $T = 0$





The regulator and its removal

Spontaneous sym. breaking @ BEC phase:

→ Goldstone mode – zero mode

- for simulations:

  need to introduce a regulator

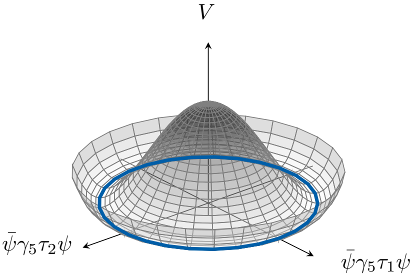
  – λ (pion source) [Kogut, Sinclair '02]
- physical results: extrapolate $\lambda \to 0$

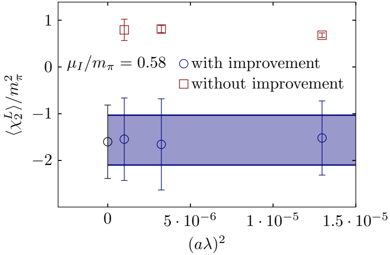
  reliable extrapolations:

  main task for analysis
- facilitated by improvement program:

  [Brandt, Endrődi, Schmalzbauer '18]
  - leading order reweighting
  - valence quark improvement
    - condensates/densities:

      works well for $T \neq 0$ & $T = 0$
    - **susceptibilities?**





Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O} \right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N \right\rangle$

Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$

- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle = \left\langle \sum_{n=0}^{N_{\max}} \frac{\varphi_n^\dagger O\varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

here: $M = D^\dagger(\mu)D(\mu) + \lambda^2$ $O_{nm} = \varphi_n^\dagger O\varphi_m$

Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$

- condensates/densities:

$$\sim \left\langle \mathsf{Tr}\big(OM^{-1}\big)\right\rangle \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O\varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

here: $M = D^\dagger(\mu)D(\mu)+\lambda^2$ $O_{nm} = \varphi_n^\dagger O\varphi_m$

Valence quark improvement for susceptibilities

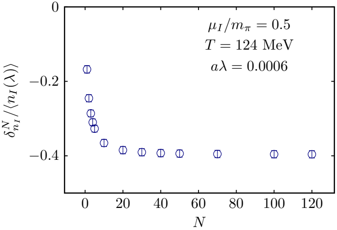
Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$

- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle \quad \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O \varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

define: $\delta_O^N = \sum_{n=0}^{N} O_{nn}\Big(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\Big)$



here: $M = D^\dagger(\mu)D(\mu) + \lambda^2$ $\qquad O_{nm} = \varphi_n^\dagger O \varphi_m$

Valence quark improvement for susceptibilities

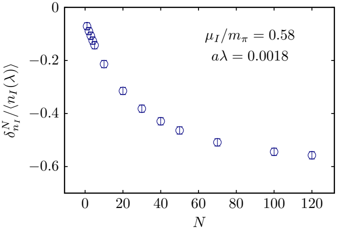
Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O} \right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N \right\rangle$

- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O \varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

define: $\delta_O^N = \sum_{n=0}^{N} O_{nn}\left(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\right)$

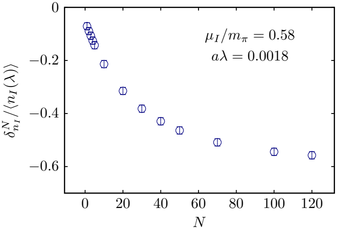


here: $M = D^\dagger(\mu)D(\mu) + \lambda^2$ $O_{nm} = \varphi_n^\dagger O \varphi_m$

Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$



- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle \quad \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O\varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

define: $\delta_O^N = \sum_{n=0}^{N} O_{nn}\Big(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\Big)$

- susceptibilities:

$$\sim \left\langle \mathrm{Tr}\big(O_1M^{-1}O_2M^{-1}\big)\right\rangle$$

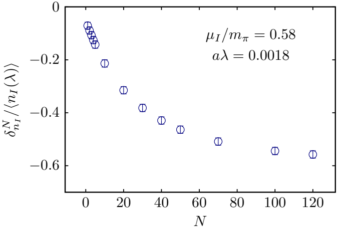
$$= \left\langle \sum_{n,m=0}^{N_{\max}} \frac{O_{1;nm}}{\xi_m^2+\lambda^2}\,\frac{O_{2;mn}}{\xi_n^2+\lambda^2}\right\rangle$$

here: $M = D^\dagger(\mu)D(\mu)+\lambda^2$ $\quad O_{nm} = \varphi_n^\dagger O\varphi_m$

Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $\quad D(\mu)$: massive Dirac operator

improvement: $\lim\limits_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim\limits_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$



- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle \quad \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O \varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

define: $\delta_O^N = \sum\limits_{n=0}^{N} O_{nn}\left(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\right)$

- susceptibilities:

$$\sim \left\langle \mathrm{Tr}\big(O_1M^{-1}O_2M^{-1}\big)\right\rangle$$

$$\approx \left\langle \sum_{n,m=0}^{N} \frac{O_{1;nm}}{\xi_m^2+\lambda^2}\frac{O_{2;mn}}{\xi_n^2+\lambda^2}\right\rangle$$

here: $M = D^\dagger(\mu)D(\mu)+\lambda^2 \qquad O_{nm} = \varphi_n^\dagger O \varphi_m$

Valence quark improvement for susceptibilities

Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $\quad D(\mu)$: massive Dirac operator

improvement: $\lim\limits_{\lambda\to0}\left\langle \hat{O}\right\rangle = \lim\limits_{\lambda\to0}\left\langle \hat{O} - \delta_O^N\right\rangle$

- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\left(OM^{-1}\right)\right\rangle \quad \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O \varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

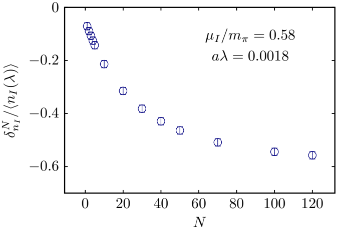
define: $\delta_O^N = \sum\limits_{n=0}^{N} O_{nn}\left(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\right)$

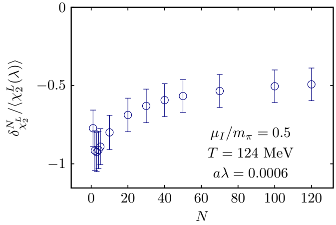
- susceptibilities:

$$\sim \left\langle \mathrm{Tr}\left(O_1 M^{-1} O_2 M^{-1}\right)\right\rangle$$

$$\approx \left\langle \sum_{n,m=0}^{N} \frac{O_{1;nm}}{\xi_m^2+\lambda^2}\frac{O_{2;mn}}{\xi_n^2+\lambda^2}\right\rangle$$

here: $\quad M = D^\dagger(\mu)D(\mu)+\lambda^2 \qquad O_{nm} = \varphi_n^\dagger O \varphi_m$





Valence quark improvement for susceptibilities

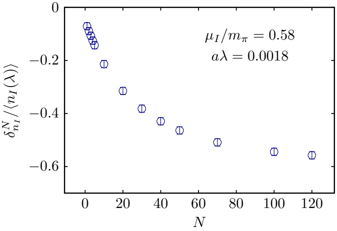
Singular values: $D^\dagger(\mu)D(\mu)\varphi_n = \xi_n^2\varphi_n$ $D(\mu)$: massive Dirac operator

improvement: $\lim_{\lambda\to 0}\left\langle \hat{O}\right\rangle = \lim_{\lambda\to 0}\left\langle \hat{O} - \delta_O^N\right\rangle$

- condensates/densities:

$$\sim \left\langle \mathrm{Tr}\big(OM^{-1}\big)\right\rangle \approx \left\langle \sum_{n=0}^{N} \frac{\varphi_n^\dagger O\varphi_n}{\xi_n^2+\lambda^2}\right\rangle$$

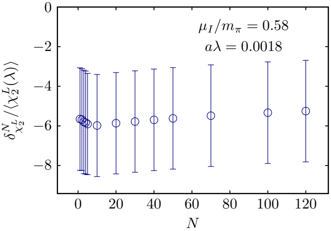
define: $\delta_O^N = \sum_{n=0}^{N} O_{nn}\left(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\right)$



- susceptibilities:

$$\sim \left\langle \mathrm{Tr}\big(O_1M^{-1}O_2M^{-1}\big)\right\rangle$$

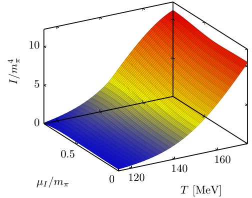
$$\approx \left\langle \sum_{n,m=0}^{N} \frac{O_{1;nm}}{\xi_m^2+\lambda^2}\frac{O_{2;mn}}{\xi_n^2+\lambda^2}\right\rangle$$



here: $M = D^\dagger(\mu)D(\mu) + \lambda^2$ $O_{nm} = \varphi_n^\dagger O\varphi_m$

Conclusions

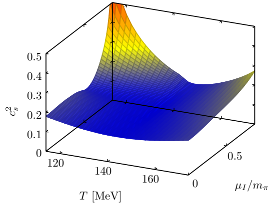
- Equation of state at $T = 0$ & $T \neq 0$
  - continuum limit in progress
  - needs $N_t = 16$ lattices



- Speed of sound

  for $T \ll T_c$ & $\mu_I \gg m_\pi/2$:
  larger than conformal limit
  [Cherman, Cohen, Nellore '09]

  → implications for EoS modelling based on neutron star radii and masses
  [e.g.: Tews *et al* '18; Annala *et al* '20]



- Taylor epansion to $\mu_L \neq 0$ from $\mu_I \neq 0$

  work in progress ...

  problem: reliable λ-extrapolations for χ_2^L