

# Inhomogeneous phases in the 3+1-dimensional mean-field Nambu-Jona-Lasinio model on the lattice

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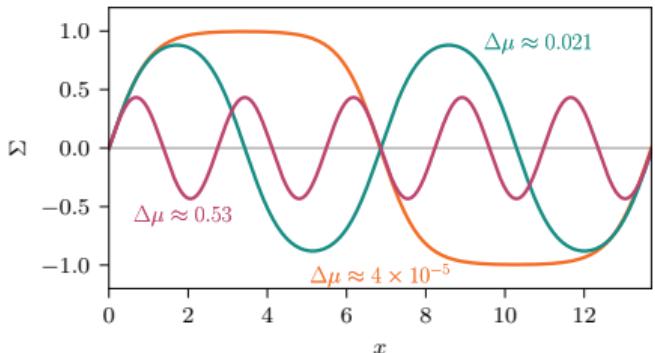
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# Motivation

- Several low-energy effective models exhibit a chiral inhomogeneous phase (IP) i.e. a **space-dependent chiral condensate**  
[ M. Buballa, S. Carignano, *Prog. Part. Nucl. Phys.*, **81**, 39–96 (2015), arXiv: 1406.1367 ]
- Indications for such phases and related moat regimes in QCD  
[ R. D. Pisarski, F. Rennecke, *Phys. Rev. Lett.*, **127**, 152302 (2021), arXiv: 2103.06890 ]  
[ W.-j. Fu et al., *Phys. Rev. D*, **101**, 054032 (2020), arXiv: 1909.02991 ]
- Some of these models suffer from non-renormalizability and an inherent **regulator dependence**
- Vast majority of investigations in **mean-field**



[ M. Thies, K. Urlichs, *Phys. Rev.*, **D67**, 125015 (2003), arXiv: hep-th/0302092 ]

[ A. Koenigstein et al., (2021), arXiv: 2112.07024 ]

# Motivation

- **Main goal:** Investigate inhomogeneous phases in 3 + 1-dimensional (Four-Fermi) models beyond mean-field via lattice Monte-Carlo simulations.

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- **But first:**
  - Is the lattice even a suitable regularization to investigate IPs in  $3 + 1$ -dimensional Four-Fermi models?
  - Are results of IPs a consistent feature independent of the regularization scheme?

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- **Main goal:** Investigate inhomogeneous phases in  $3 + 1$ -dimensional (Four-Fermi) models beyond mean-field via lattice Monte-Carlo simulations.
- **But first:**
  - Is the lattice even a suitable regularization to investigate IPs in  $3 + 1$ -dimensional Four-Fermi models?
  - Are results of IPs a consistent feature independent of the regularization scheme?
- Start with **stability analysis** of the  $3 + 1$ -dimensional mean-field Nambu-Jona-Lasinio (NJL) model on the **lattice**

$$S = \int d^3x \int_0^\beta \left\{ \bar{\psi}(\not{d} + \gamma_0\mu + m_0)\psi + G \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\boldsymbol{\tau}\psi)^2 \right] \right\}$$

# 3 + 1-dimensional mean-field NJL model

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↓  
Bosonization, integration over fermions

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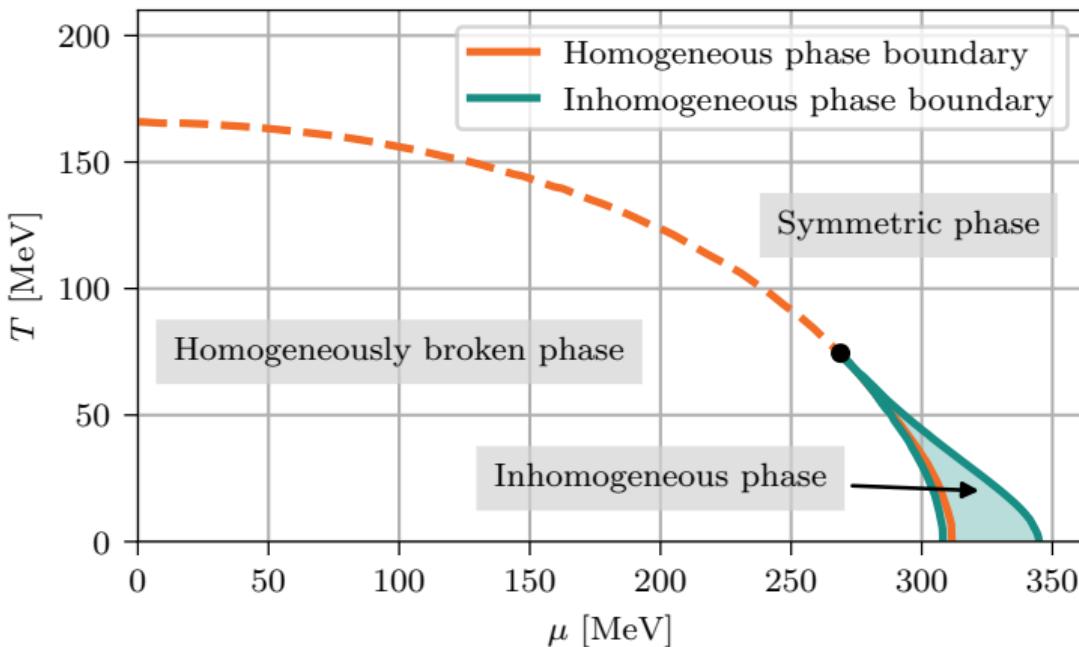
- **Mean-field** approximation: only field configurations that minimize  $S_{\text{eff}}$  contribute  
⇒ path-integral is reduced to a minimization problem

# 3+1-dimensional mean-field NJL model

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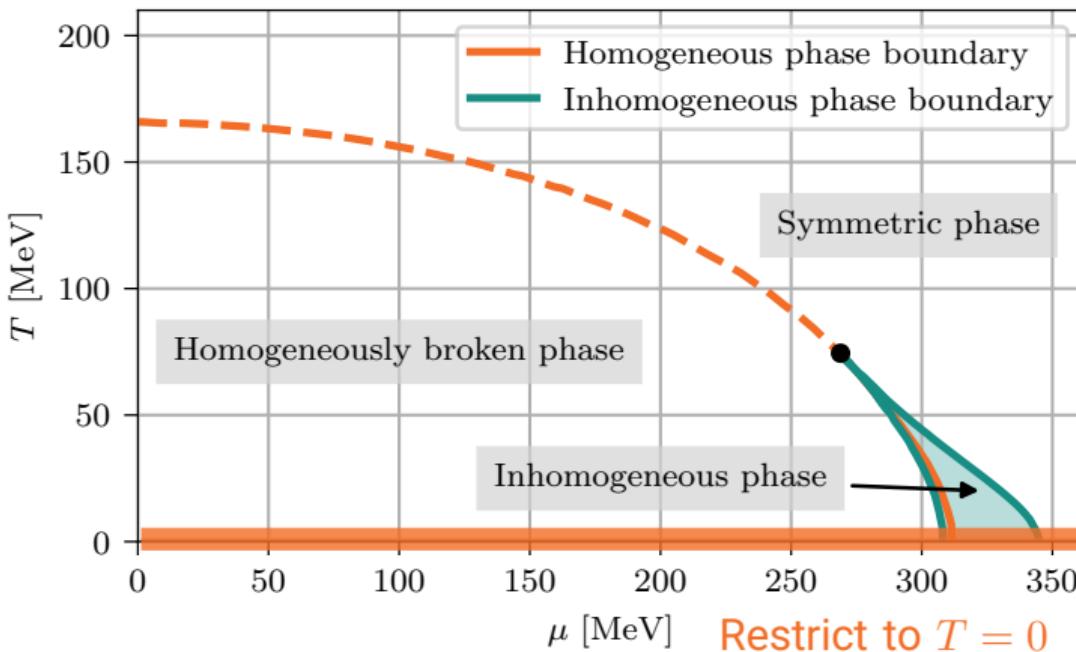
- 3 parameters: the coupling  $G$ , the bare fermion mass  $m_0$  and the regulator  $\Lambda$
- Parameters are tuned such that certain observable assume physically motivated values [S. P. Klevansky, *Rev. Mod. Phys.*, **64**, 649–708 (1992)]
  - Pion mass  $m_\pi = 0$  (chiral limit)  $\Rightarrow m_0 = 0$
  - Pion decay constant  $f_\pi = 88$  MeV
  - Constituent quark mass  $M_0$  in the range of 200 MeV – 500 MeV  
 $\Rightarrow$  small  $M_0$  corresponds to large regulators  $\Lambda$

# Phase diagram of the 3+1-d mean-field NJL model



[ D. Nickel, *Phys. Rev. D.*, **80**, 074025 (2009), arXiv: 0906.5295 ]

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# How to detect an inhomogeneous phase?

- How to detect an inhomogeneous phase in the  $3 + 1$ -dimensional NJL model?
- Two choices:
  - Use explicit **ansatz** for the chiral condensate / minimize on the lattice  
⇒ Difficult, sometimes impossible and often numerically expensive
  - Analyze the **stability** of the homogeneous minimum under **inhomogeneous perturbations**  
⇒ Flexible and cheap ⇒ better suited for our investigation

# Regularization for an investigation of IPs

- Possible regularization scheme choices
  - Most common: **Pauli-Villars**  
e.g. [ D. Nickel, *Phys. Rev. D.*, **80**, 074025 (2009), arXiv: 0906.5295 ] [ M. Buballa et al., *The Eur. Phys. J. Special Top.*, **229**, 3371–3385 (2020), arXiv: 2006.02133 ]
  - **Momentum cutoff** (*might* have some conceptual problems)
  - Successful applications of energy cutoffs and dim. regularization when using ansätze  
[ P. Adhikari, J. O. Andersen, *Phys. Rev. D.*, **95**, 036009 (2017), arXiv: 1608.01097 ] [ D. Ebert et al., *Phys. Rev. D.*, **84**, 025004 (2011), arXiv: 1102.4079 ]
- How do **lattice regularizations** fit in?

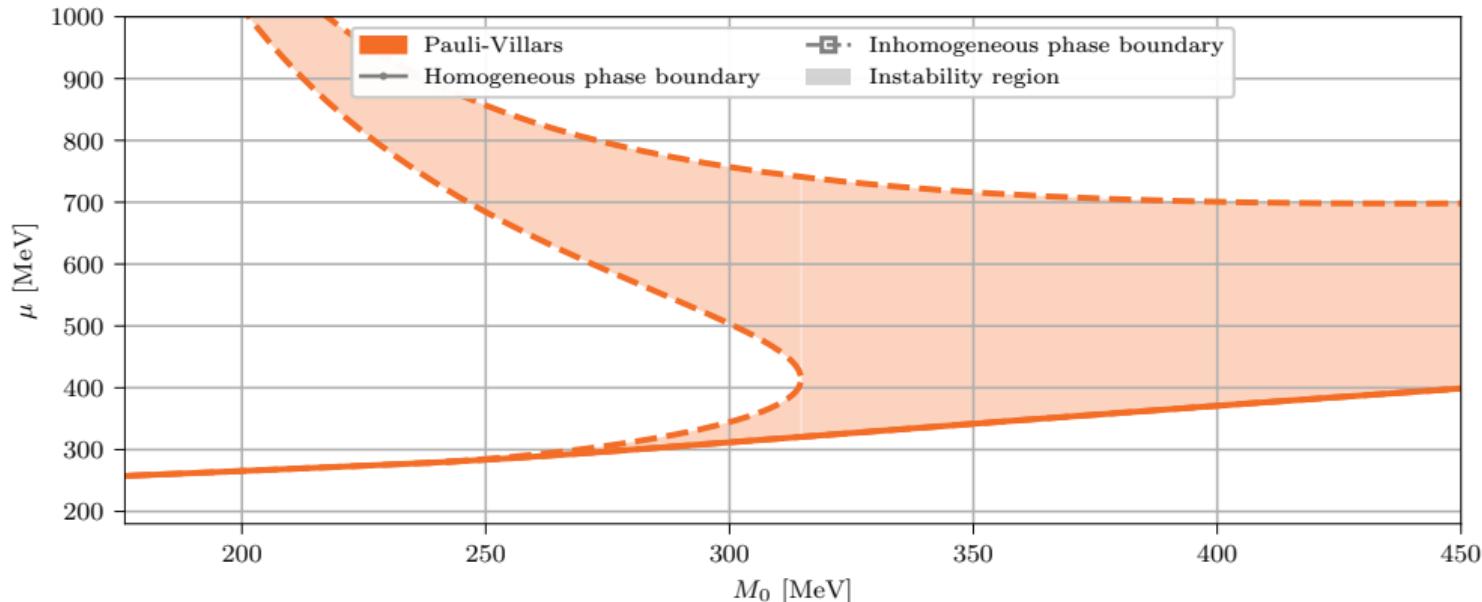
# Lattice regularizations

Investigate three different lattice regularizations

- SLAC fermions
  - Linear dispersion relation, thus conceptually very similar to a sharp cutoff in the continuum
- Two variations of a hybrid discretization with naive fermions in the spatial direction and SLAC fermions in the temporal direction

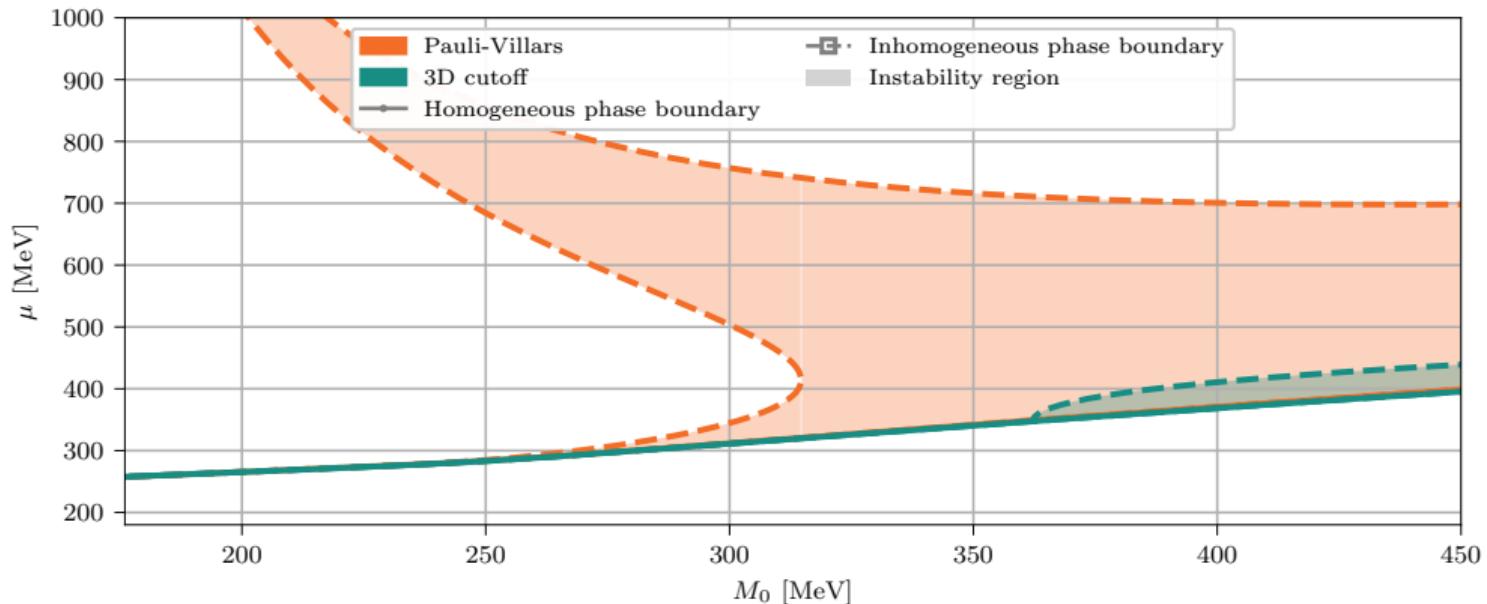
# Results

# Quark mass scan

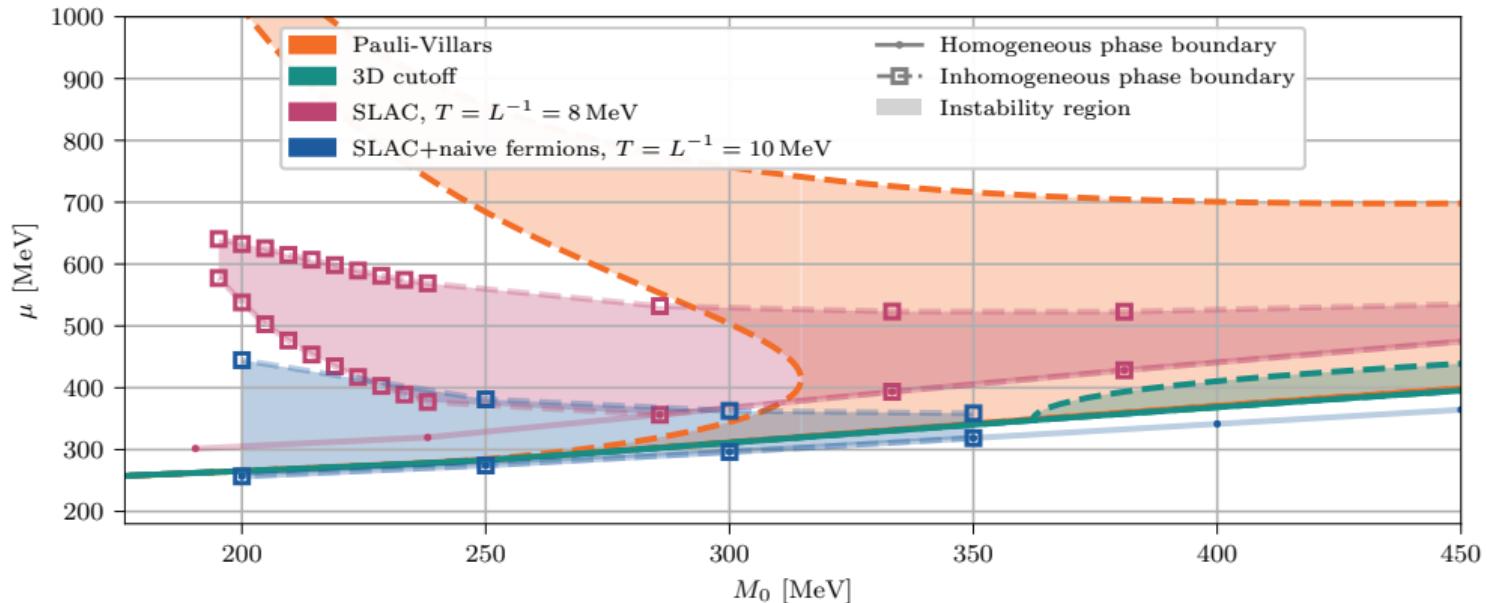


- Instability detection restricted to  $\mu \leq \Lambda$
- [ D. Nickel, *Phys. Rev. D.*, **80**, 074025 (2009), arXiv: 0906.5295 ]

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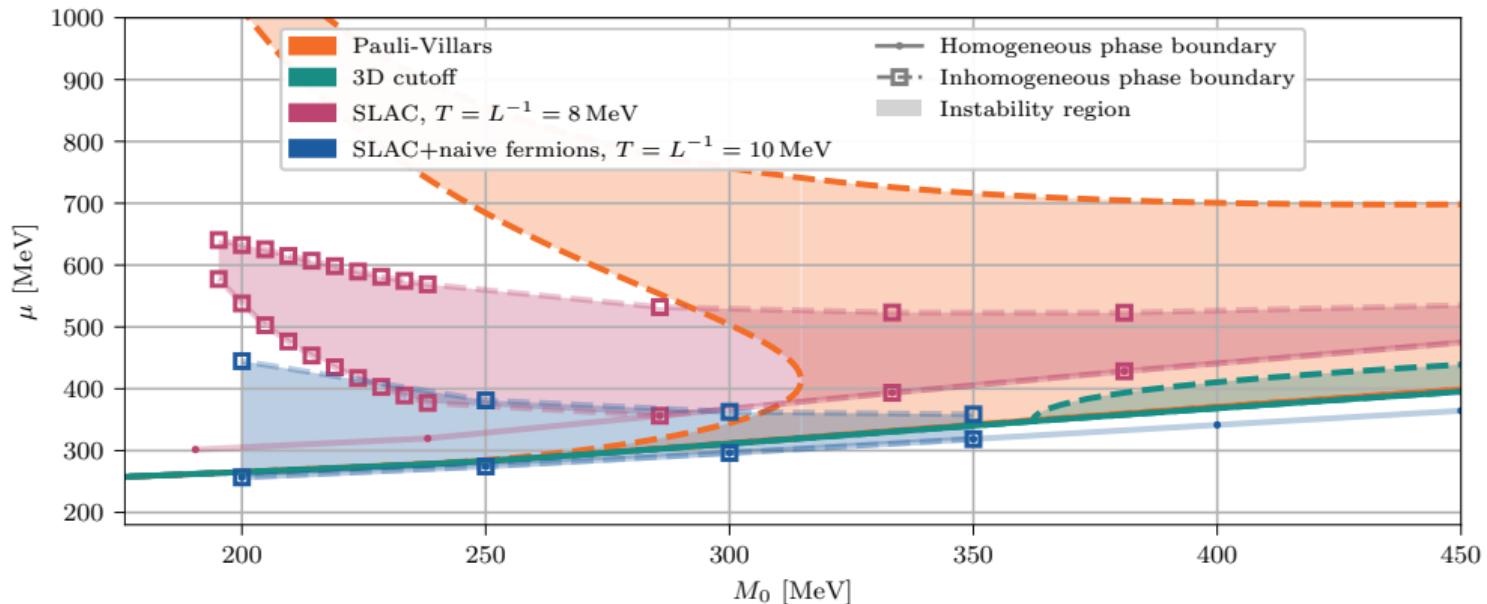


# Quark mass scan



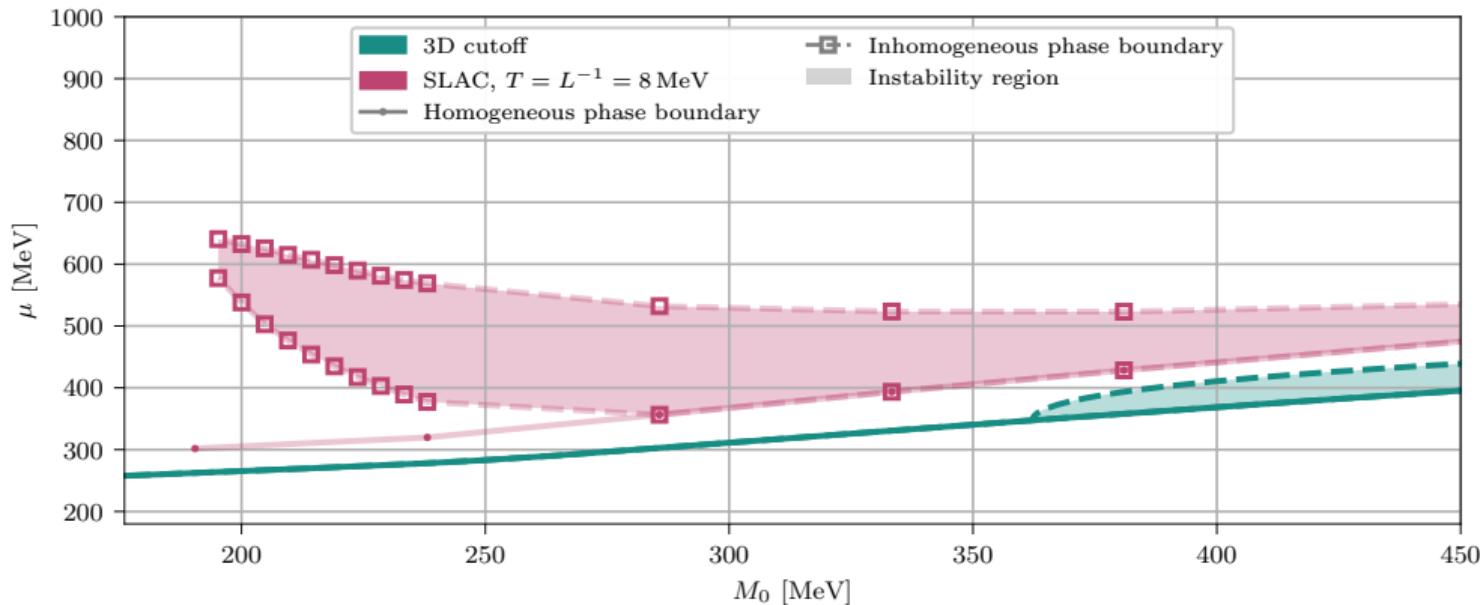
- The lattice results are calculated at  $T = L^{-1} = 8 \text{ MeV} - 10 \text{ MeV}$
- One of the naive discretizations **does not exhibit an instability**

# Quark mass scan



- No single point where all regularizations show an instability

# SLAC vs. 3D momentum cutoff



- Why does SLAC show such a different behavior than the momentum cutoff?

# SLAC vs. 3D momentum cutoff

- Differences between the regularizations:
  - I The lattice results are at **finite temperature and volume**
  - II The SLAC fermions **regulate the temporal direction** – the momentum cutoff does not
  - III The cutoff region of SLAC fermions is **cubic** and that of the 3D momentum cutoff is **spherical**.

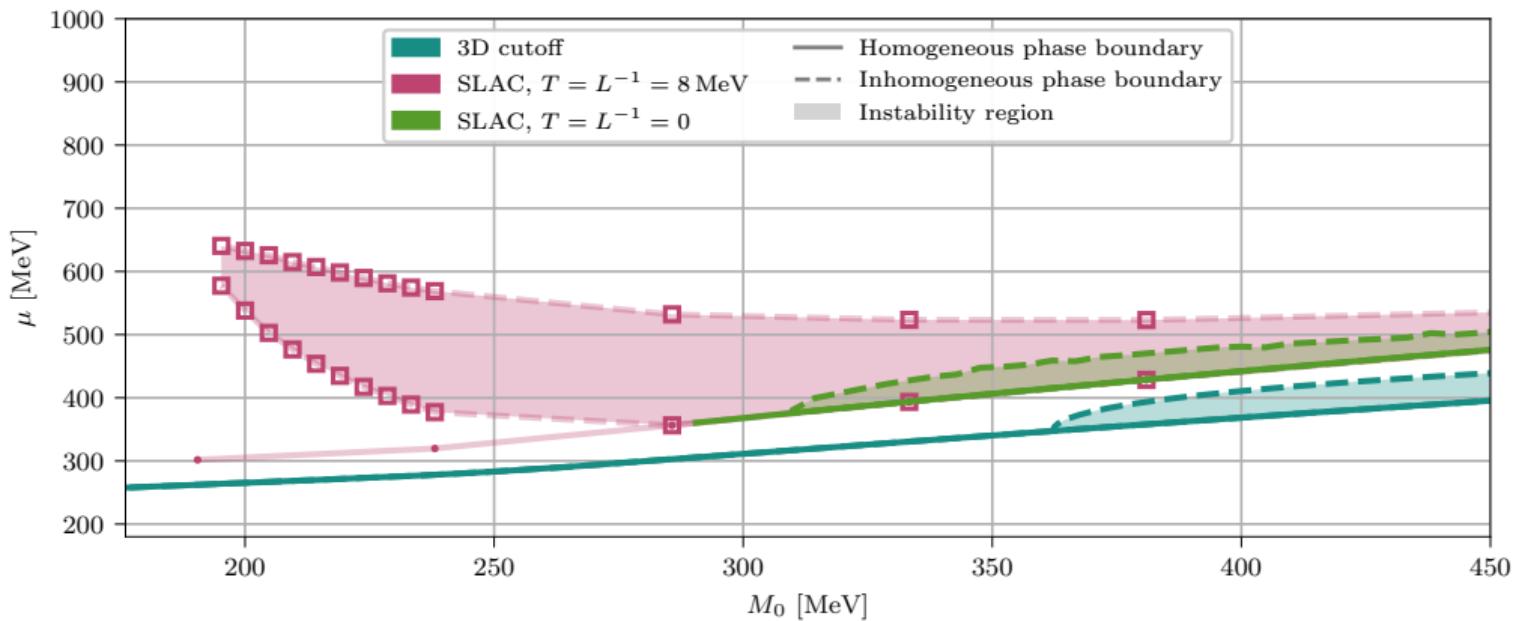
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- only need to compute a 1-loop diagram via **momentum sums** in stability analysis
- continue sums to integrals  
⇒ **exact infinite volume and  $T = 0$  on the lattice**

# SLAC vs. 3D momentum cutoff



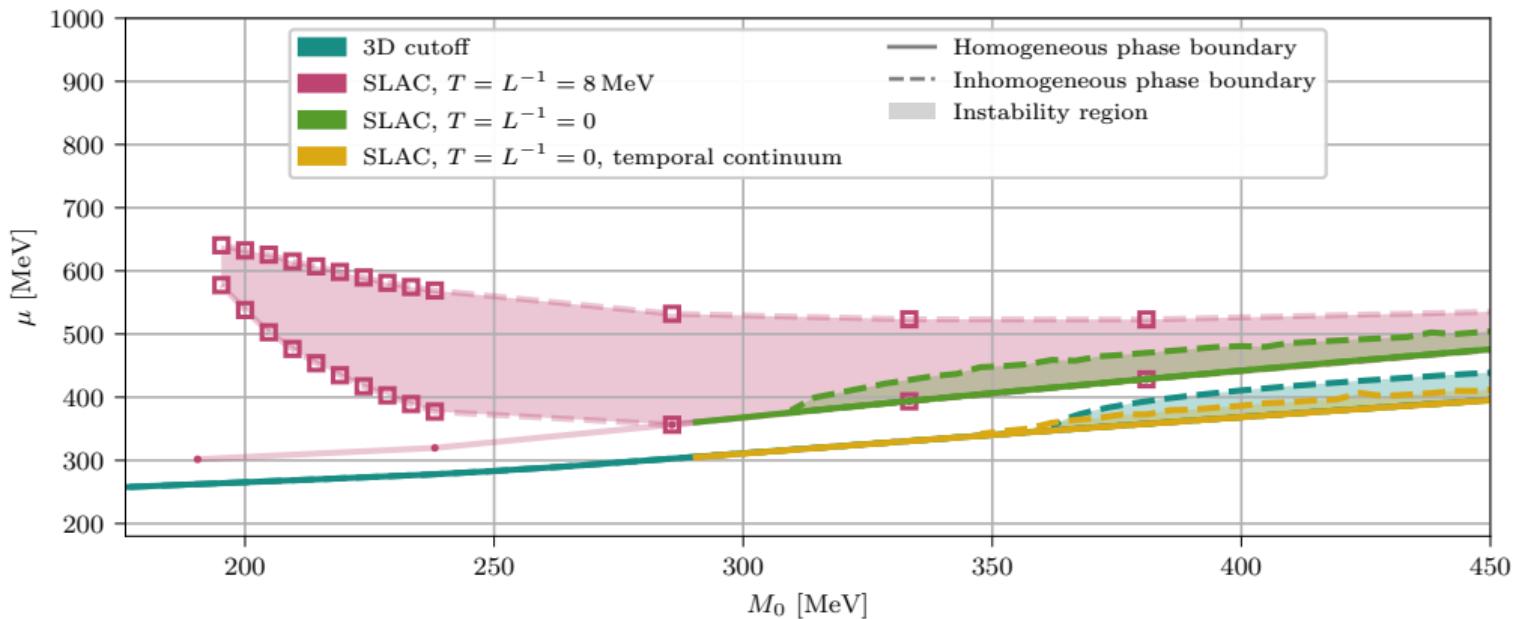
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  - I The lattice results are at finite temperature and volume
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  - III The cutoff region of SLAC fermions is cubic and that of the 3D momentum cutoff is spherical
- introduce anisotropic lattice
- perform **continuum limit** in **temporal** direction

# SLAC vs. 3D momentum cutoff

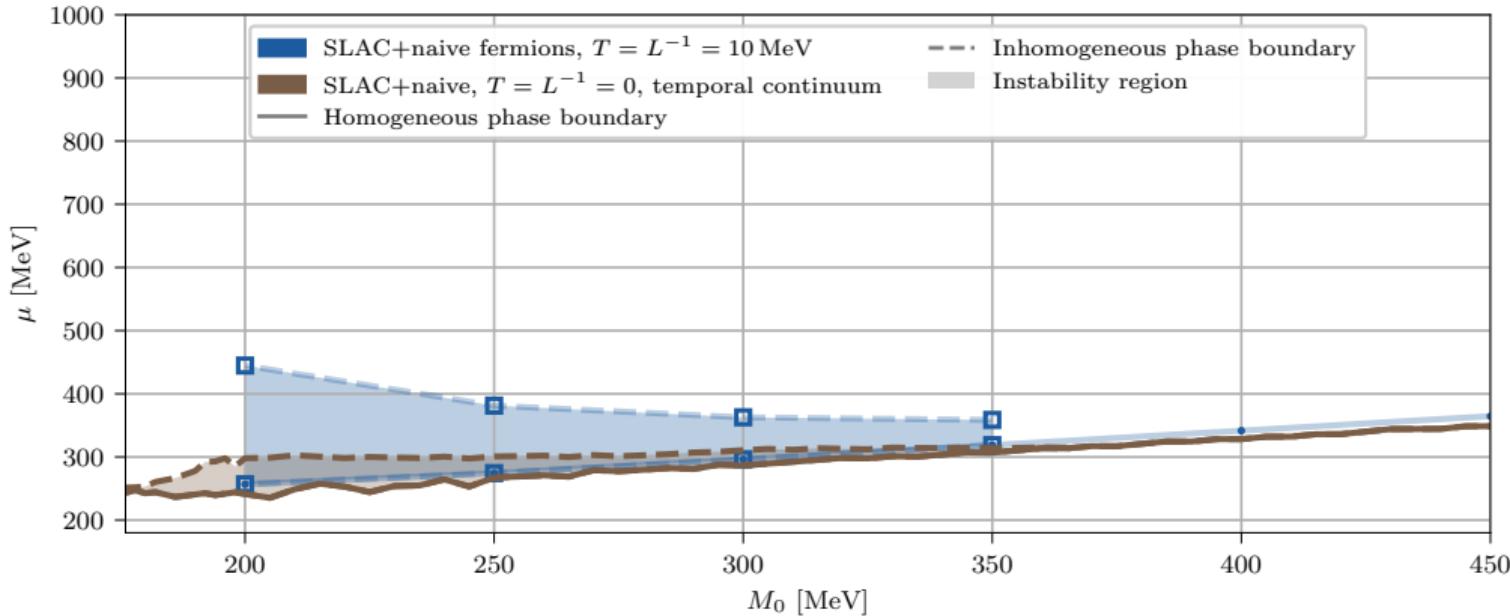


Remaining differences are due to the differing cutoff region shapes

# Naive fermions at infinite volume and temporal continuum

What about the naive fermions?

# Naive fermions at infinite volume and temporal continuum



(Due to time constraints some integrals needed for the results are evaluated stochastically and thus the phase boundaries are not smooth)

# Conclusions and Outlook

## Conclusions:

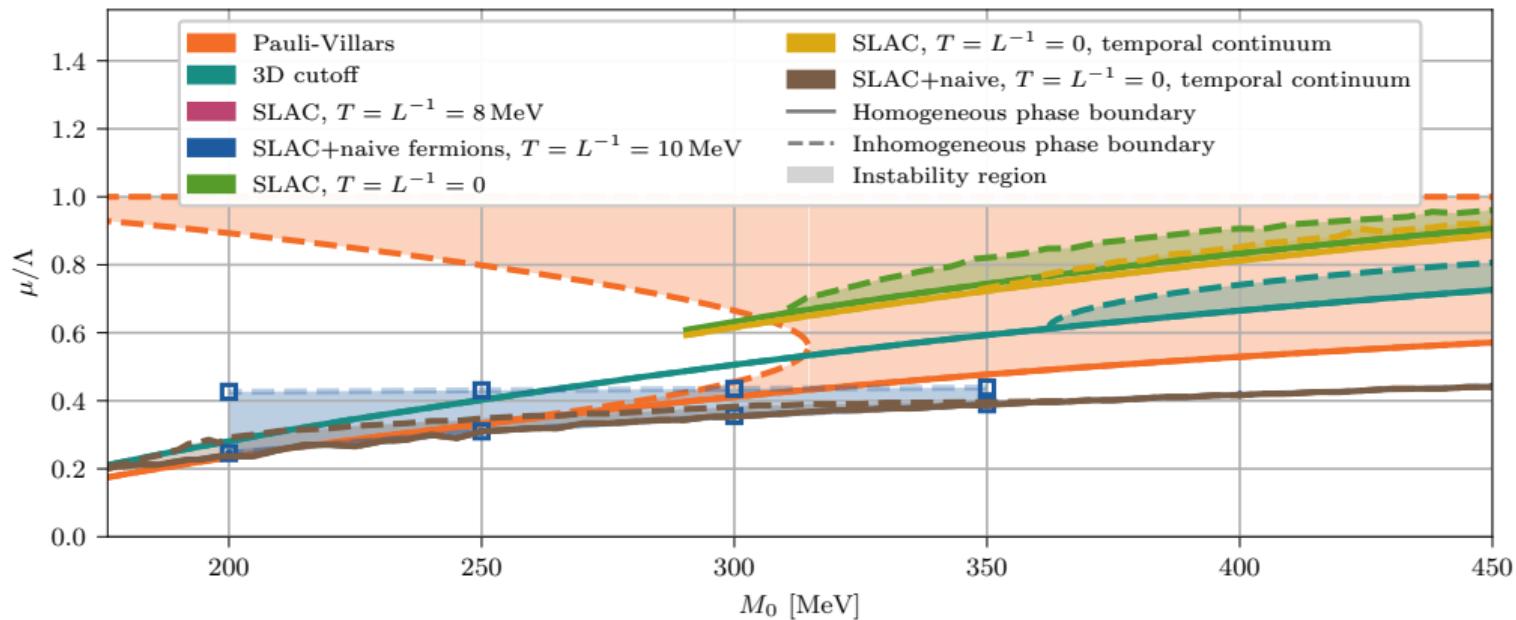
- No single point in  $\mu - M_0$  plane where all regularizations show an instability
- A lattice investigation of inhomogeneous phases in the 3 + 1-dimensional NJL model is
  - at best not straightforward and expensive
  - at worst not sensible due to larger problems
- Some problems not discussed,
  - Scales for “interesting” parameter regions are very/too close to the cutoff
  - baryon density saturation
  - artifactual minimizing configurations

## Outlook:

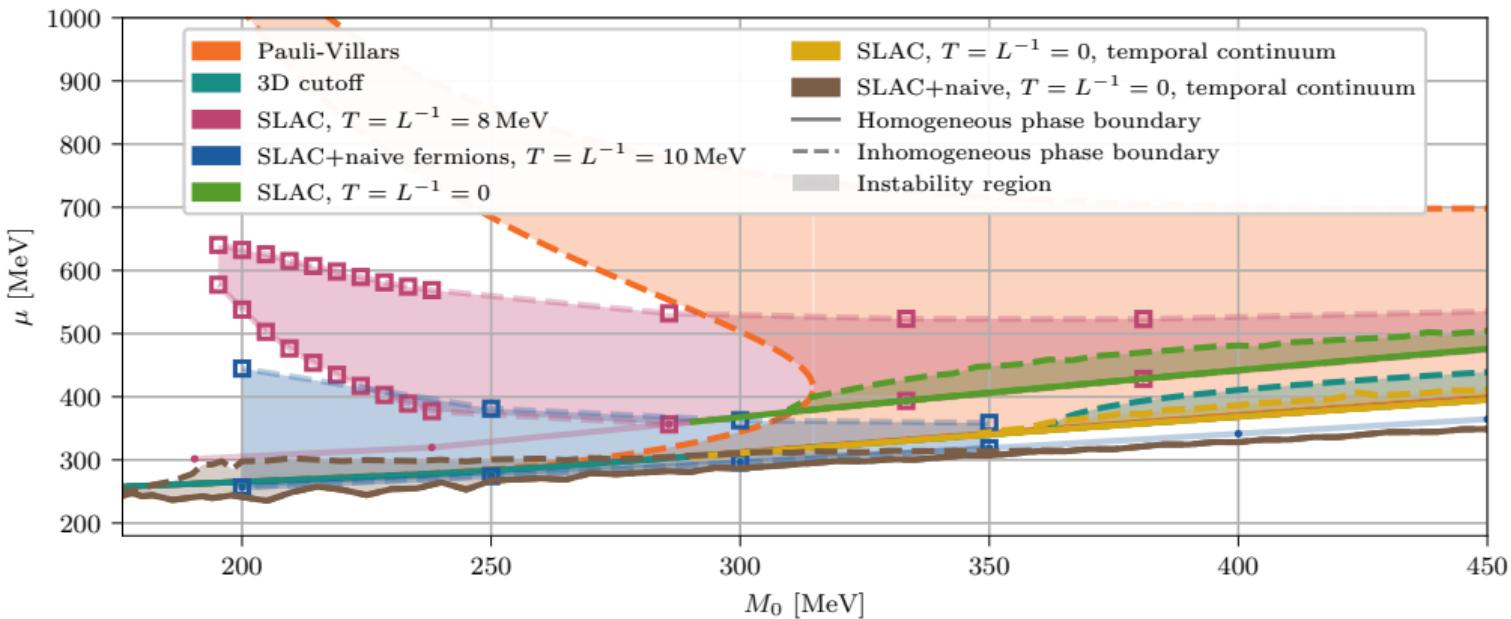
- Further understand cutoff regularizations in the context of IPs
- Investigate the Quark-Meson model

# Appendix

# Quark mass scan - Cutoff units



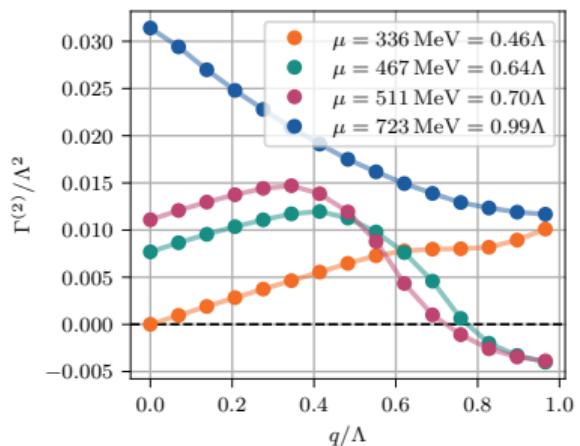
# Quark mass scan - Cutoff units



# SLAC in infinite volume vs. 3D momentum cutoff

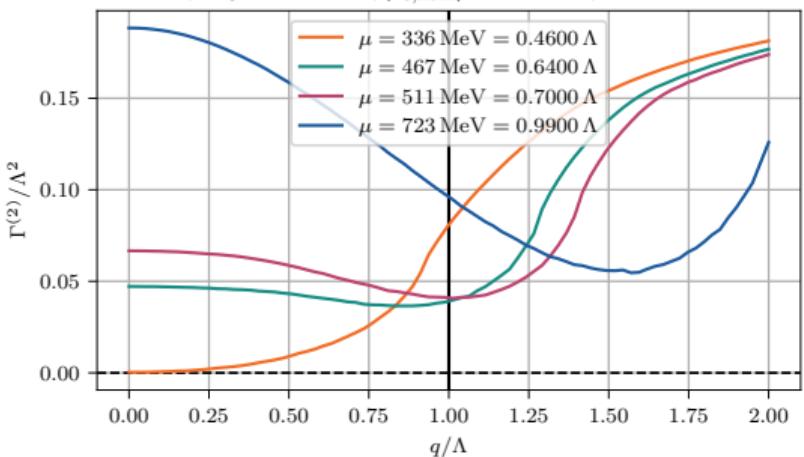
## SLAC in finite volume

SLAC,  $T \approx 8$  MeV,  $M_0 = 250$  MeV,  $m_\pi = 0$  MeV



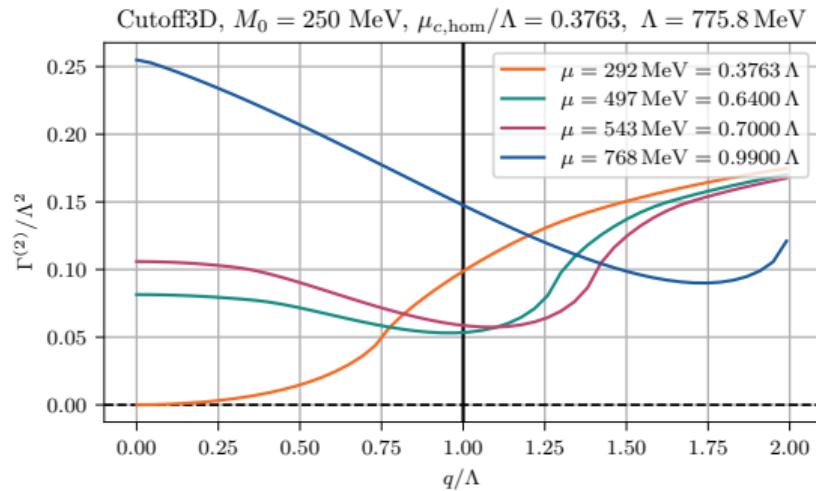
## SLAC in infinite volume

SLAC,  $M_0 = 250$  MeV,  $\mu_{c,hom}/\Lambda = 0.4619$ ,  $\Lambda = 730.3$  MeV

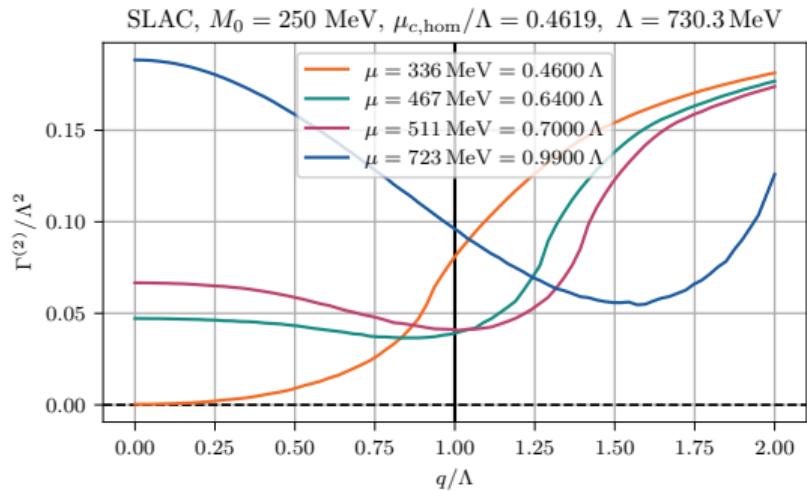


# SLAC in infinite volume vs. 3D momentum cutoff

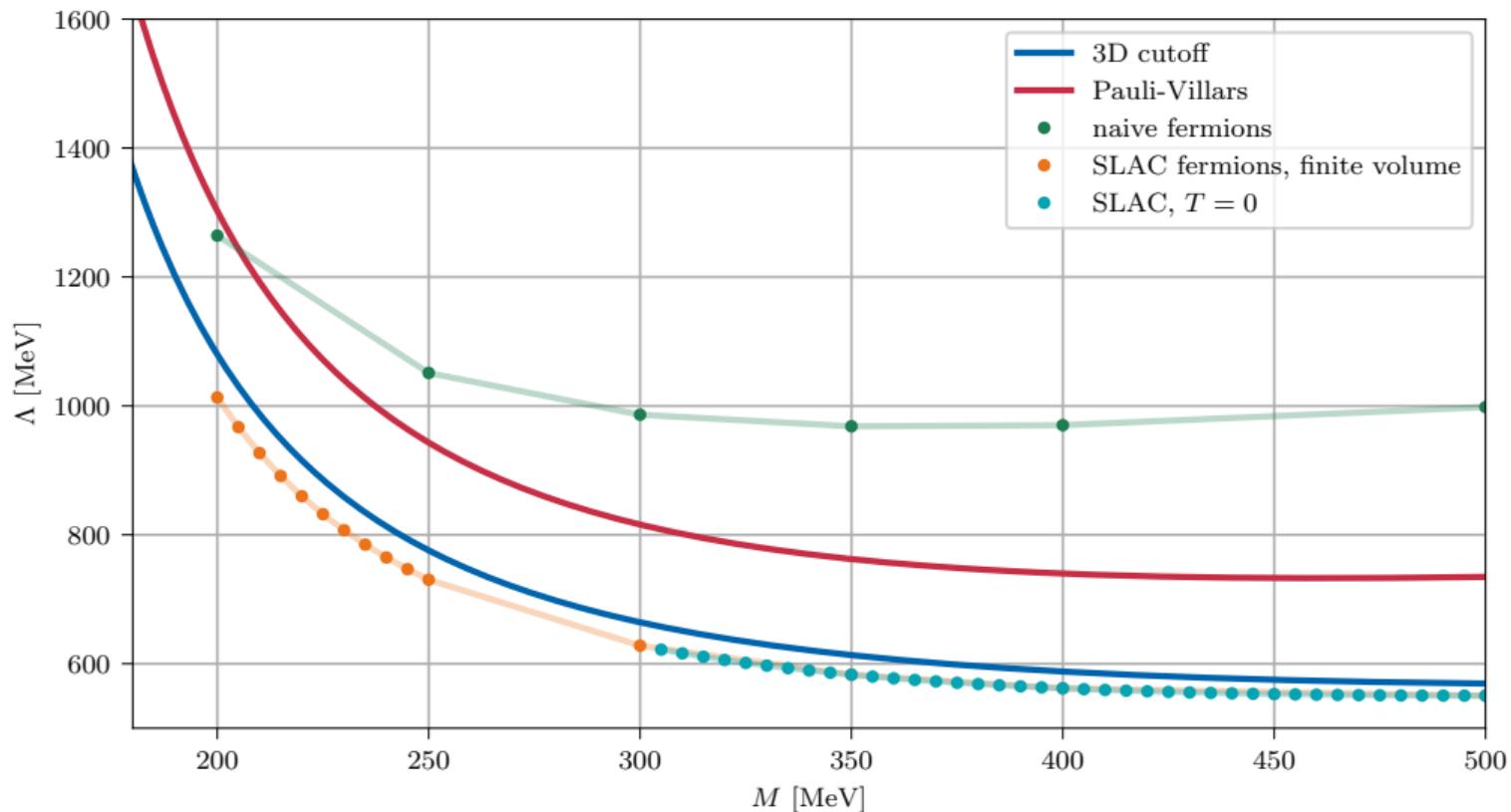
## 3D Cutoff



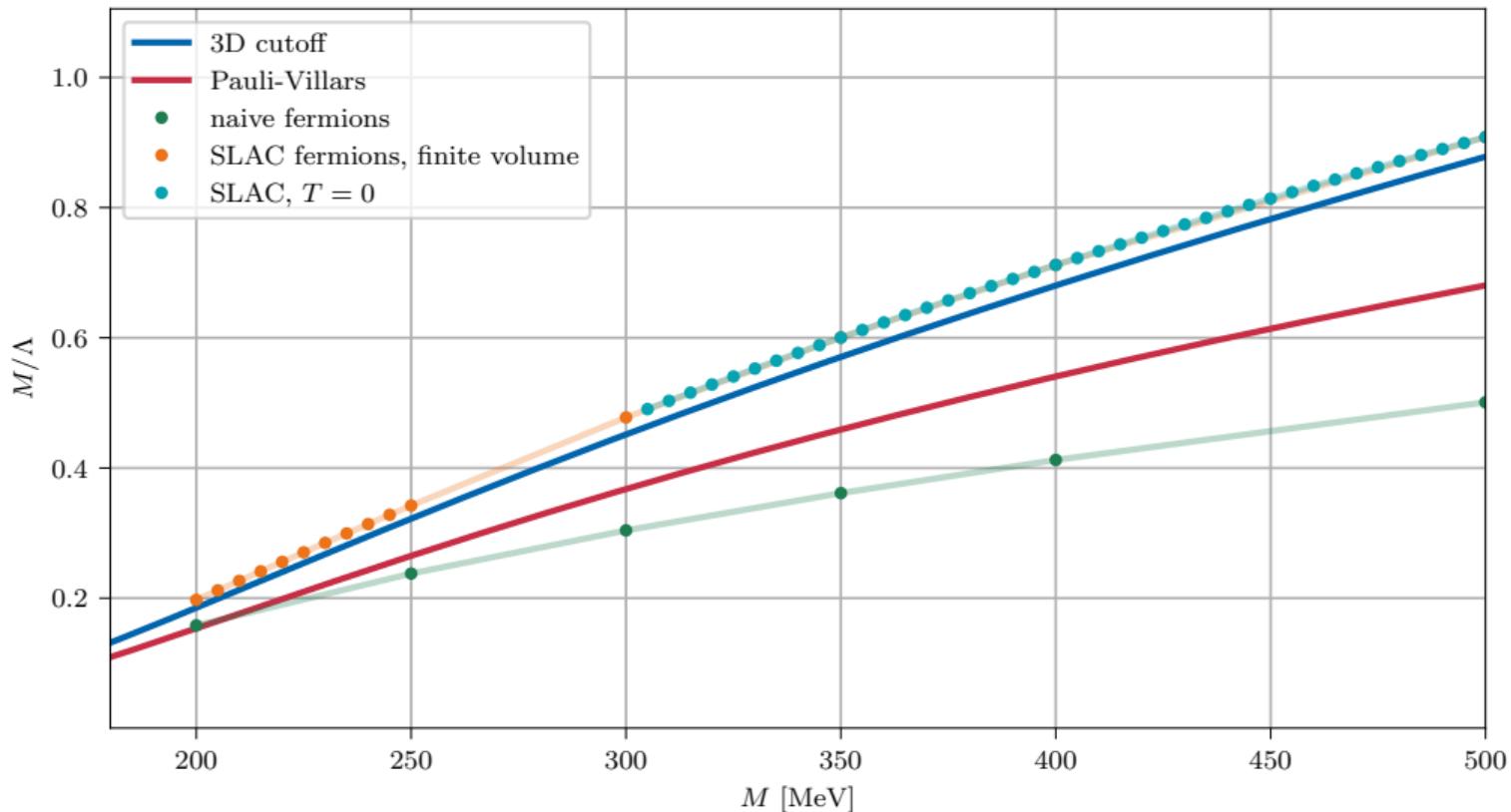
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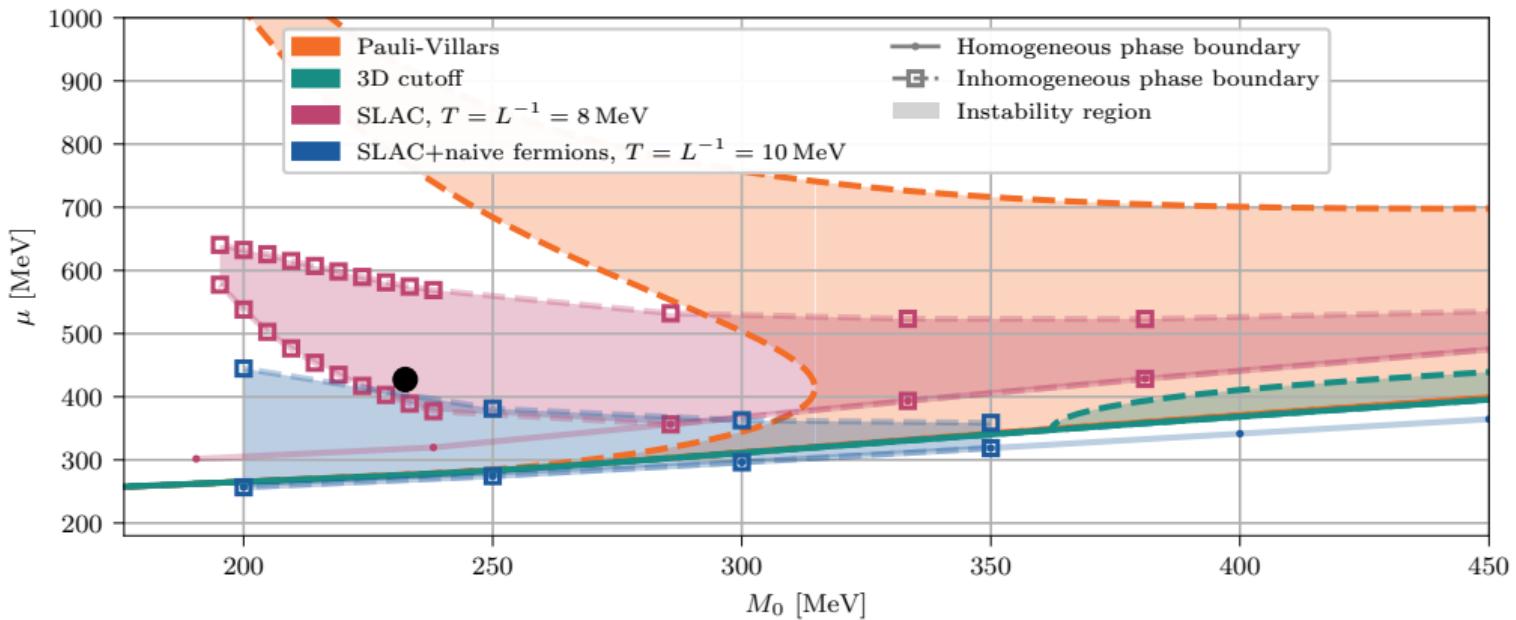
# Quark mass vs Cutoff



# Quark mass vs Quark mass in Cutoff

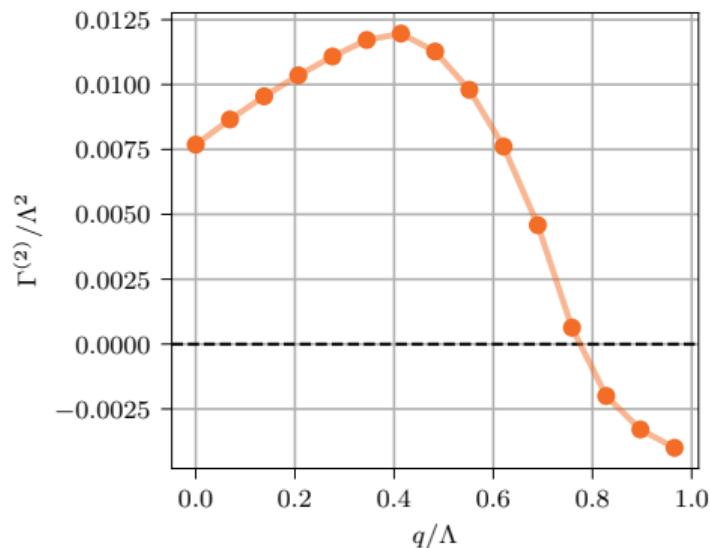


# Inhomogeneous field configurations

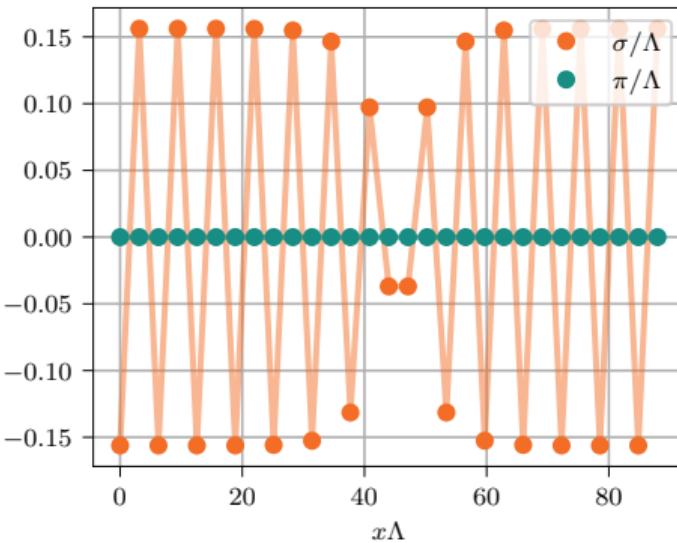


# Inhomogeneous field configurations

SLAC,  $T \approx 8$  MeV,  $\mu = 464.93$  MeV,  
 $M_0 = 250$  MeV,  $f_\pi = 92.4$  MeV,  $m_\pi = 0$  MeV



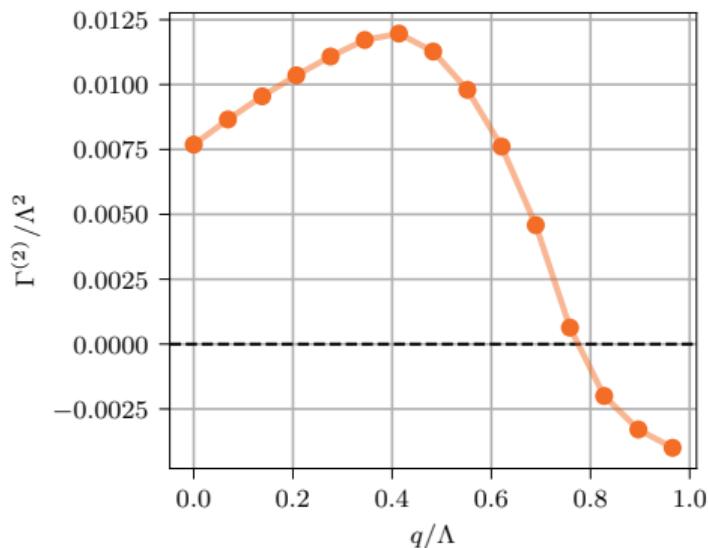
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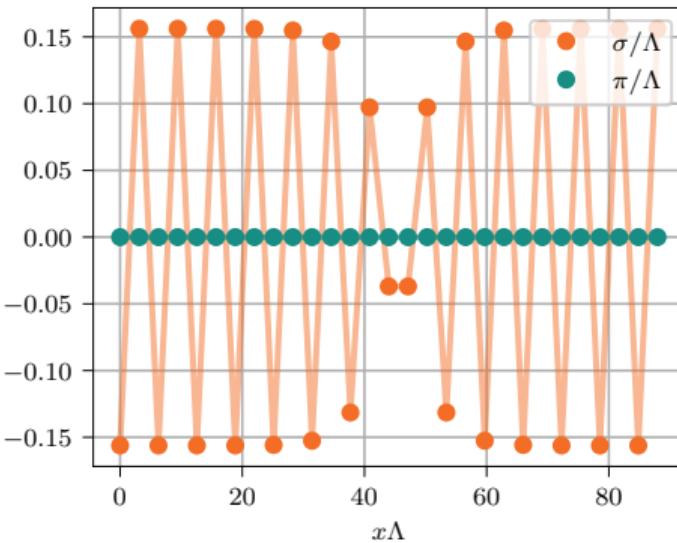
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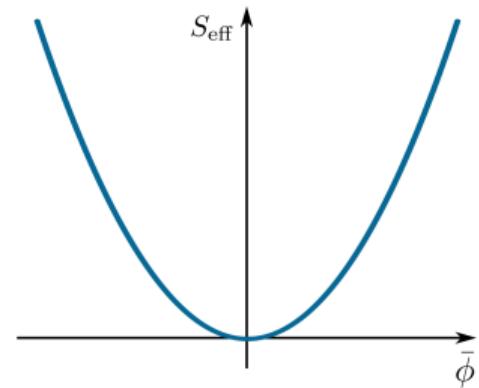
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# Stability analysis

- Homogeneous fields

$$\phi(x) = \bar{\phi}$$

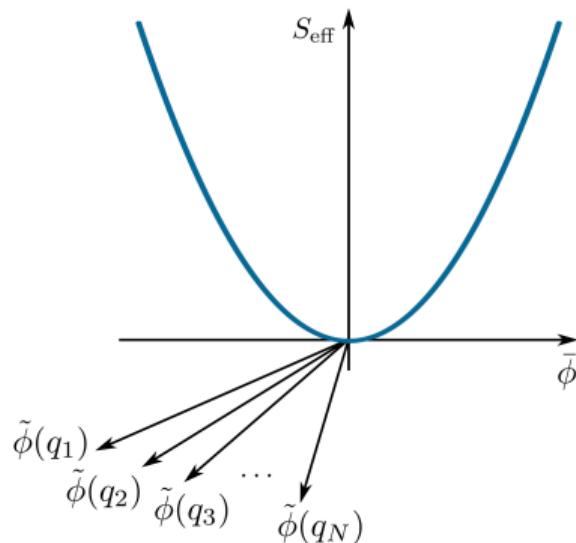
- Minimum is easy to obtain.



# Stability analysis

- In general fields have full space dependence

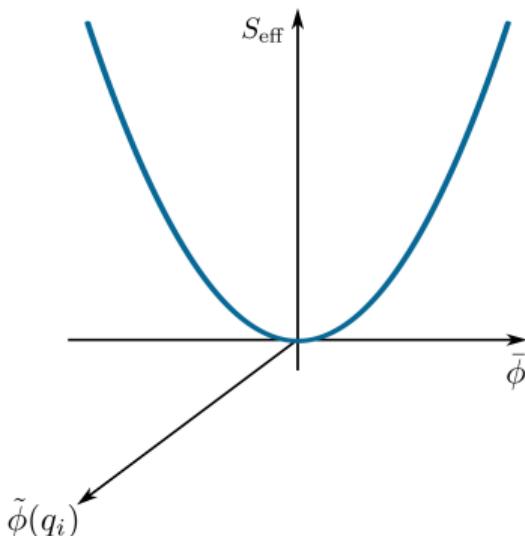
$$\begin{aligned}\phi(x) &= \bar{\phi} + \phi_s(x) \\ &= \bar{\phi} + \oint_j \tilde{\phi}_s(q_j) e^{ixq_j}\end{aligned}$$



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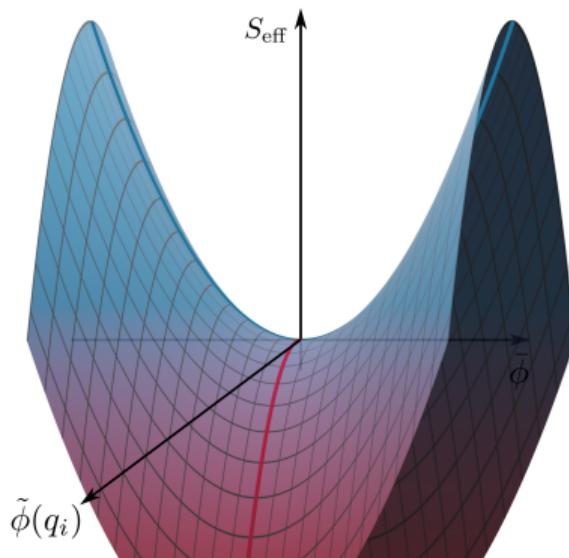


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- Former homogeneous minimum might only be **saddle point**
- Full dependence of  $S_{\text{eff}}$  on  $\phi(x)$  extremely difficult or impossible

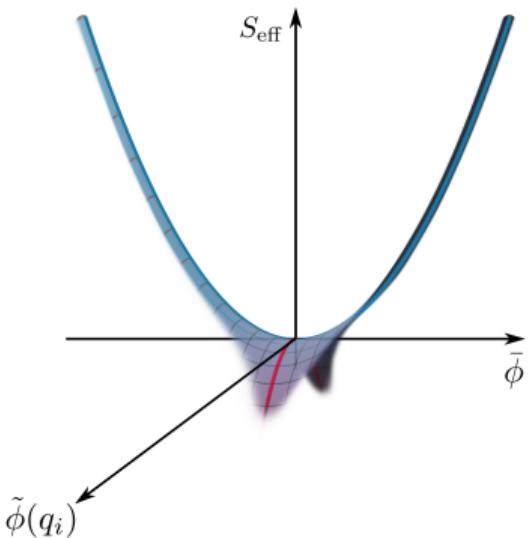


# Stability analysis

- Consider only inhomogeneous perturbations

$$\begin{aligned}\phi(x) &= \bar{\phi} + \delta\phi_s(x) \\ &= \bar{\phi} + \sum_j \delta\tilde{\phi}_s(q_j) e^{ixq_j}\end{aligned}$$

- investigate curvature at homogeneous minimum



- Curvature in direction  $\delta\tilde{\phi}(\mathbf{q})$  is given by the **bosonic two-point function**  $\Gamma_\phi^{(2)}(\mathbf{q})$
- Simple quantity in the mean-field approximation

$$\Gamma_\phi^{(2)}(\mathbf{q}) = \frac{1}{2G} + \text{Diagram}$$

The diagram illustrates the bosonic two-point function  $\Gamma_\phi^{(2)}(\mathbf{q})$ . It consists of a horizontal dotted line with arrows pointing right, labeled  $\phi$  below it. At each end of the line is a black dot. From each dot, a curved arrow points clockwise around a circle. The top arc is labeled  $\mathbf{q}$  above the line, and the bottom arc is labeled  $\mathbf{p} + \mathbf{q}$  below the line. The center of the circle is labeled  $\psi$ .