

Eta pole contributions to HLbL at the physical point

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Flavor Singlet Project for ETMC

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Project overview

Goal

Computing $\mathcal{F}_{P \rightarrow \gamma^* \gamma^*}$, $P = \pi_0, \eta, \eta'$ to determine the corresponding contributions to HLbL in the muon $g - 2$.

Using

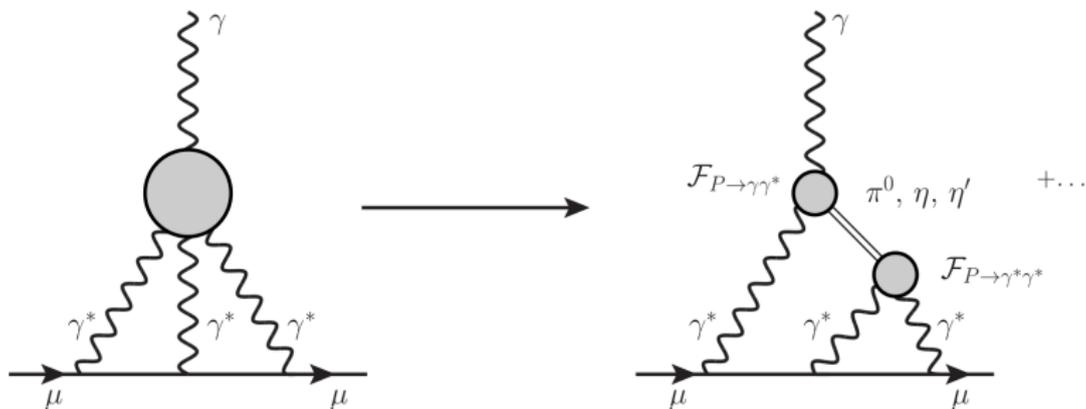
- Twisted mass clover improved lattice QCD at maximal twist
- Four dynamical flavors ($N_f = 2 + 1 + 1$) [C. Alexandrou et al., [arXiv:2104.06747](https://arxiv.org/abs/2104.06747)]

- Analysis planned on

ensemble	$L^3 \cdot T / a^4$	m_π [MeV]	a [fm]	$a \cdot L_x$ [fm]	$m_\pi \cdot L_x$
cB072.64	$64^3 \cdot 128$	140.2	0.080	5.09	3.62
cC06.80	$80^3 \cdot 160$	136.7	0.068	5.46	3.78
cD054.96	$96^3 \cdot 192$	140.8	0.057	5.46	3.90

- Ensembles at the physical point (m_s also physical to within a few percent)

Hadronic light-by-light scattering



- Governed by rank-four hadronic vacuum polarization tensor.
- Numerically dominant role played by the π^0 pole, followed by η and η' poles.
- Nonperturbative information is encapsulated in the pole masses and the transition form factors $\mathcal{F}_{P \gamma^* \gamma^*}$.

Following [M. Knecht, A. Nyffeler, Phys. Rev. D65, 073034 (2002)], the transition form factors are defined via the matrix element

$$\begin{aligned} M_{\mu\nu}(p, q_1) &= i \int d^4x e^{iq_1x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | P(p) \rangle \\ &= \varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2). \end{aligned}$$

Defining

$$\tilde{A}_{\mu\nu}(\tau) = \langle 0 | T \{ j_\mu(\vec{q}_1, \tau) j_\nu(\vec{p} - \vec{q}_1, 0) \} | P(p) \rangle,$$

the matrix element is recovered by integration:

$$M_{\mu\nu}^E = \int_{-\infty}^{\infty} d\tau e^{\omega_1\tau} \tilde{A}_{\mu\nu}(\tau), \quad i^{n_0} M_{\mu\nu}^E(p, q_1) = M_{\mu\nu}(p, q_1).$$

On the lattice, starting from the amplitude

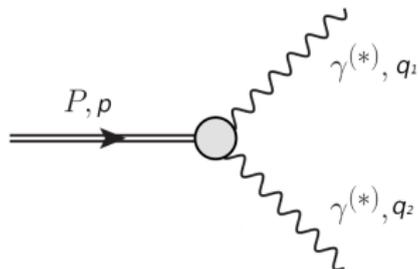
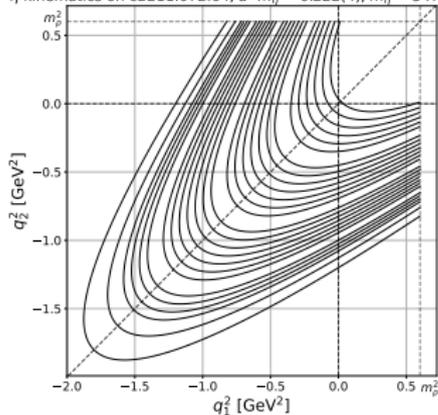
$$C_{\mu\nu}(\tau, t_P) = a^6 \sum_{\vec{x}, \vec{z}} \langle j_\mu(\vec{x}, t_i) j_\nu(\vec{0}, t_f) P^\dagger(\vec{z}, t_0) e^{i\vec{p}\vec{z}} e^{-i\vec{x}\vec{q}_1} \rangle,$$

one constructs

$$\tilde{A}_{\mu\nu}(\tau) = \frac{2E_P}{Z_P} \lim_{t_P \rightarrow \infty} e^{E_P(t_f - t_0)} C_{\mu\nu}(\tau, t_P).$$

Kinematics

η kinematics on cB211.072.64, $a \cdot m_\eta = 0.222(4)$, $m_\eta = 547(9)$ MeV



Pseudoscalar at rest, i.e. $\vec{p} = \vec{0}$

$$\Rightarrow q_1^2 = \omega_1^2 - \vec{q}_1^2$$

$$q_2^2 = (m_P - \omega_1)^2 - \vec{q}_1^2$$

Threshold for hadron production

$$\Rightarrow q_{1,2}^2 < m_V^2 = \min(m_\rho^2, 4m_\pi^2)$$

$$-\sqrt{m_V^2 + \vec{q}_1^2} + m_P < \omega_1 < \sqrt{m_V^2 + \vec{q}_1^2}$$

Define

$$\tilde{A}(\tau) = im_P^{-1} \varepsilon_{ijk} \frac{\vec{q}_1^i}{\vec{q}_1^2} \tilde{A}_{jk}(\tau)$$

Current operators and isospin combinations

- Also consider strange contributions in the electromagnetic current, i.e.

$$j_\mu(x) = \underbrace{\frac{2}{3}\bar{u}\gamma_\mu u(x) - \frac{1}{3}\bar{d}\gamma_\mu d(x)}_{=j'_\mu(x)} - \underbrace{\frac{1}{3}\bar{s}\gamma_\mu s(x)}_{=j_\mu^s} = \frac{1}{6}j_\mu^{l,(0,0)} + \frac{1}{2}j_\mu^{l,(1,0)} + j_\mu^s.$$

- Decomposition into definite isospin for the light contribution yields

$$C_{\mu\nu}^{l-current} = \frac{1}{4} \langle \eta j_\mu^{l,(0,0)} j_\nu^{l,(0,0)} \rangle + \frac{1}{36} \langle \eta j_\mu^{l,(1,0)} j_\nu^{l,(1,0)} \rangle$$

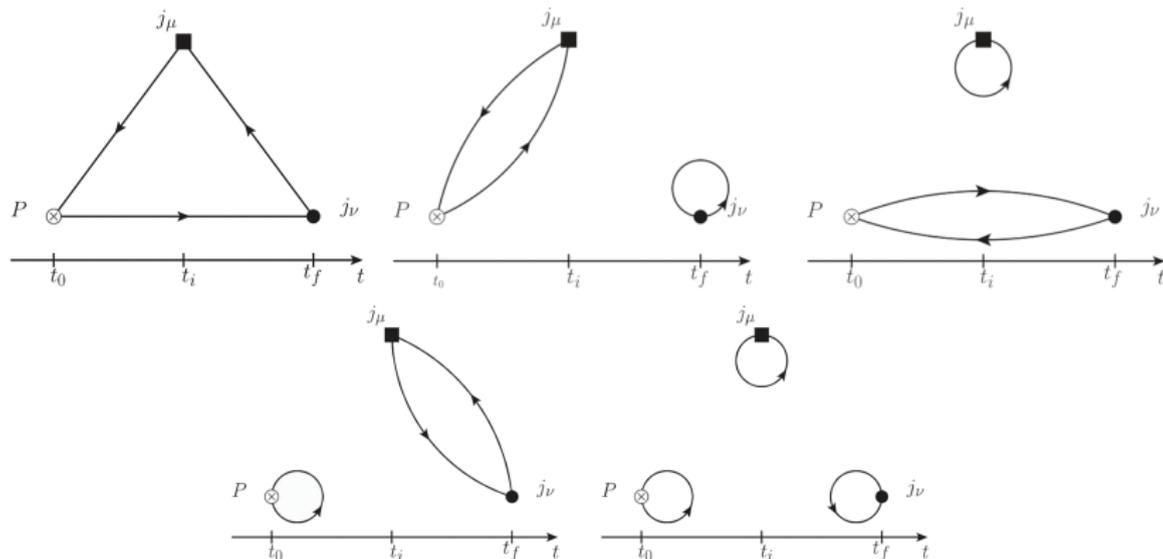
- Osterwalder-Seiler is used for introducing the strange quark. [R. Frezzotti, G. C. Rossi, JHEP 10 (2004) 070]
- Mixing not considered at the moment:

$$\eta \approx \eta_8 \propto \bar{u}u + \bar{d}d - 2\bar{s}s$$

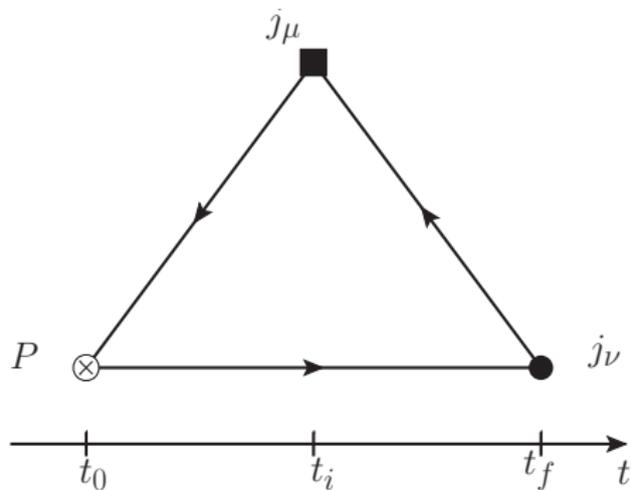
$$\eta' \approx \eta_1 \propto \bar{u}u + \bar{d}d + \bar{s}s$$

Considered diagrams

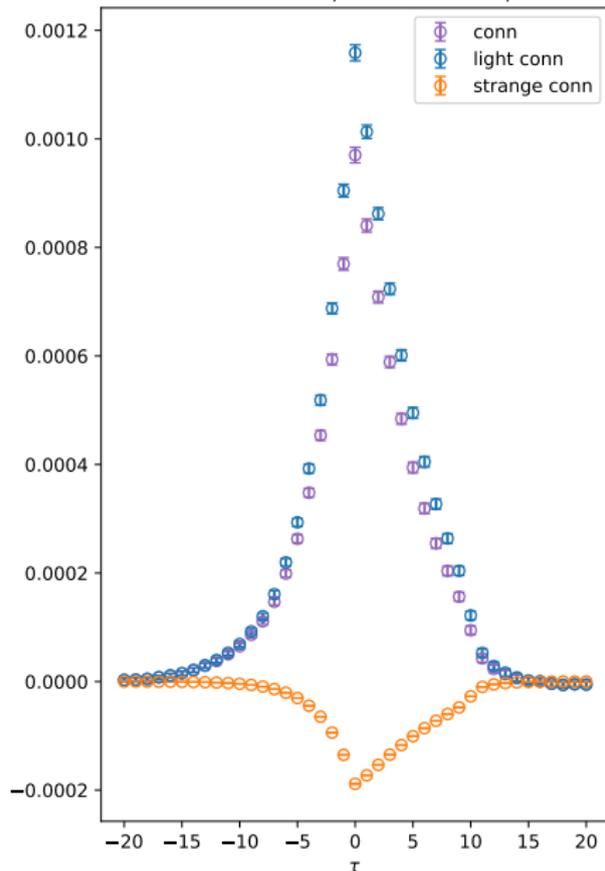
The amplitude $C_{\mu\nu}$ contains connected, vector current disconnected, pseudoscalar disconnected and doubly disconnected Wick contractions.



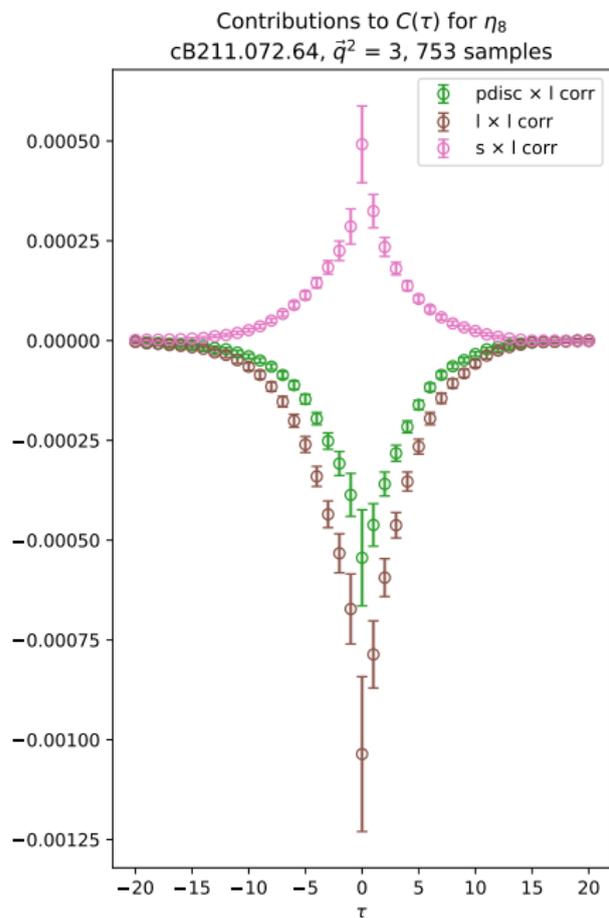
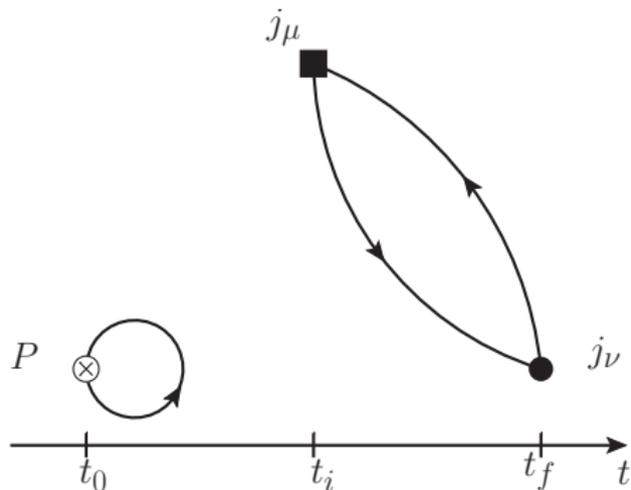
Considered diagrams



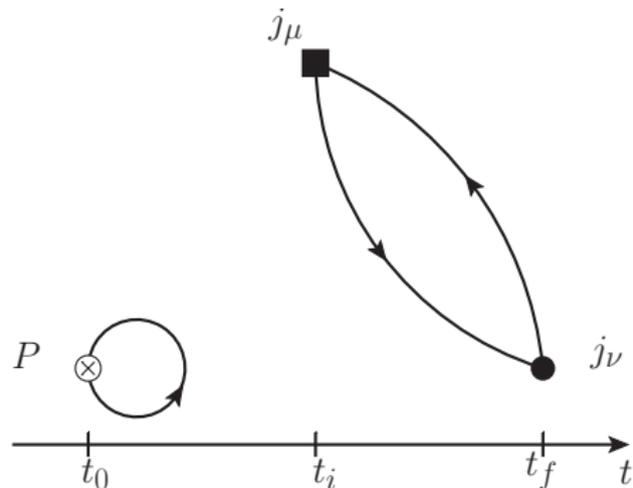
Contributions to $C(\tau)$ for η_8
cB211.072.64, $\bar{q}^2 = 3$, 753 samples



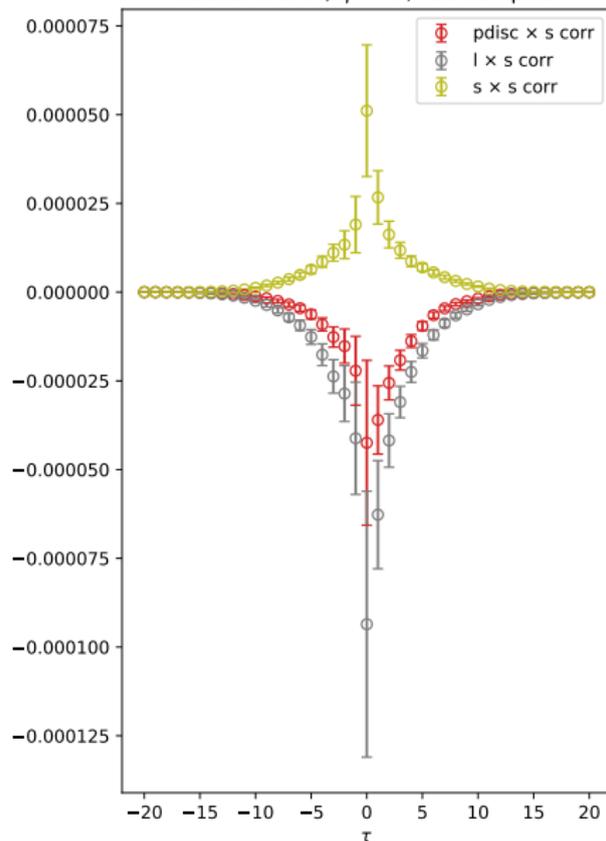
Considered diagrams



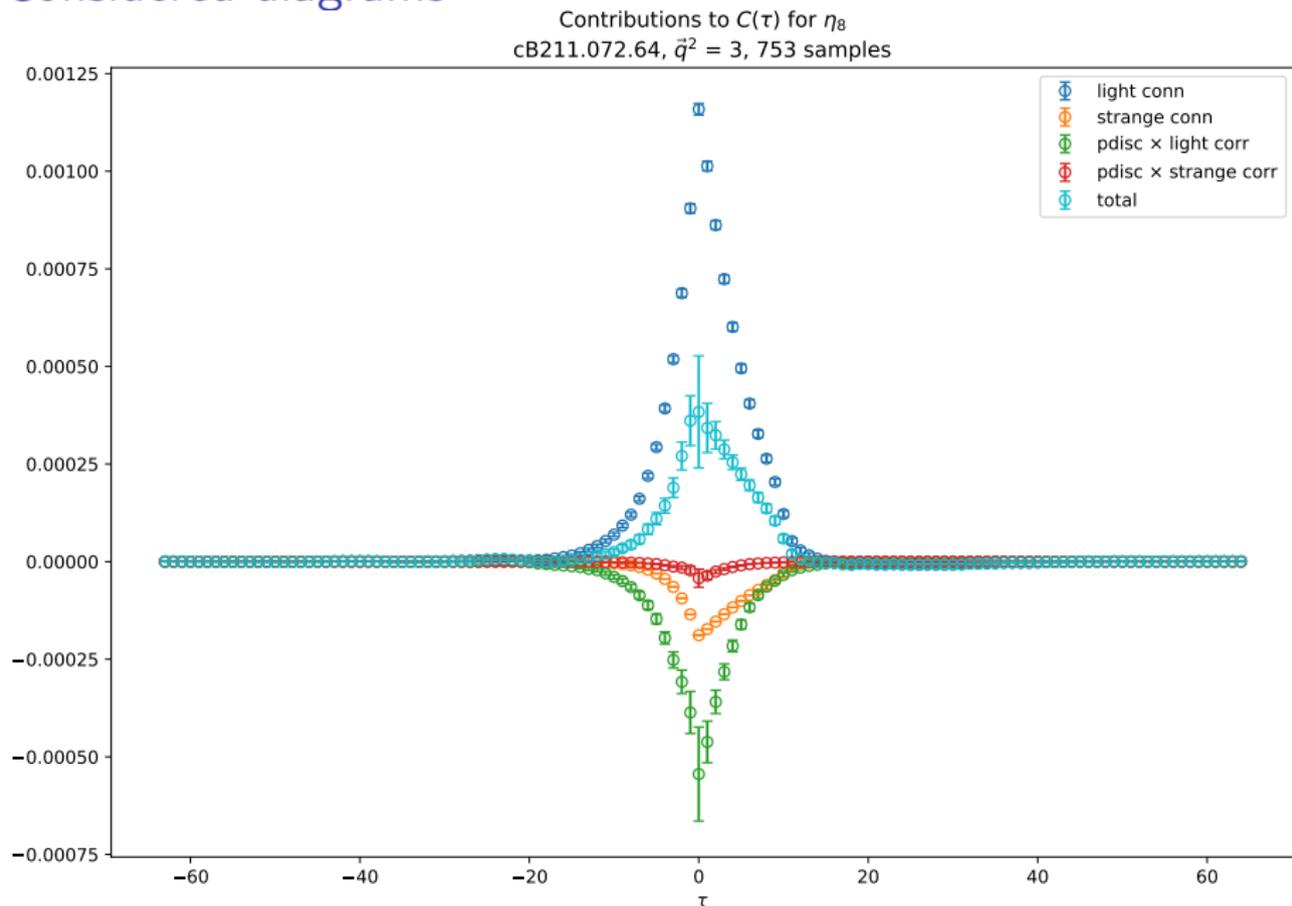
Considered diagrams



Contributions to $C(\tau)$ for η_8
cB211.072.64, $\bar{q}^2 = 3, 753$ samples



Considered diagrams



Tail fits

We need

$$\mathcal{F}(q_1^2, q_2^2) = \int_{-t_c}^{t_c} d\tau \tilde{A}^{(latt.)}(\tau) e^{\omega_1 \tau} + \int_{\pm t_c}^{\pm \infty} d\tau \tilde{A}^{(fit)}(\tau) e^{\omega_1 \tau}.$$

Following [Gerardin et al., Phys. Rev. D94, 074507 (2016) and refs. therein], consider

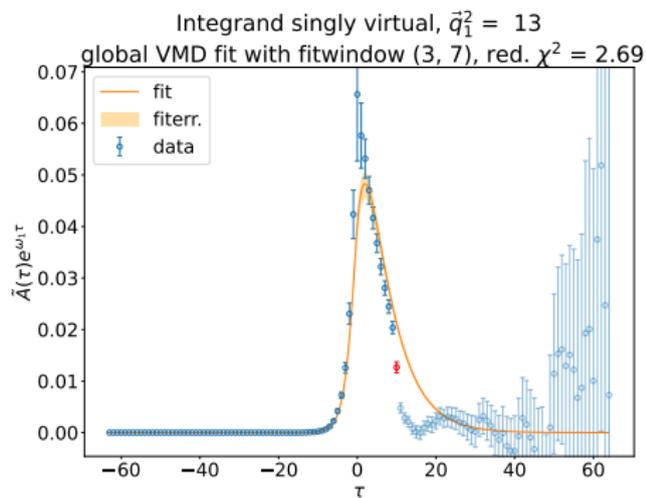
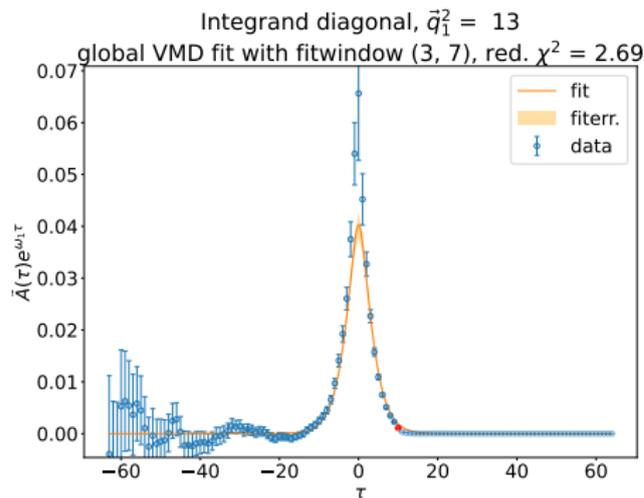
- Vector meson dominance (VMD) model:

$$\mathcal{F}_{P\gamma^*\gamma^*}^{VMD}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \rightarrow \tilde{A}^{(fit, VMD)}(\tau).$$

- Lowest meson dominance (LMD) model:

$$\mathcal{F}_{P\gamma^*\gamma^*}^{LMD}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \rightarrow \tilde{A}^{(fit, LMD)}(\tau).$$

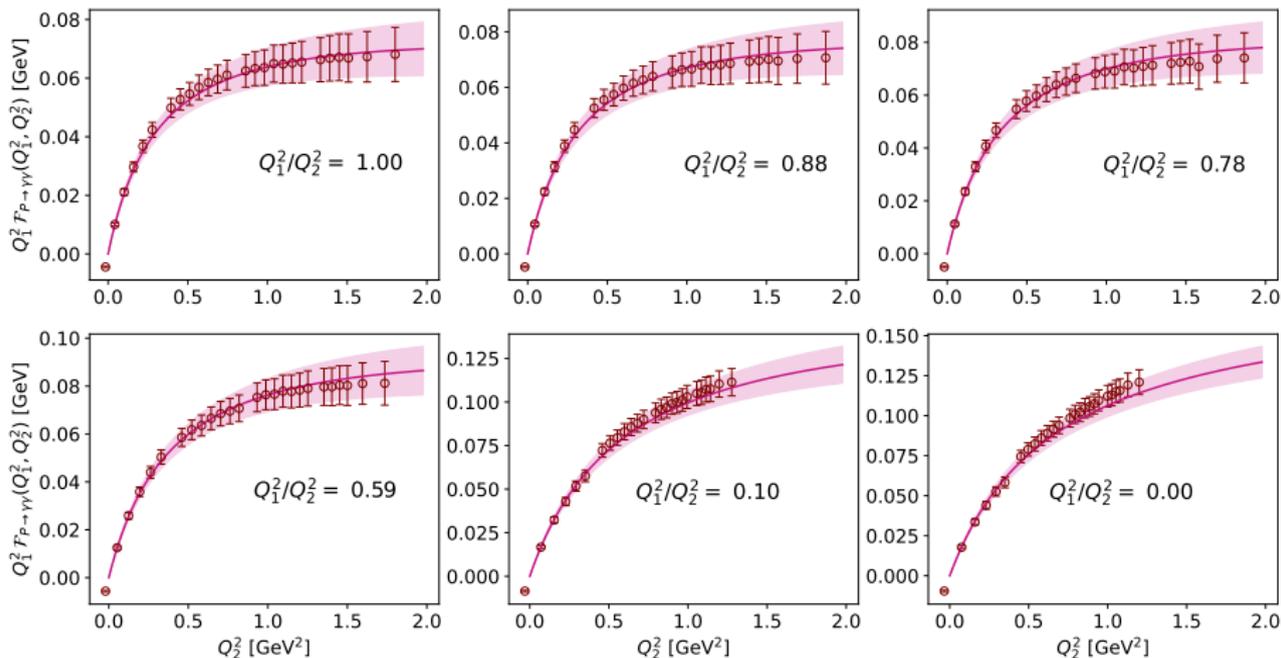
Integrands



- Diagonal kinematics: $q_1^2 = q_2^2 \Rightarrow \omega_1 = m_{\eta_8}/2$.
- Singly virtual kinematics: $q_1^2 = 0 \Rightarrow \omega_1 = |\vec{q}_1|$.
- Global fit to model, i.e. simultaneous for all orbits $1 \leq \vec{q}^2(2\pi/L_x)^2 \leq 32$.

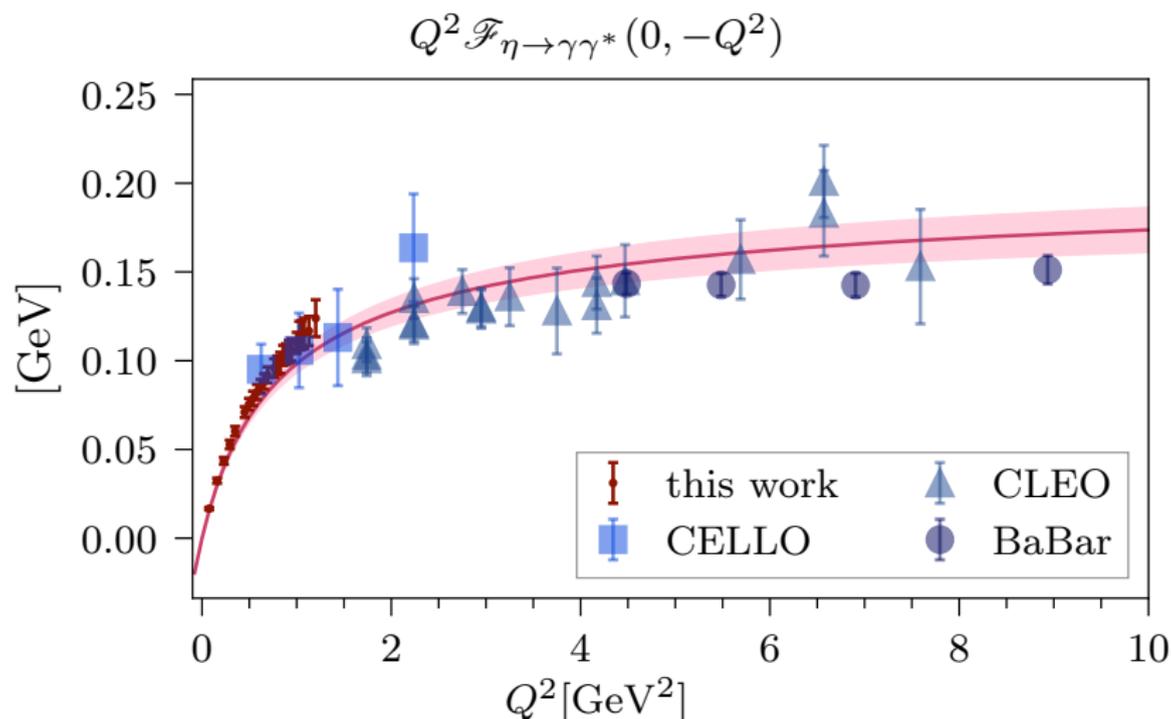
Example TFFs

η TFFs cB211.072.64, global_VMD, $t_{seq} = 10$, $(t_{min}, t_{max}) = (3, 7)$, $t_c = 8$, NOCOV



z-Expansion fit: [Gerardin et al., Phys. Rev. D100, 034520 (2019) and refs. therein]

Comparison to experimental data

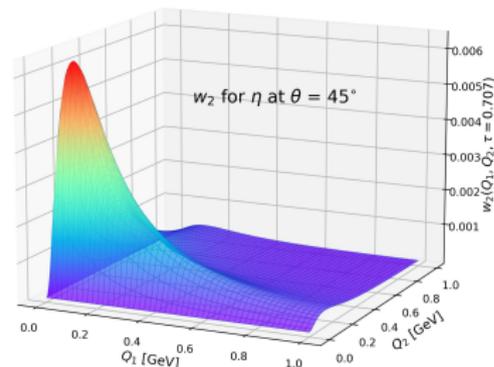
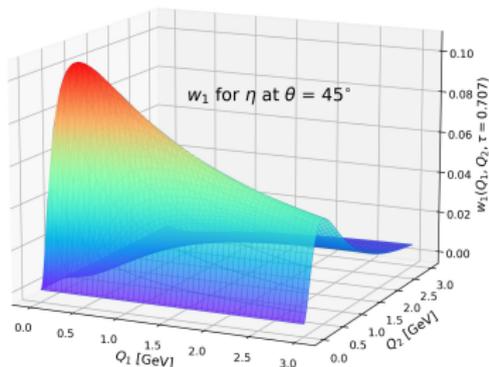


[H. J. Behrend et al. (CELLO), Z. Phys. C49, 401 (1991)] [J. Gronberg et al. (CLEO), Phys. Rev. D57, 33 (1998)] [B. Aubert et al. (BABAR), Phys. Rev. D80, 052002 (2009)] [P. del Amo Sanchez et al. (BABAR), Phys. Rev. D84, 052001 (2011)]

Pseudoscalar pole contribution

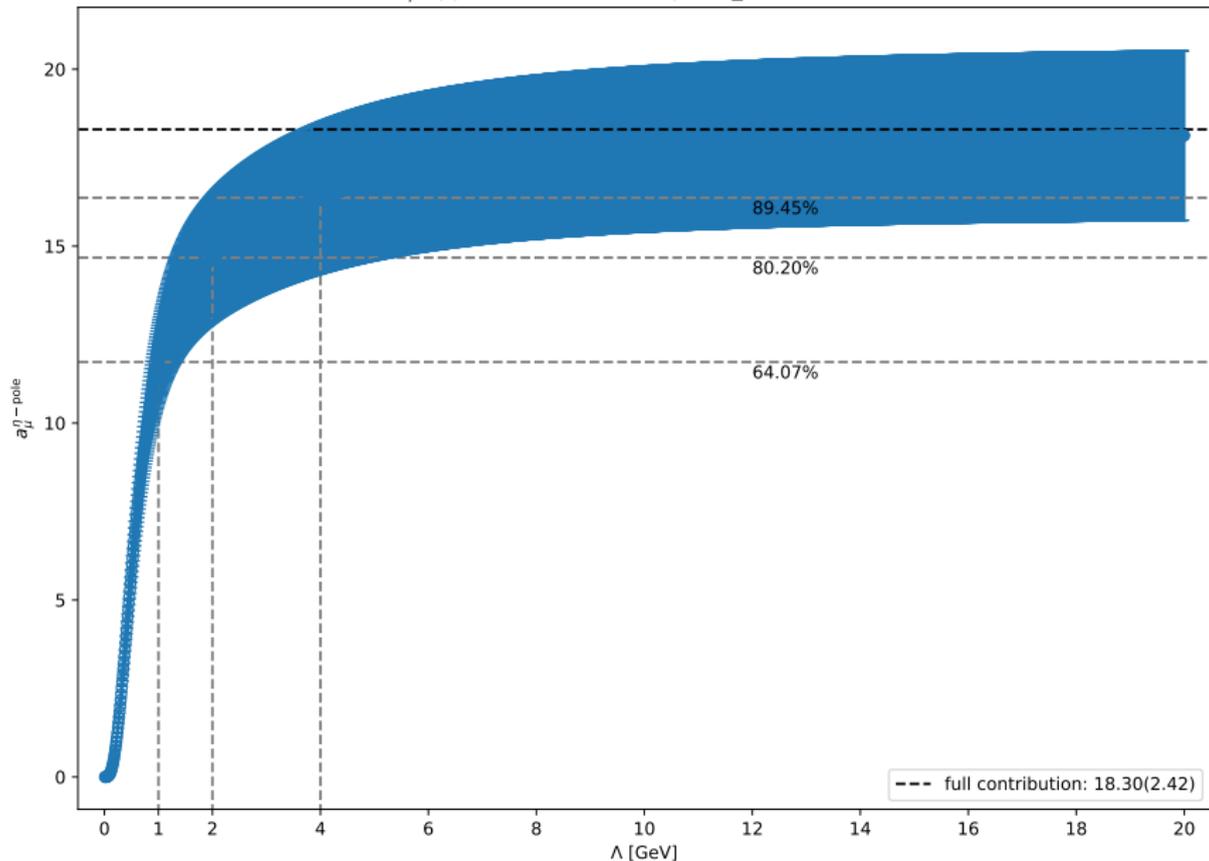
3d integral representation [A. Nyffeler, Phys. Rev. D94, 053006 (2016) and refs. therein]

$$a_{\mu}^{P\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^{\infty} dQ_1 \int_0^{\infty} dQ_2 \int_{-1}^{+1} d\tau \left[w_1(Q_1, Q_2, \tau) \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1, Q_2, \tau) \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0) \right]$$

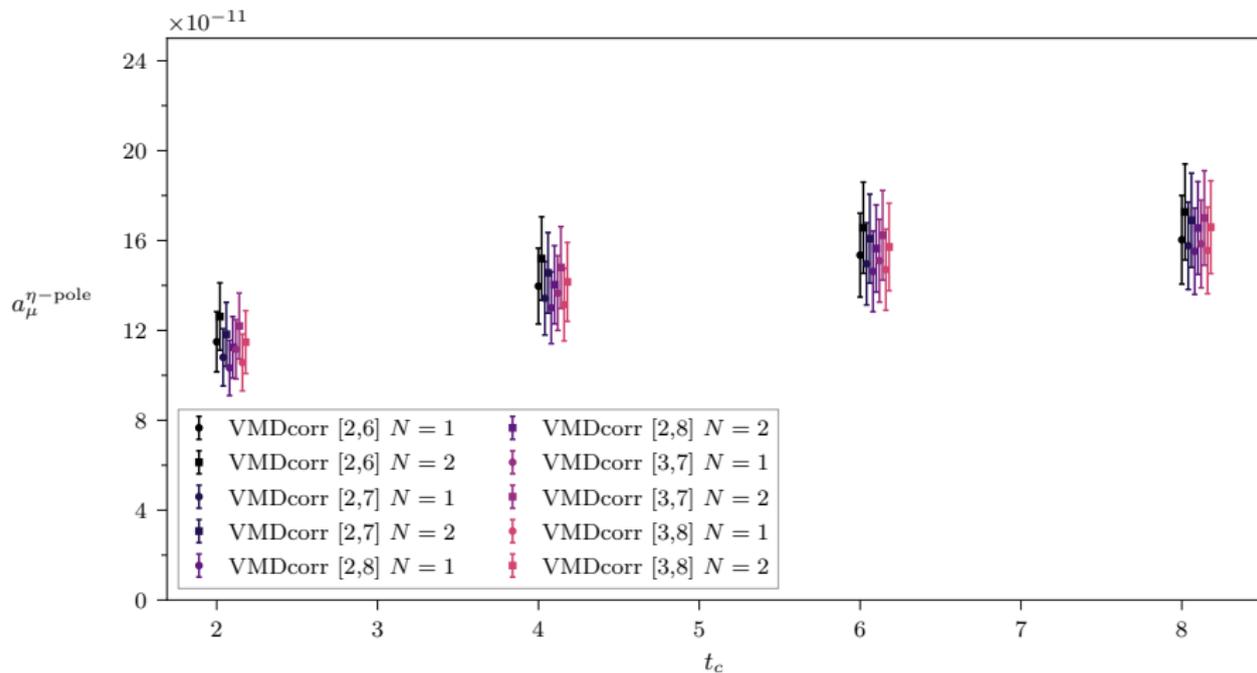


Saturation

Contribution to $a_{\mu, \eta}$ pole, cB211.072.64, global_VMD, $t_{min}, t_{max} = (3, 7), t_c = 8$



Eta pole contribution to a_μ



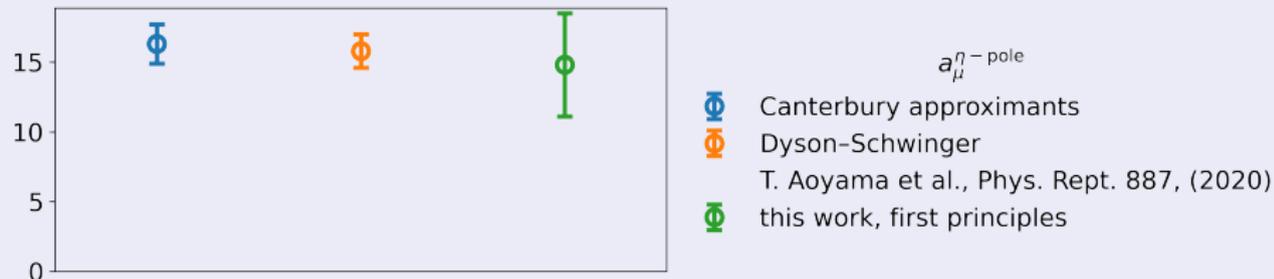
AIC model averaging gives a **preliminary**

$$a_\mu^{\eta\text{-pole}} = 14.8(1.8)_{stat} (2.0)_{sys} (2.6)_{t_{seq}} [3.7]_{tot} \times 10^{-11}.$$

Conclusion & Outlook

Summary

Our setup allows the first determination of the eta pole contribution to a_μ from lattice QCD directly at the physical point. We get promising preliminary results, compatible with other calculations, e.g.



Next steps

- More precise characterization of the systematic error.
- Analysis on the other physical point ensembles, continuum limit estimation.

BACKUP - Ensembles

ensemble	$L^3 \cdot T/a^4$	β	c_{SW}	M_π [MeV]	a [fm]	$a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	κ_{crit}
cA53.24	$24^3 \cdot 48$	1.726	1.74	350	0.0927	0.0053	0.1408	0.1521	0.1400645
cA40.24	$24^3 \cdot 48$	1.726	1.74	301	0.0927	0.0040	0.1408	0.1521	0.1400645
cA30.32	$32^3 \cdot 64$	1.726	1.74	260	0.0927	0.0030	0.1408	0.1521	0.1400645
cA12.48	$48^3 \cdot 96$	1.726	1.74	165	0.0927	0.0012	0.1408	0.1521	0.1400645
cB25.48	$48^3 \cdot 96$	1.778	1.69	255	0.0800	0.0025	0.1247	0.1315	0.1394267
cB140.64	$64^3 \cdot 128$	1.778	1.69	190	0.0800	0.00140	0.1247	0.1315	0.1394267
cB072.64	$64^3 \cdot 128$	1.778	1.69	135	0.0800	0.00072	0.1247	0.1315	0.1394265
cC06.80	$80^3 \cdot 160$	1.836	1.645	135	0.069	0.0006	0.1060	0.1135	0.1387510

Table: Parameter values for the gauge configurations available through (and under production by) ETMC with $N_f = 2 + 1 + 1$ Wilson clover twisted mass quark flavours. For each ensemble we provide the volume, the gauge coupling β , the clover coefficient c_{SW} , the pion mass M_π and the lattice spacing a in physical units, the bare twisted mass values $a\mu_\ell$, $a\mu_\sigma$, $a\mu_\delta$, and the hopping parameter κ_{crit} .

BACKUP - Clover improved tmLQCD action [C. Alexandrou et al.,

Phys. Rev. D98, 054518 (2018)] $S = S_g + S_{tm}^l + S_{tm}^h$

Iwasaki improved gauge action for S_g :

$$S_g = \frac{\beta}{3} \sum_x \left(b_0 \sum_{\substack{\mu, \nu=1 \\ 1 \leq \mu < \nu}}^4 \{1 - \text{Re Tr}(U_{x, \mu, \nu}^{1 \times 1})\} + b_1 \sum_{\substack{\mu, \nu=1 \\ \mu \neq \nu}}^4 \{1 - \text{Re Tr}(U_{x, \mu, \nu}^{1 \times 2})\} \right)$$

Light up and down doublet:

$$S_{tm}^l = \sum_x \bar{\chi}_l(x) \left[D_W(U) + \frac{i}{4} c_{SW} \sigma^{\mu\nu} \mathcal{F}^{\mu\nu}(U) + m_l + i \mu_l \tau^3 \gamma^5 \right] \chi_l(x)$$

Heavy quark action, non-degenerate strange and charm quarks*:

$$S_{tm}^h = \sum_x \bar{\chi}_h(x) \left[D_W(U) + \frac{i}{4} c_{SW} \sigma^{\mu\nu} \mathcal{F}^{\mu\nu}(U) + m_h - \mu_\delta \tau_1 + i \mu_\sigma \tau^3 \gamma^5 \right] \chi_h(x)$$

*In practise, we use $m_h = m_l$, this constitutes an additional $\mathcal{O}(a^2)$ lattice artefact which is small for the strange quark and practically vanishes for the charm quark.

BACKUP - Extraction of the form factors

We follow the conventions from [A. Gérardin, H. B. Meyer and A. Nyffeler, Phys. Rev. D94, 074507 (2016)]

- The relevant hadronic quantity is the rank-four hadronic vacuum polarization tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 d^4x_2 d^4x_3 \left[e^{i(q_1x_1 + q_2x_2 + q_3x_3)} \cdot \langle 0 | T \{ j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0) \} | 0 \rangle \right]$$

- Each pseudoscalar pole $P \in \{\pi_0, \eta, \eta'\}$ contributes to the amplitude via one-particle-reducible single pseudoscalar exchanges

$$\Pi_{\mu\nu\lambda\rho}^{(P)}(q_1, q_2, q_3) = \left[i \frac{\mathcal{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) \mathcal{F}_{P\gamma^*\gamma^*}(q_3^2, (q_1 + q_2 + q_3)^2)}{(q_1 + q_2)^2 + M_P^2} \cdot \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \epsilon_{\lambda\rho\gamma\delta} q_3^\gamma (q_1 + q_2)^\delta \right] + (\text{crosses})$$

BACKUP - z-expansion

Extract $\mathcal{F}_{\pi_0\gamma^*\gamma^*}$ over the whole kinematical range using the modified z-expansion [Gerardin et al., Phys. Rev. D100, 034520 (2019) and refs. therein]:

$$P(Q_1^2, Q_2^2)\mathcal{F}_{\pi_0\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{m,n=0}^N c_{nm} \left(z_1^n - (-1)^{N+n+1} \frac{n}{N+1} z_1^{N+1} \right) \left(z_2^m - (-1)^{N+m+1} \frac{m}{N+1} z_2^{N+1} \right)$$

- $z_k = z_k(Q_k^2)$, $P(Q_1^2, Q_2^2)$ a polynomial in four-momenta
- Sampling in momentum plane by fixing Q_2^2/Q_1^2 using continuous free parameter ω .
- Correlated order $N \in 1, 2, 3$ fits.

BACKUP - z-expansion - detailed

Extract $\mathcal{F}_{\pi_0\gamma^*\gamma^*}$ over the whole kinematical range using the modified z-expansion [Gerardin et al., Phys. Rev. D100, 034520 (2019)]:

- $z_k = \frac{\sqrt{t_c + Q_k^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_k^2} + \sqrt{t_c - t_0}}, k \in \{1, 2\}$
- $P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2}$
- Map branch cut starting at $t_c = 4m_\pi^2$ onto unit circle $|z_k| = 1$.
- Free parameter t_0 , optimal choice $t_0 = t_c(1 - \sqrt{1 + Q_{max}^2/t_c})$.

$$P(Q_1^2, Q_2^2)\mathcal{F}_{\pi_0\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{m,n=0}^N c_{nm} \left(z_1^n - (-1)^{N+n+1} \frac{n}{N+1} z_1^{N+1} \right) \left(z_2^m - (-1)^{N+m+1} \frac{m}{N+1} z_2^{N+1} \right)$$

- Sampling in momentum plane by fixing Q_2^2/Q_1^2 using continuous free parameter ω .
- Correlated order $N \in 1, 2, 3$ fits.

BACKUP - Projection on η

By the following argument, we still extract $a_{\mu}^{\eta\text{-pole}}$ easily: Note that η is the ground state in the η/η' system and η_8 the ground state in the η_8/η_1 system, and that η/η' are linear combinations of η_8/η_1 . Explicitly,

$$\eta = \cos \theta_P \cdot \eta_8 + \sin \theta_P \cdot \eta_1. \quad (1)$$

For asymptotically large Euclidean time the ground state is uniquely the state of lowest mass, which survives. Thus

$$\begin{aligned} \langle 0 | \mathcal{O}(t) | \cos \theta_P \cdot \eta_8 + \sin \theta_P \cdot \eta_1 \rangle &\stackrel{t \rightarrow \infty}{\simeq} \cos \theta_P \langle 0 | \mathcal{O}(0) | \eta_8 \rangle \cdot e^{-m_{\eta_8} t} \\ &+ \sin \theta_P \langle 0 | \mathcal{O}(0) | \eta_1 \rangle \cdot e^{-m_{\eta_1} t} \end{aligned} \quad (2)$$

So with ground state projection, after dividing out the overlap, we directly obtain \tilde{A} for η (of course $\langle 0 | \mathcal{O}(0) | \eta_8 \rangle$ and $\langle 0 | \mathcal{O}(0) | \eta \rangle$ differ, but $\langle 0 | \mathcal{O}(0) | \eta_8 \rangle$ vanishes in the division).