

# An update on RI/IMOM schemes

Nicolas Garron (Liverpool Hope and University of Liverpool )

In collaboration with

Caroline Cahill, Martin Gorbahn, John Gracey, Paul Rakow

Lattice 2022 - Bonn, Germany

08/08/22

## In collaboration with

- Caroline Cahill, Martin Gorbahn, John Gracey, Paul Rakow (Liverpool)
- Thanks to RBC-UKQCD !
- Thanks to Holger Perlt (Leipzig) and QCDSF
- Thanks to CalLat (Henry Jose Monge Camacho, Amy Nicholson, André Walker-Loud)

Proceedings and preprint (submitted to PRD)

- <https://arxiv.org/abs/2202.04394>
- <https://arxiv.org/abs/2112.11140>

# The RBC & UKQCD collaborations

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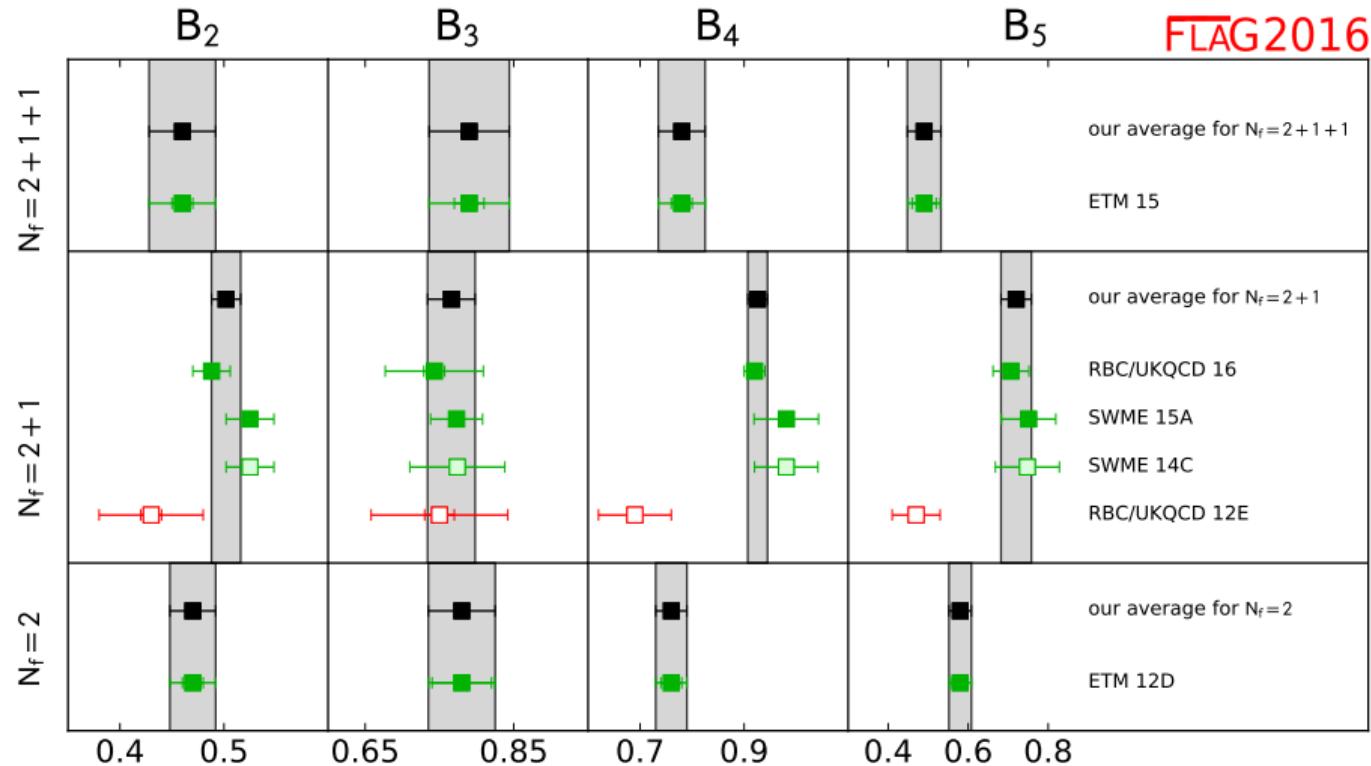
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## On the importance of the renormalisation scheme

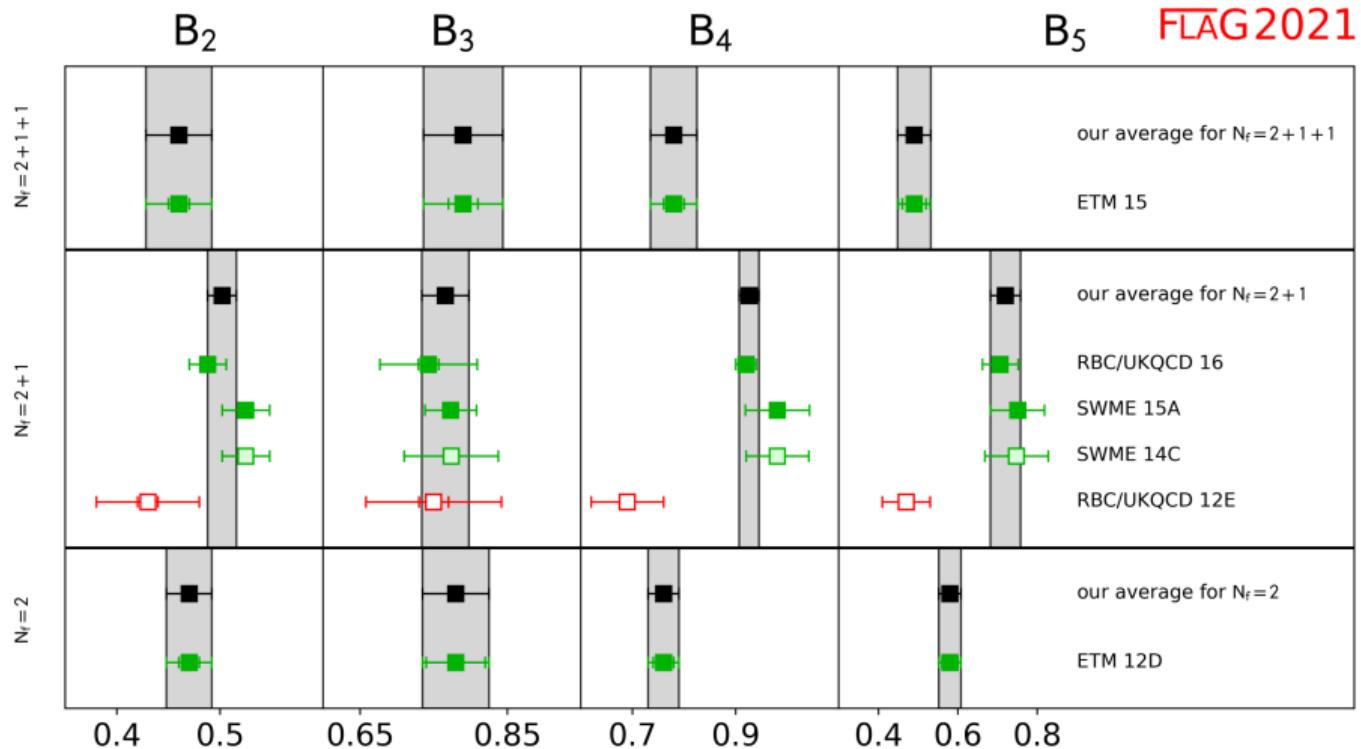
based on RBC-UKQCD 2010-now

. . . [NG Hudspith Lytle'16] , [Boyle NG Hudspith Lehner Lytle '17] [... Kettle, Khamseh, Tsang 17-19]

# BSM kaon mixing - Results

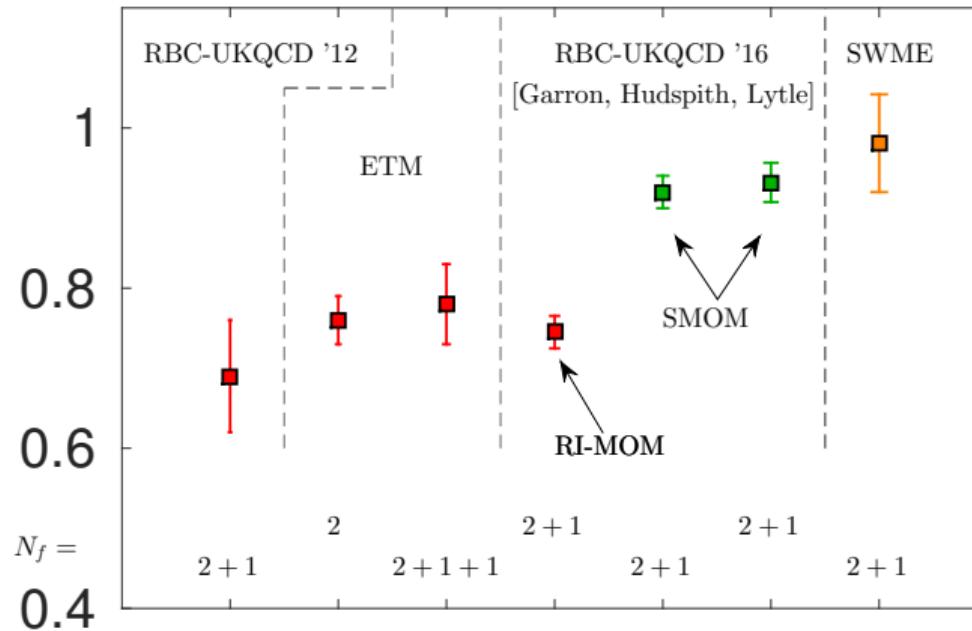


# BSM kaon mixing - Results



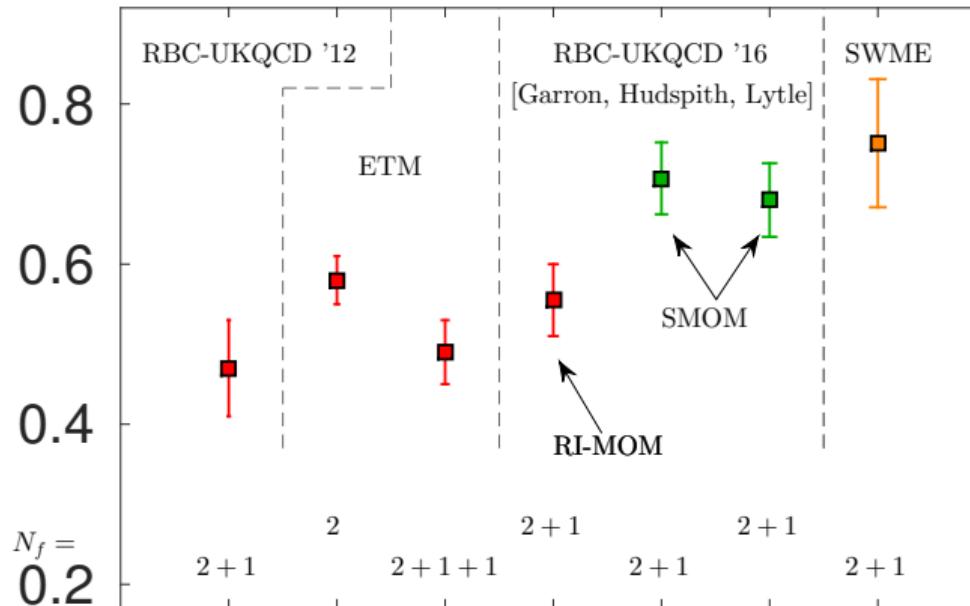
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$B_4$

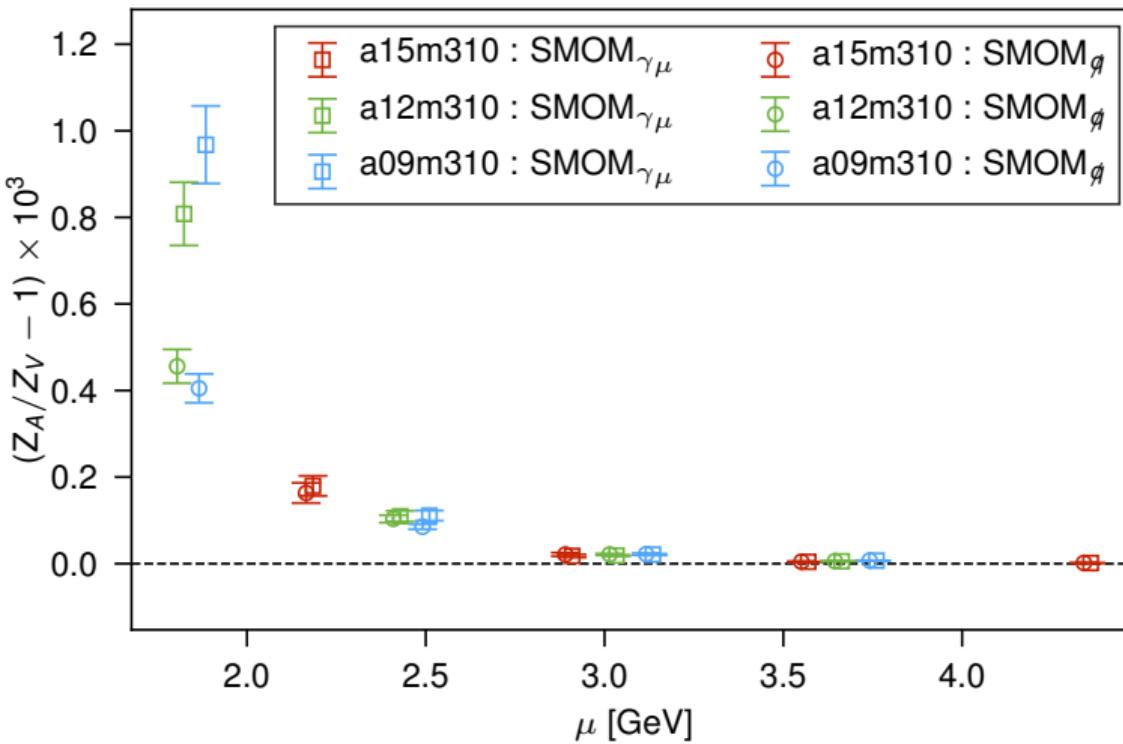


# BSM kaon mixing - Results

$B_5$



## Another example $Z_A/Z_V$



[C Chang, A Nicholson, E Rinaldi, E Berkowitz, N G, D Brantley, H Monge-Camacho, C Monahan, C Bouchard, M Clark, B Joó, T Kurth, K Orginos, P Vranas, A Walker-Loud, Nature 558 (2018)]

## Non Perturbative Renormalisation (NPR)

# Reminders - general strategy

- 1 First step: remove the divergences

For a generic composite operator  $Q^{\text{bare}}(a)$  which renormalises multiplicatively, determine the  $Z$ -factor such that

$$Q^{\text{scheme}}(\mu, a) = Z^{\text{scheme}}(\mu, a) Q^{\text{bare}}(a)$$

has well-defined continuum limit.

This step can be done non-perturbatively.

- 2 Second step: match to phenomenology (e.g.  $\overline{\text{MS}}$ ), This step has to be done in (continuum) perturbation theory .

$$Q_i^{\text{scheme}}(\mu, 0) \longrightarrow Q^{\overline{\text{MS}}}(\mu) = (1 + r_1 \alpha_S(\mu) + r_2 \alpha_S^2(\mu) + \dots) Q^{\text{scheme}}(\mu, 0)$$

# Non-Perturbative Renormalisation (NPR)

There are two popular methods for the non-perturbative determination of the  $Z$ -factors

- Schrödinger Functional (SF)
- Rome-Southampton: RI/MOM, RI/MOM', RI/SMOM, RI/mSMOM, ....

Here I am talking about extensions of the latter.

# The Rome Southampon method

[Martinelli et al '95]

Original setup [Martinelli et al '95].

Consider for example a quark bilinear build from quark propagators.

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- Gauge fixed (Landau).

Prescription: one requires some amputated Green function(s) to be finite.

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Several extensions and improvement, most notably

- Non-exceptional kinematics (SMOM) [RBC, RBC-UKQCD, Sturm et al., Lehner and Sturm, Almeida and Sturm, Gorbahn and Jäger, Gracey, ...]
- Momentum sources [Göckeler et al. '98 QCDSF]
- Twisted boundary conditions [many references !]
- Massive momentum scheme [Boyle, Del Debbio and Khamseh, 2016]
- Step scaling [Alpha, RBC-UKQCD, ...]

## Example: quark bilinear

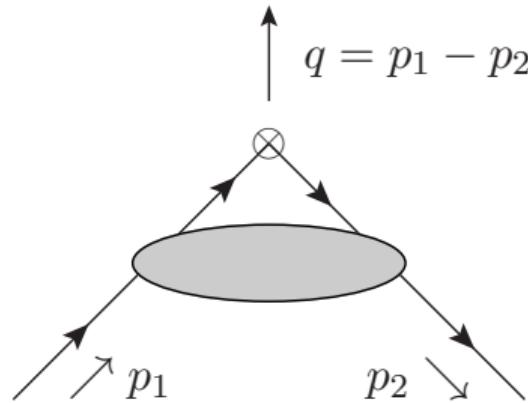
Consider a quark bilinear  $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$ , where  $\Gamma = 1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu \gamma_5, \gamma_5$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle G_2(x_2, 0) \Gamma G_1(0, x_1) \rangle$$

In Fourier space  $G(p) = \sum_x G(x, 0) e^{ip \cdot x}$  and  $G(-p) = \gamma_5 G(p)^\dagger \gamma_5$

$$V(p_2, p_1) = \langle G_2(-p_2) \Gamma G_1(p_1)^\dagger \rangle$$



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Amputated Green function

$$\Pi(p_2, p_1) = \langle G_2(p_2)^{-1} \rangle \langle G_2(p_2) \Gamma G_1(p_1)^\dagger \rangle \langle (G_2(p_1)^\dagger)^{-1} \rangle$$

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Rome Southampton original scheme (RI/MOM),  $p_1 = p_2 = p$  and  $\mu = \sqrt{p^2}$

$$\frac{Z_\Gamma}{Z_q}(\mu, a) \times \lim_{m \rightarrow 0} \text{Tr}(\Gamma \Pi(p, p))_{\mu^2 = p^2} = \text{Tree}$$

# Rome-Southampton windows

Ideally, in order to keep the discretisation effects under control [G. Martinelli et al 95]

$$\mu \ll 1/a$$

and to apply perturbation theory

$$\Lambda_{\text{QCD}} \ll \mu$$

In practice, might be tight if  $1/a \sim 2 \text{ GeV}$

## Rome-Southampton windows

Imagine you have computed the hadronic matrix elements on a coarse lattice

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$$Z(a_{\text{coarse}}, \mu) = \lim_{a_{\text{fine}} \rightarrow 0} \{ Z(a_{\text{fine}}, \mu) Z^{-1}(a_{\text{fine}}, \mu_0) \} \times Z(a_{\text{coarse}}, \mu_0)$$

Where

- $\mu_0$  is a lower scale, eg  $\mu_0 \sim 1 \text{ GeV}$
- the running is computed on finer lattices and extrapolated to the continuum

[Arthur and Boyle '10], [Arthur, Boyle, N.G., Kelly, Lytle '11]

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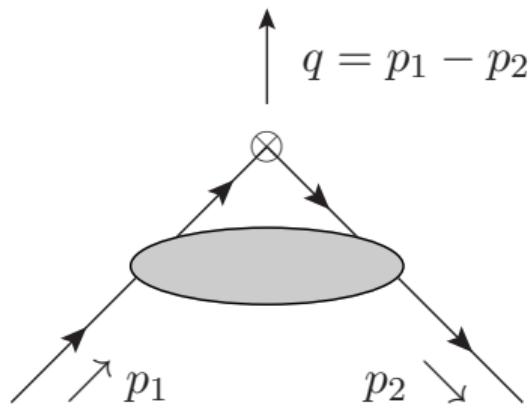
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But still have to deal with IR effect in  $Z(a, \mu_0)$

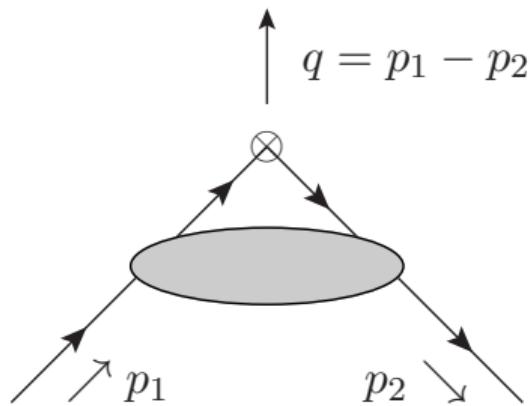
RI/MOM vs RI/SMOM

# Kinematics



- In the original RI/MOM setup,  $p_1 = p_2 \Rightarrow q = 0$  and  $\mu = \sqrt{p_1^2}$ .  
Lead to IR poles, for example in  $1/\mu^2$

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- In RI/SMOM we have

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Improved IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, ...]

## SMOM and IMOM

## More MOM schemes

Renormalisation scale is  $\mu$ , given by the choice of kinematics

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- We are now studying a generalisation (see also [Bell and Gracey, Perlt])

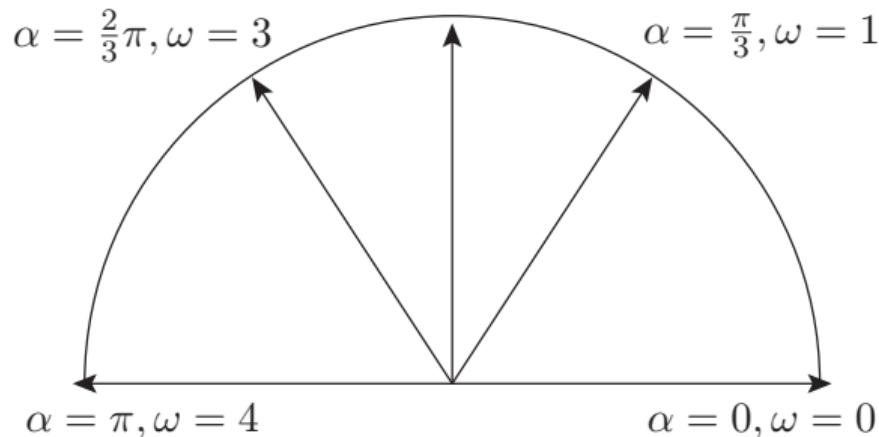
$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2, \quad (p_1 - p_2)^2 = \omega \mu^2 \text{ where } \omega \in [0, 4]$$

Note that  $\omega = 0 \leftrightarrow RI/MOM$  and  $\omega = 1 \leftrightarrow RI/SMOM$

# IMOM schemes

$\alpha$  is the angle between  $p_1$  and  $p_2$

$$\alpha = \frac{\pi}{2}, \omega = 2$$



$$\omega = 2(1 - \cos \alpha)$$

# Implementation (1)

We want to achieve  $p_1^2 = p_2^2 \equiv \mu^2$ ,  $q^2 = (p_1 - p_2)^2 = \omega\mu^2$ ,

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One possibility, for example [QCDSF'17]

$$p_1 = \frac{2\pi}{L} (m, m, m, m) , \quad p_2 = \frac{2\pi}{L} (-m, -m, -m, m)$$

$$\Rightarrow q = \frac{2\pi}{L} (2m, 2m, 2m, 0)$$

gives

$$\mu^2 = \left(\frac{2\pi}{L}\right)^2 4m^2 , \text{ and } q^2 = 3\mu^2$$

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The number of  $-$  signs in  $p_2$  gives the value of  $\omega = 0, 1, \dots, 4$ .

## Implementation (2)

Another possibility is to take advantage of twisted boundary conditions, say take

$$p_1 = \frac{2\pi}{L} (l, 0, 0, 0) \quad p_2 = \frac{2\pi}{L} (m, n, 0, 0)$$

$$\Rightarrow q = \frac{2\pi}{L} (l - m, -n, 0, 0)$$

And for each pair of desired  $(\mu, \omega)$ , just need to solve

$$\begin{aligned}\mu &= 2\pi/L \\ l^2 &= m^2 + n^2 \\ \omega l^2 &= (l - m)^2 + n^2\end{aligned}$$

# Definitions

We call  $\Lambda_\Gamma$  the projected-amputated Green function, normalised by its tree value

For example  $\Lambda_S = \frac{1}{12} \text{Tr}(\Pi_S)$ .

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We define  $Z_m = 1/Z_S$  and compute the Z-factors for the scalar density

$$\left( \frac{Z_S}{Z_q}(\mu, \omega) \right)^{\text{IMOM}} \times (\Lambda_S)_{q^2=\omega\mu^2} = 1$$

For  $Z_q$  we use the vector current

$$\left( \frac{Z_V}{Z_q}(\mu, \omega) \right)^{\text{IMOM}-\gamma_\mu} \times (\Lambda_V^{(\gamma_\mu)})_{q^2=\omega\mu^2} = 1$$

and

$$\left( \frac{Z_V}{Z_q}(\mu, \omega) \right)^{\text{IMOM}-\not{q}} \times (\Lambda_V^{(\not{q})})_{q^2=\omega\mu^2} = 1$$

# Projectors

The difference between (I)MOM –  $\gamma_\mu$  and (I)MOM –  $\not{q}$  lies in the projector

$$\Lambda_V^{(\gamma_\mu)} = \frac{1}{48} \text{Tr} (\gamma_\mu \Pi_{V^\mu})$$

$$\Lambda_V^{(\not{q})} = \frac{q^\mu}{12q^2} \text{Tr} (\not{q} \Pi_{V^\mu})$$

# Ward-Takahashi identities (I)

In the continuum we have

$$\begin{aligned} q_\mu \Pi_{V^\mu}(p_1, p_2) &= -i(G^{-1}(p_2) - G^{-1}(p_1)) , \\ q_\mu \Pi_{A^\mu}(p_1, p_2) &= 2im\Pi_P(p_1, p_1) - i(\gamma_5 G^{-1}(p_2) + G^{-1}(p_1)\gamma_5) , \end{aligned}$$

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and using the decomposition

$$G^{-1}(p) = i\cancel{p}(1 + \Sigma^V) + m(1 + \Sigma^S) ,$$

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and using the decomposition

$$G^{-1}(p) = i\cancel{p}(1 + \Sigma^V) + m(1 + \Sigma^S) ,$$

leads to

$$\Lambda_V^{(\not{q})} = \frac{i}{12} \text{Tr} (1 + \Sigma^V) ,$$

and therefore expect  $Z_V^{(\not{q})}$  to be  $\omega$ -independent.

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The strategy might depend on the discretisation of the Dirac operator.

If chiral symmetry is explicitly broken, one can impose the VWI and use it as a renormalisation condition.

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Here we employ Domain-Wall fermions with good chiral-flavour symmetry. We can therefore use the VWI as a consistency check of our strategy.

## Numerical results

# Simulation

We use RBC-UKQCD ensembles, IW, 2+1 Domain-Wall fermions

We have two lattice spacings:

$$a^{-1} = 1.785(5) \text{ GeV } (24^3) \quad (1)$$

$$a^{-1} = 2.383(9) \text{ GeV } (32^3), \quad (2)$$

sea quark masses,  $am = 0.005, 0.010, 0.020$  for the  $24^3 \times 64 \times 16$  lattice

and  $am = 0.004, 0.006, 0.008$  for the  $32^3 \times 64 \times 16$  lattice.

We take the chiral limit on each lattice spacing using the values

$$am_{res} = 0.003152(43) \text{ } (24^3), \quad (3)$$

$$am_{res} = 0.0006664(76) \text{ } (32^3). \quad (4)$$

Our values for  $Z_V$  are

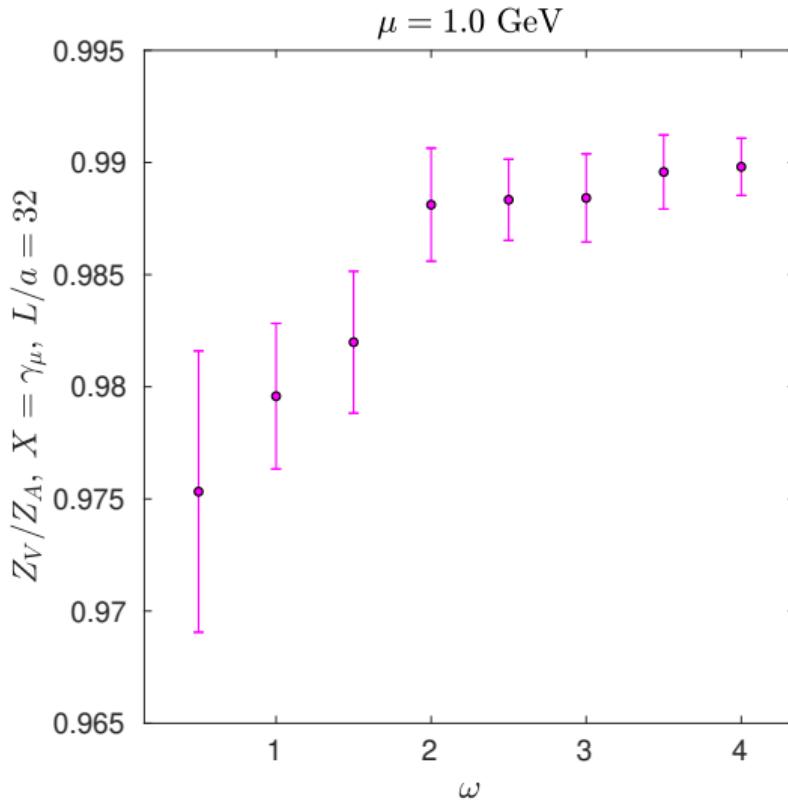
$$Z_V = Z_A = 0.71651(46) \text{ } (24^3), \quad (5)$$

$$Z_V = Z_A = 0.74475(12) \text{ } (32^2). \quad (6)$$

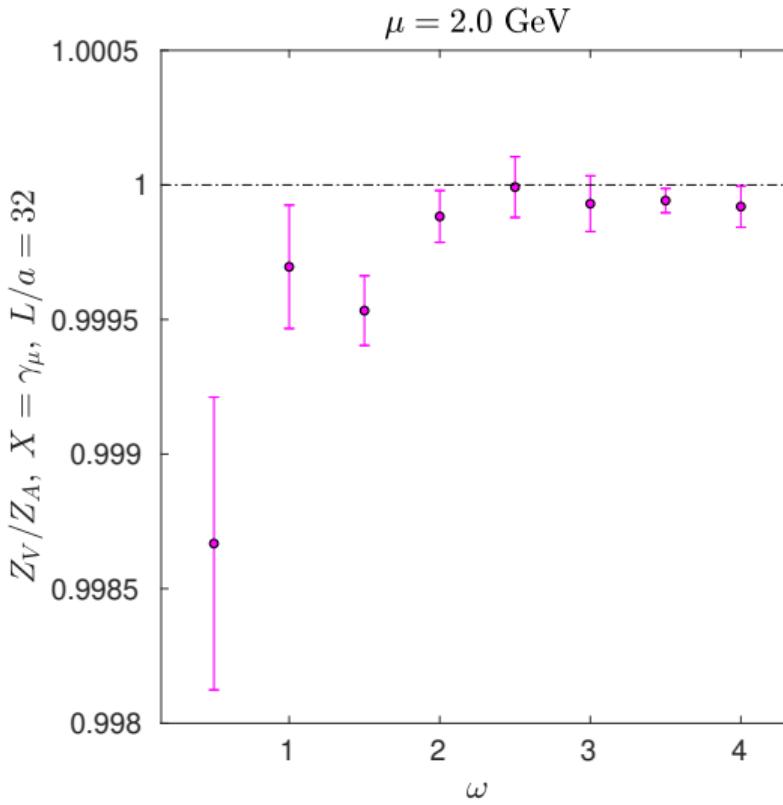
## Study of the systematic effects

- Chiral symmetry breaking effects
- Vector Ward Identity
- Discretisation effects

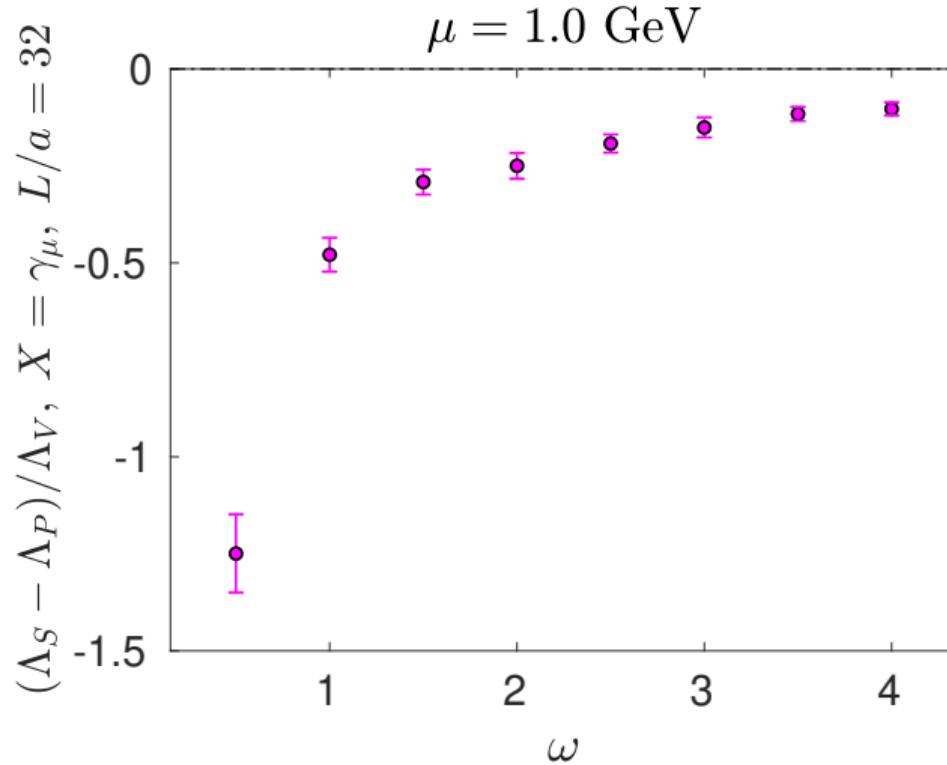
# Results for $Z_V/Z_A$



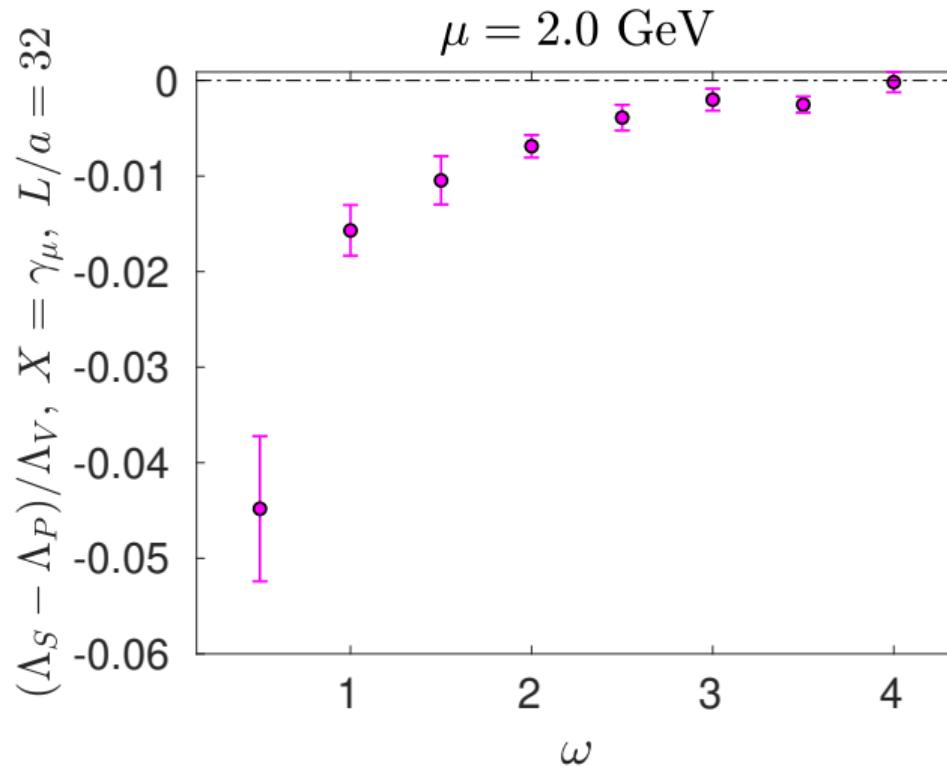
# Results for $Z_V/V_A$



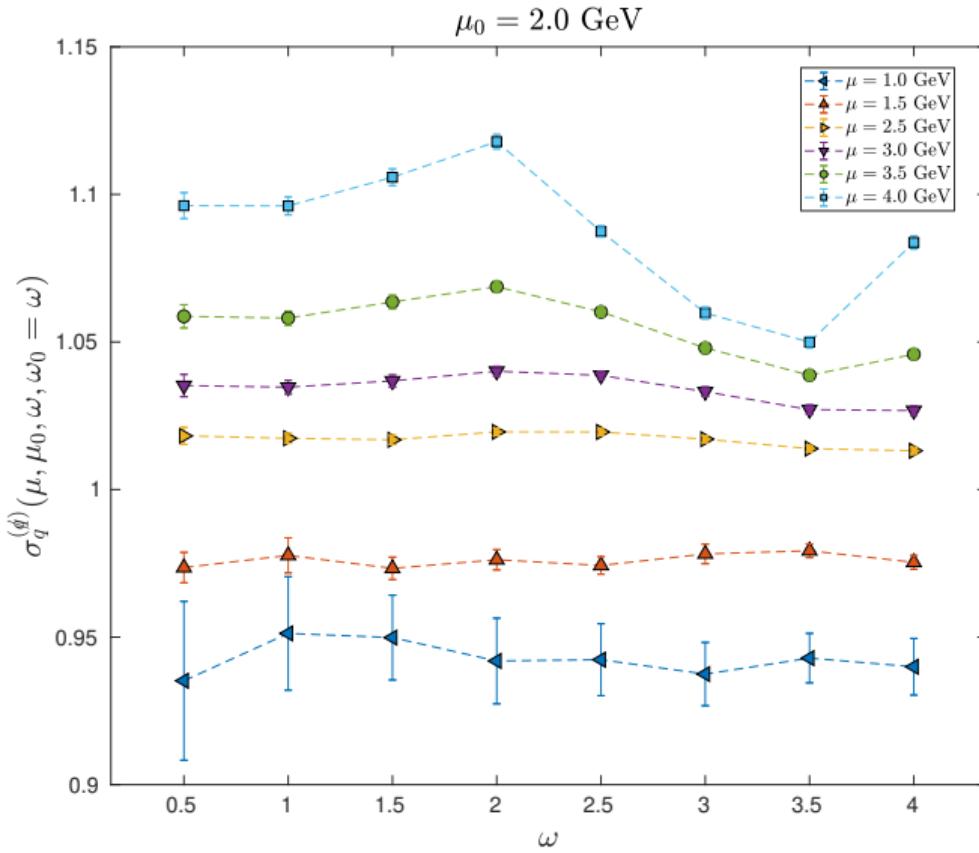
# Results for $(\Lambda_S - \Lambda_P)/\Lambda_V$



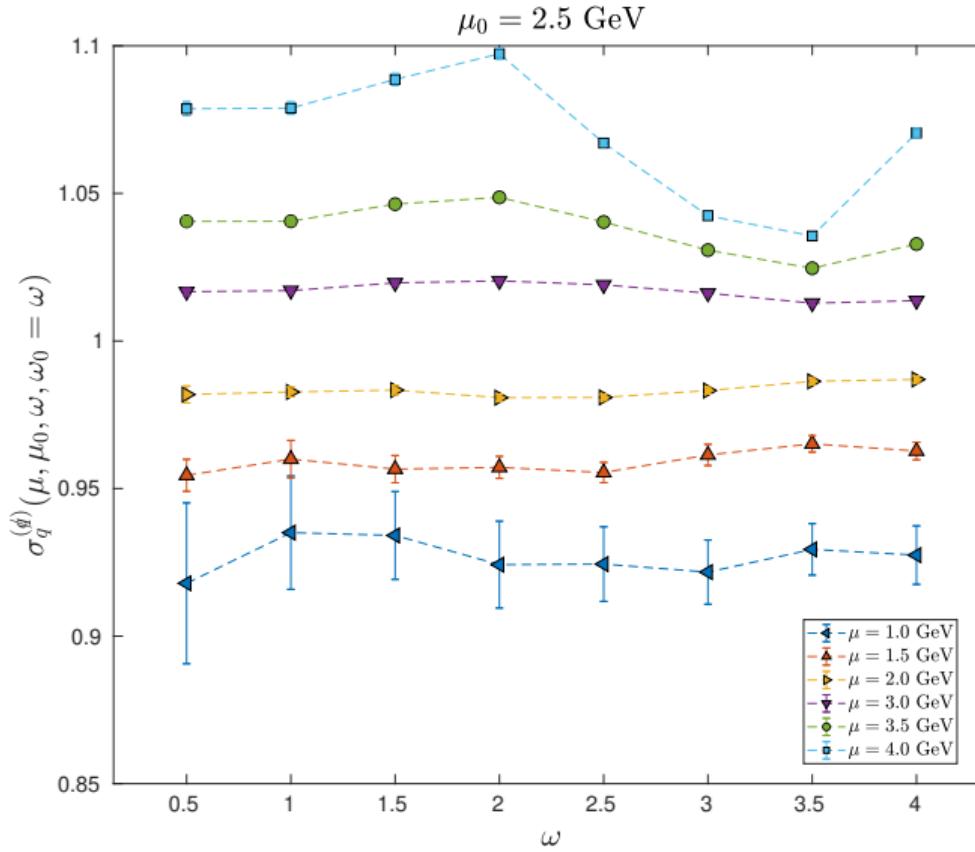
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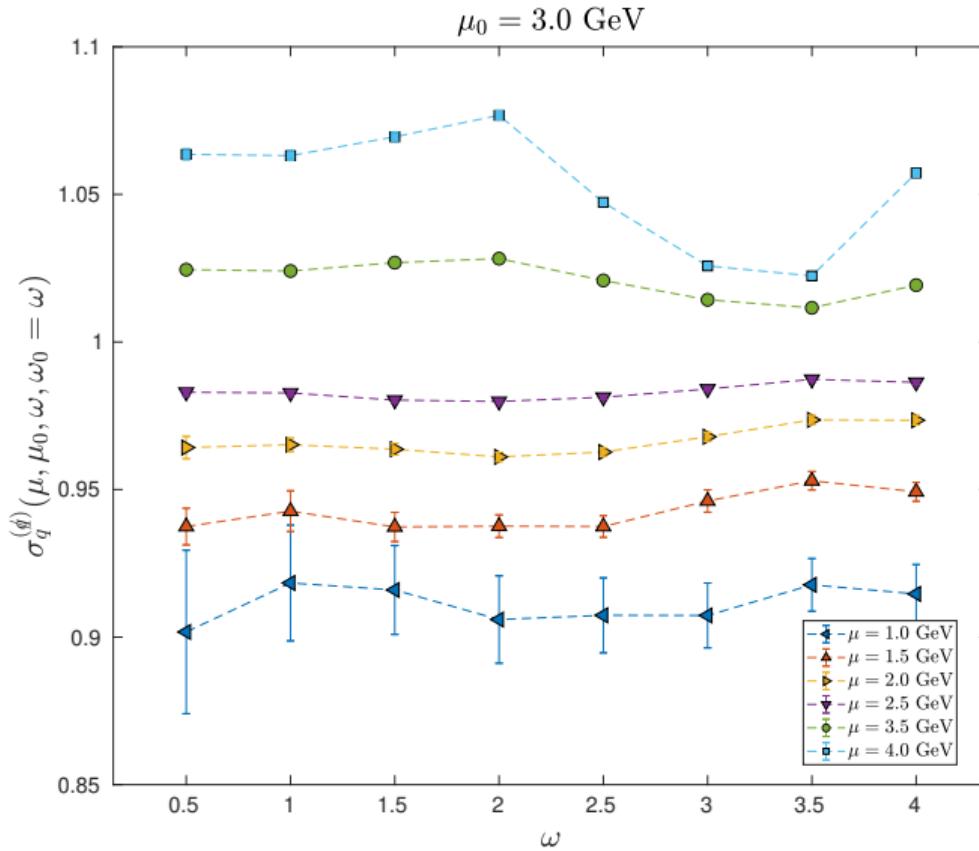
# Results for $\sigma_q^{(\phi)}$



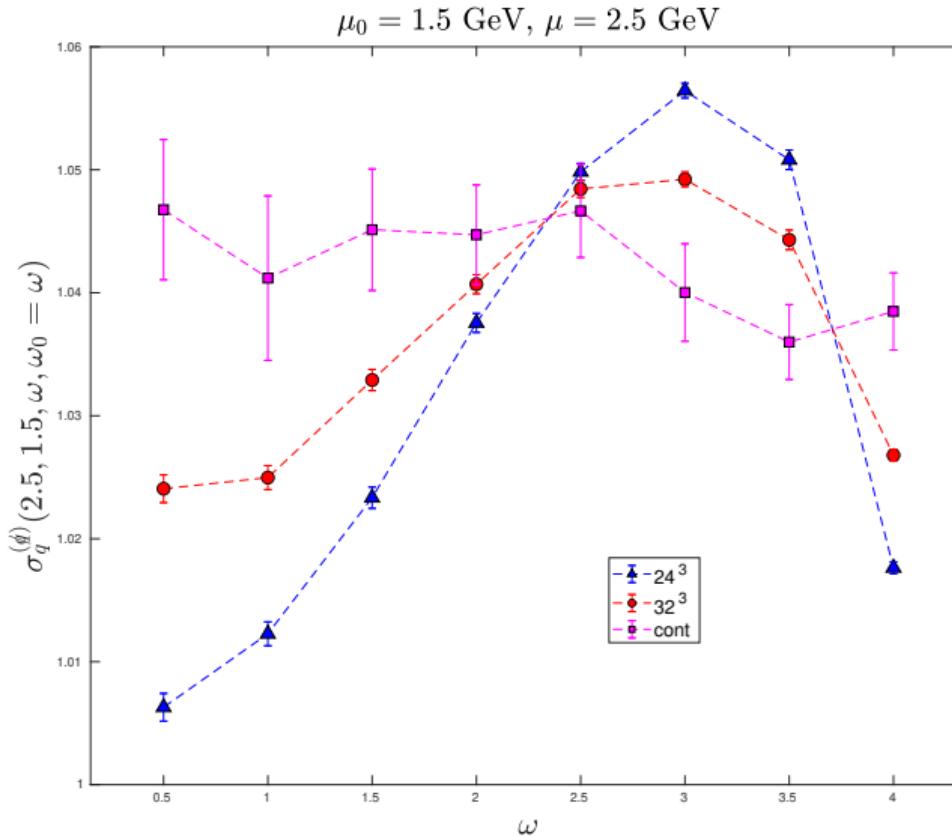
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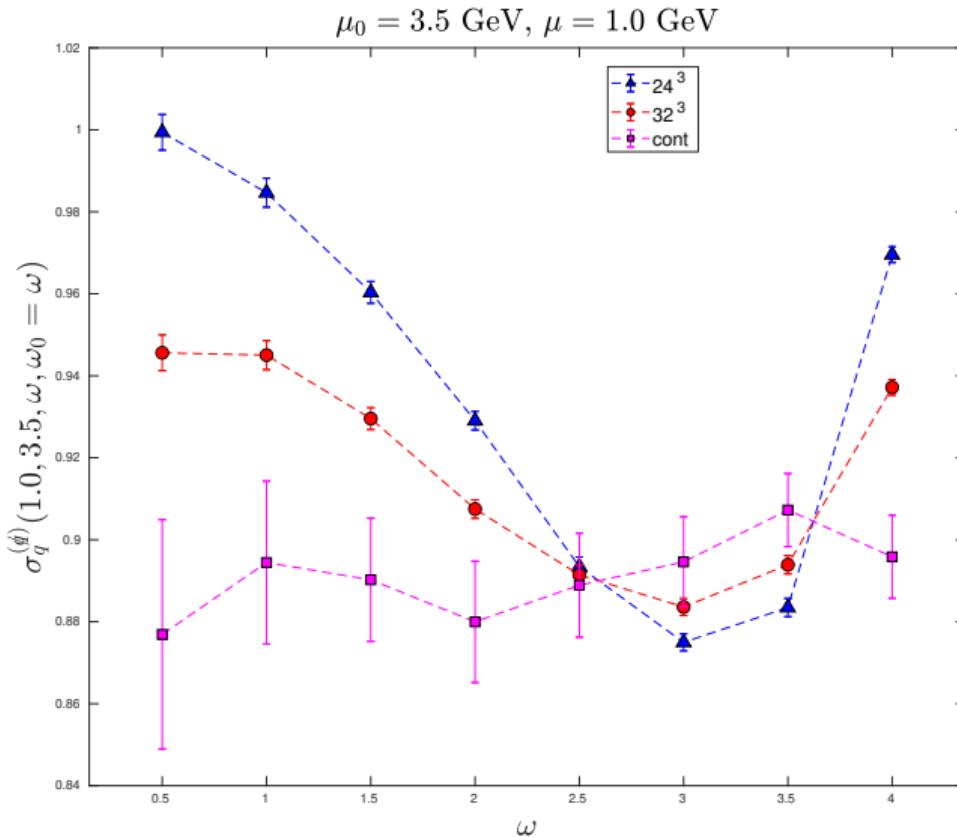
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# Results for $\sigma_q^{(\phi)}$ vs lattice spacing



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## Conclusions and outlook

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- Can we argue that IMOM schemes are better than SMOM schemes ?
- We have some evidences that  $\omega \sim 2$  exhibit better IR behaviour and smaller lattice artefacts than  $\omega = 1$  (for some quantities)
- But this argument is probably quantity dependent

# Conclusions and outlook

So what?

- Proof of concept, first simulation of  $\omega \neq 0, 1$
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- Perturbative matching factor to  $\overline{\text{MS}}$  at NNLO

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Proceedings and preprint (submitted to PRD)

- <https://arxiv.org/abs/2202.04394>
- <https://arxiv.org/abs/2112.11140>

## Backup



# Backup slides

- Perturbative vs non-perturbative results
- Poles subtraction
- More about  $Z_q$

# Results

Non-perturbative scale evolution (running), taking the continuum limit

$$\sigma(\mu, \omega, \mu_0, \omega_0) = \lim_{a^2 \rightarrow 0} \frac{Z(\mu, \omega)}{Z(\mu_0, \omega_0)}$$

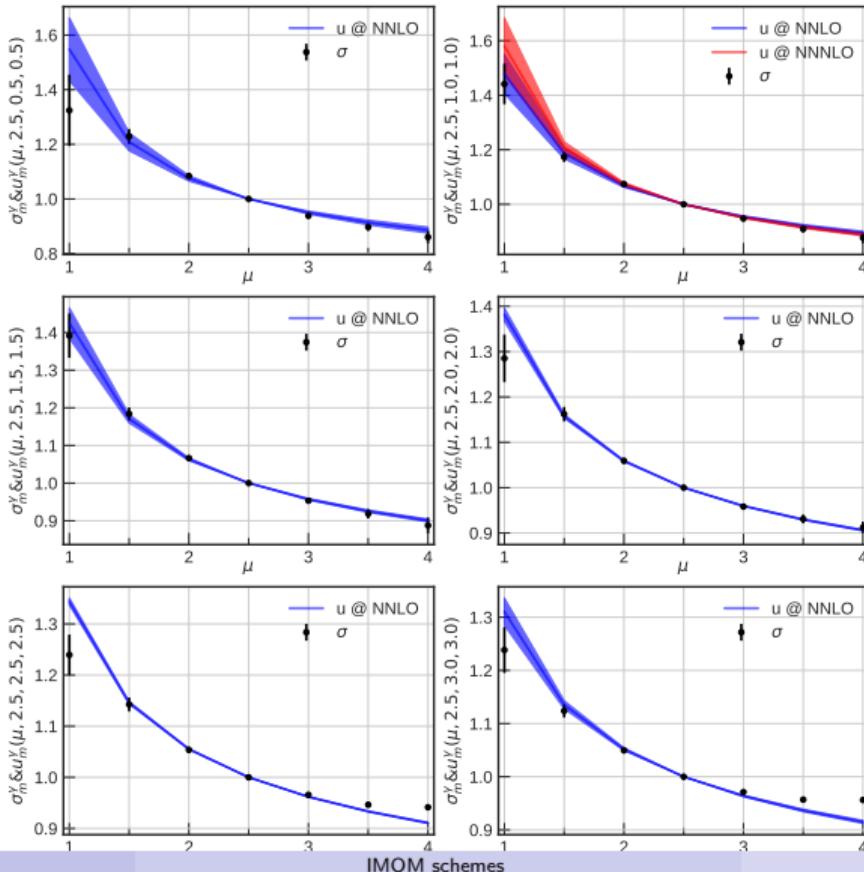
We have computed the perturbative prediction at NNLO,  $U(\mu, \omega, \mu_0, \omega_0)$

In the next slides, I show some plots for the ratios

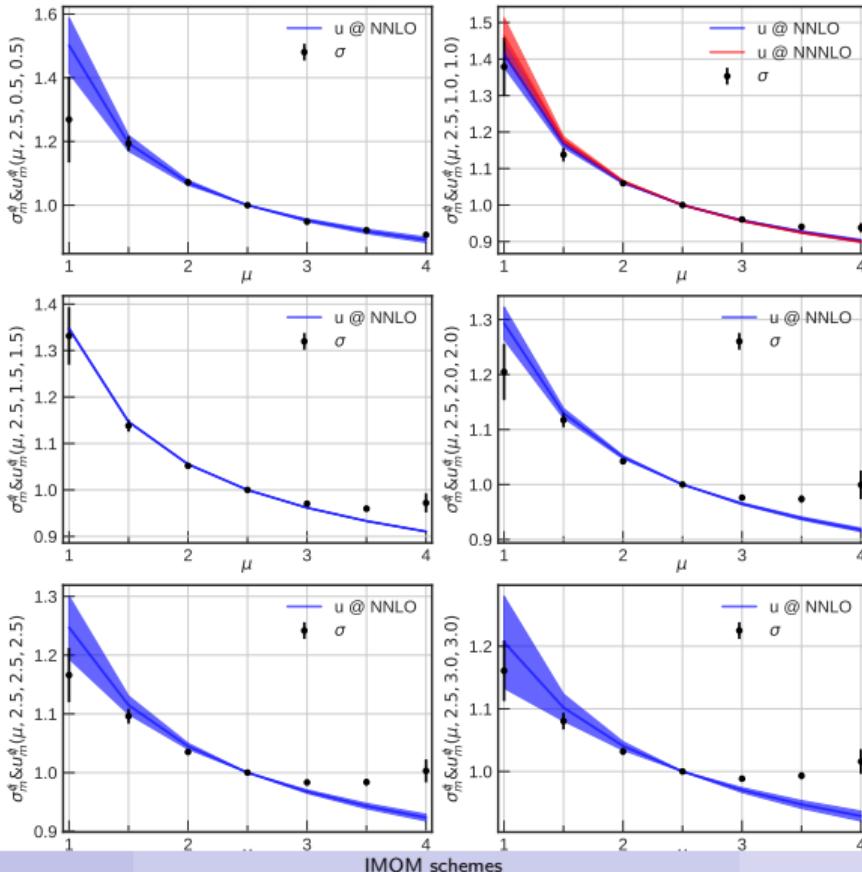
$$\frac{\sigma(\mu, \omega, \mu_0, \omega_0)}{U(\mu, \omega, \mu_0, \omega_0)}$$

for fixed  $\mu_0, \omega_0$  and various order in PT

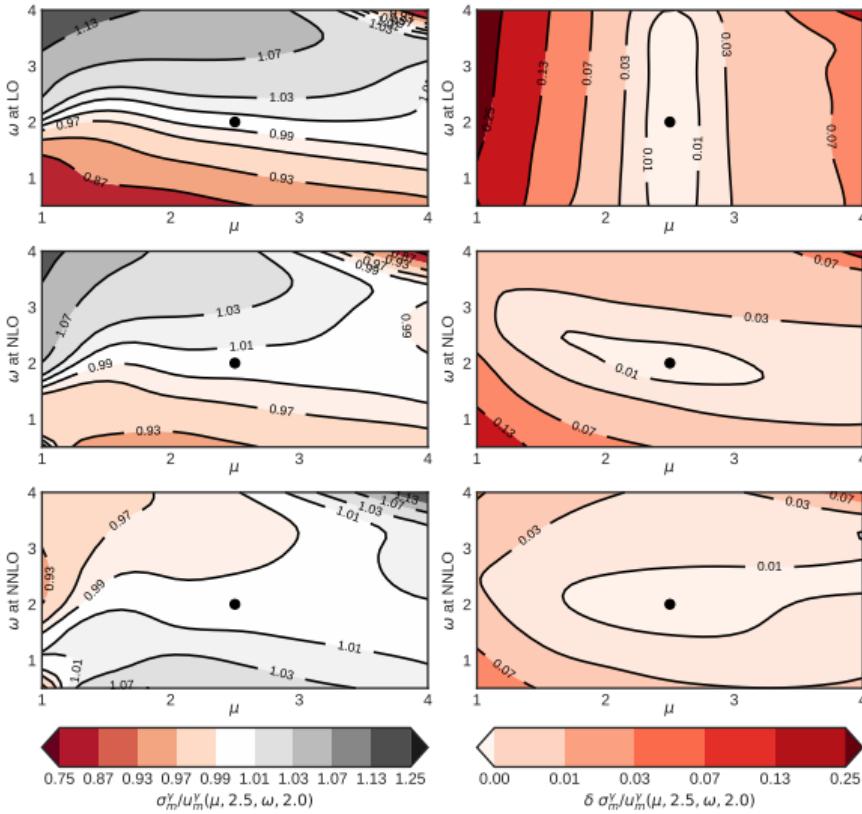
# Results for $Z_m^{(\gamma_\mu)}$



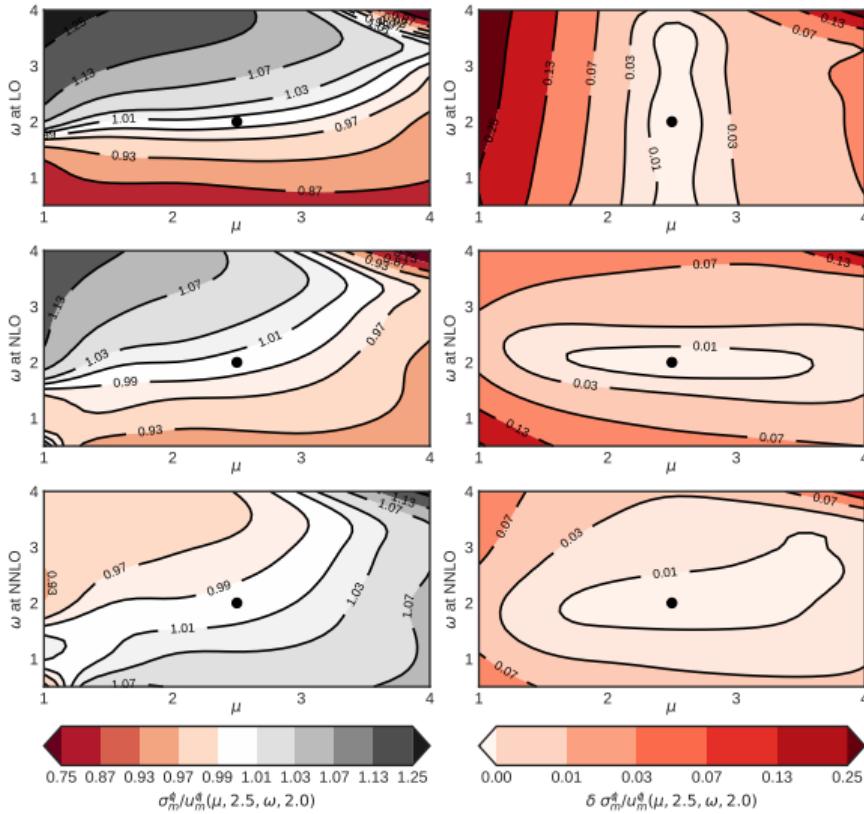
# Results for $Z_m^{(\phi)}$



# Results for $Z_m^{(\gamma\mu)}$



# Results for $Z_m^{(\phi)}$



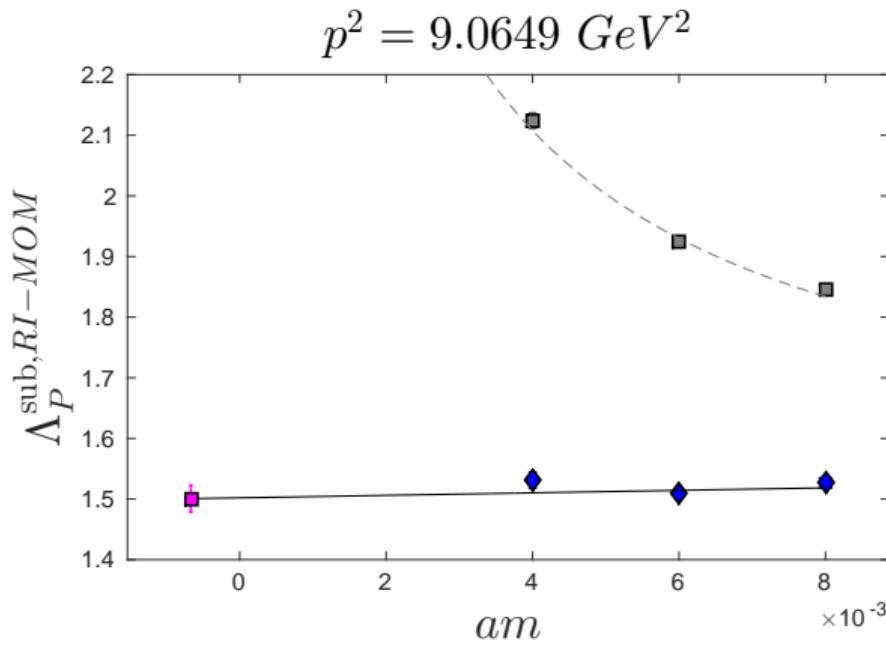
## Pole subtraction

- The Green functions might suffer from IR poles,  $\sim 1/p^2$ , or  $\sim 1/m_\pi^2$  which can pollute the signal
- In principle these poles are suppressed at high  $\mu$  but they appear to be quite important at  $\mu \sim 3$  GeV for some quantities which allow for pion exchanges
- The traditional way is to “subtract” these contamination by hand

# Pole subtraction

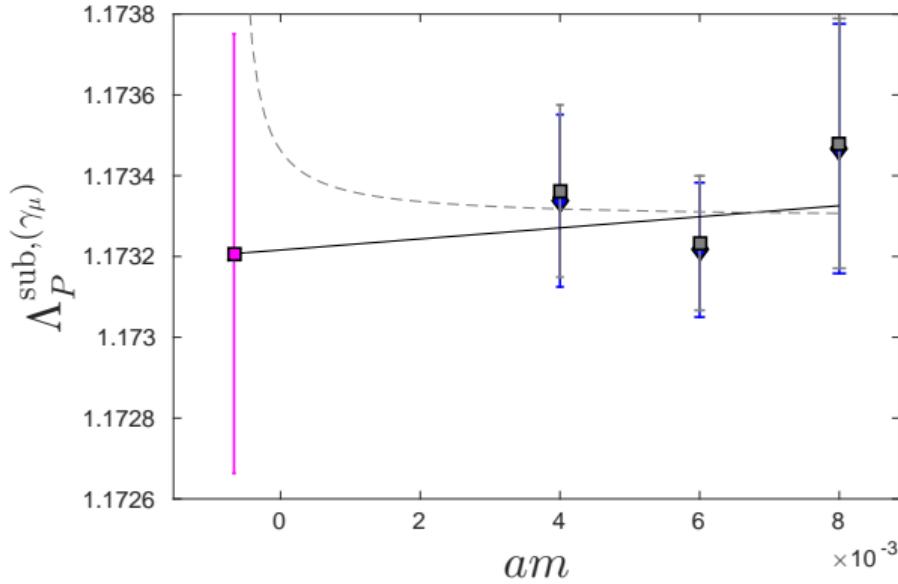
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- In principle these poles are suppressed at high  $\mu$  but they appear to be quite important at  $\mu \sim 3$  GeV for some quantities which allow for pion exchanges
- The traditional way is to “subtract” these contamination by hand
- However these contaminations are highly suppressed in a SMOM scheme, with non-exceptional kinematics
- We argue that this pion pole subtractions is difficult to control and that **schemes with exceptional kinematics should be discarded**

# Pole subtraction (I)



# Pole subtraction (I)

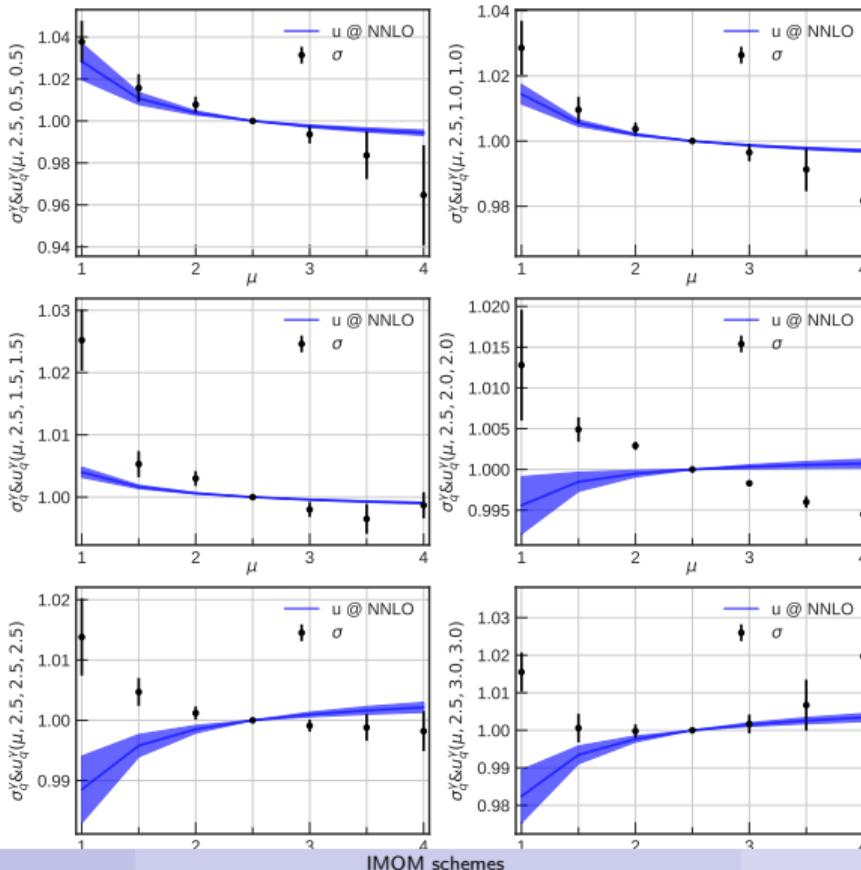
$$p^2 = 9.0649 \text{ GeV}^2$$



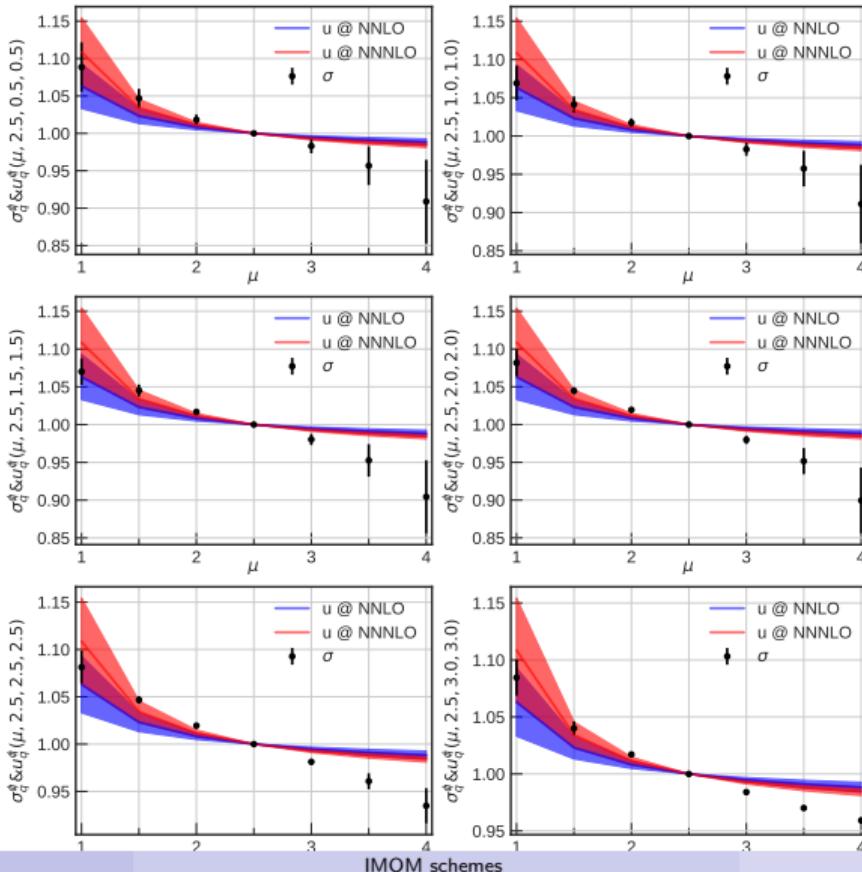
## Pole subtraction (II)

In [Boyle, NG, Hudspith, Lehner, Lytle, '17 (1708.03552)] we did a careful study and argued that the disagreement observed between different computations is due to the renormalisation procedure. We found that the pole-subtraction procedure is prone to systematic errors which are difficult to control

# Results for $Z_q^{(\gamma_\mu)}$



# Results for $Z_q^{(\phi)}$



What is going on for  $Z_q$  ?

# Results for $Z_q$

Scheme	LO	NLO	NNLO	NNNLO	NP
$\overline{\text{MS}}$	1.0	1.0048	1.0062	1.0064	
$\overline{\text{MS}} \leftarrow \gamma_\mu$	1.0	1.0069	1.0078	N.A.	
$\overline{\text{MS}} \leftarrow \not{q}$	1.0	1.0195	1.0175	1.0146	
$\gamma_\mu$	1.0	1.0017	1.0020	N.A.	1.0037(20)
$\not{q}$	1.0	1.0048	1.0081	1.0113	1.0195(25)

**Table:** Running between 2 and 2.5 GeV for the quark wave function in  $\overline{\text{MS}}$  and in the SMOM schemes  $\gamma_\mu(\omega = 1)$  and  $\not{q}$ . In this case the running is known at NNNLO.

# Results for $Z_q$

Scheme	NLO-LO	NNLO-NLO	NNNLO-NNLO
$\overline{\text{MS}}$	0.0048	0.0013	0.0003
$\gamma_\mu$	0.0017	0.0003	
$\not{q}$	0.0048	0.0033	0.0032

**Table:** Study of the convergence of the perturbative series for running of the quark wave function between 2 and 2.5 GeV in  $\overline{\text{MS}}$ , SMOM- $\gamma_\mu$  and  $\not{q}$ .

# Results for $Z_m$

Scheme	NLO-LO	NNLO-NLO	NNNLO-NNLO
$\overline{\text{MS}}$	-0.0081	-0.0015	-0.0002
$\gamma_\mu$	-0.0126	-0.0054	-0.0040
$\not{q}$	-0.0096	-0.0026	-0.0017

**Table:** Study of the convergence of the perturbative series for running of the quark mass between 2 and 2.5 GeV in  $\overline{\text{MS}}$ , SMOM- $\gamma_\mu$  and  $\not{q}$ .

# Results for $Z_m$

Scheme	LO	NLO	NNLO	NNNLO	NP
$\overline{\text{MS}}$	0.9537	0.9456	0.9441	0.9439	
$\overline{\text{MS}} \leftarrow \gamma_\mu$	0.9537	0.9350	0.9389	0.9426	
$\overline{\text{MS}} \leftarrow \not{q}$	0.9537	0.9451	0.9462	0.9475	
$\gamma_\mu$	0.9537	0.9411	0.9357	0.9318	0.9307(62)
$\not{q}$	0.9537	0.9441	0.9415	0.9400	0.9436(46)

Table: Running between 2 and 2.5 GeV for the quark mass.

# Results for $Z_q^{(\gamma_\mu)}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.972(8)	0.993(4)	1.008(4)	1.014(8)	1.023(14)	1.040(26)
1.0	0.976(8)	0.994(3)	1.004(2)	1.007(5)	1.012(8)	1.021(15)
1.5	0.978(4)	0.998(2)	1.003(1)	1.005(2)	1.006(3)	1.004(3)
2.0	0.990(7)	0.998(2)	1.003(0)	1.005(1)	1.007(1)	1.008(1)
2.5	0.987(5)	0.997(2)	1.001(1)	1.002(2)	1.002(3)	1.003(4)
3.0	0.985(4)	0.999(2)	1.000(2)	0.998(4)	0.993(9)	0.978(18)
3.5	0.989(5)	1.001(2)	0.997(2)	0.993(6)	0.982(13)	0.959(27)
4.0	0.990(5)	0.999(1)	0.994(3)	0.983(8)	0.957(22)	0.887(60)

# Results for $Z_q^{(\phi)}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.935(27)	0.974(7)	1.018(7)	1.035(16)	1.059(31)	1.096(56)
1.0	0.951(19)	0.978(6)	1.017(6)	1.035(14)	1.058(28)	1.096(51)
1.5	0.950(15)	0.973(4)	1.017(4)	1.037(11)	1.064(25)	1.106(49)
2.0	0.942(15)	0.976(3)	1.020(3)	1.040(8)	1.069(20)	1.118(45)
2.5	0.942(14)	0.974(3)	1.019(1)	1.039(3)	1.060(9)	1.087(20)
3.0	0.937(12)	0.978(4)	1.017(2)	1.033(2)	1.048(2)	1.060(2)
3.5	0.943(10)	0.979(3)	1.014(2)	1.027(3)	1.039(3)	1.050(2)
4.0	0.940(10)	0.975(4)	1.013(3)	1.027(8)	1.046(18)	1.084(40)