Towards the phase diagram of cold and dense heavy QCD

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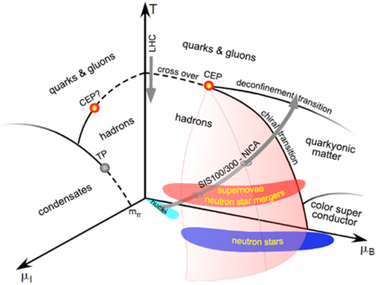
Motivation

- Why effective theory?
  - ✓ At Re $\mu_B \neq 0$ sign problem appears.
  - ✓ Theory reproduces main characteristics of LQCD in a relevant parameter region.
- Why mean-field approximation (MFA)?
  - ✓ Allows an analytic access.
  - ✓ Delivers correct results for many lattice systems.[a]
- In the end: **Implementing** μ_B and μ_I into the MFA-theory (cf. Fig. on the right).[b]



[a] Linda E. Reichl. *A Modern Course in Statistical Physics. Fourth revised, Edition*. Wiley-VCH, Verlag Gmbh, 2016.
[b] *QCD-phase diagram*. https://www.cbm.gsi.de/physics. Accessed: 2022-05-05.

Effective Action

- Starting point:

$$\mathcal{Z} = \prod_{f=1}^{N_f} \int [\mathrm{d}U_\mu] \det Q^f \, e^{-S_g} \tag{1}$$

$$= \prod_{f=1}^{N_f} \int [\mathrm{d}U_0] \int [\mathrm{d}U_i] \det Q^f e^{-S_g}$$

$$\Rightarrow \mathcal{Z}_{\text{eff}} = \int [\mathrm{d}U_0] \, e^{-S_{\text{eff}}}$$

- Effective action:

$$\boxed{-S_{\text{eff}} = \ln \int [\mathrm{d}U_i] \left[\det Q^f \, e^{-S_g}\right]} \tag{2}$$

- Advantage: We can define $\int [\mathrm{d}U_0] \to \int [\mathrm{d}L]$ as an integration measure, where $L(\vec{\mathbf{x}}) = \mathrm{Tr}\, W(\vec{\mathbf{x}}) = \mathrm{Tr} \prod_{\tau=0}^{N_\tau - 1} U_0(\vec{\mathbf{x}}, \tau)$.

Deriving of the Actions

Fermion Action

- WD-Action:

$$S_W\left[\psi, \overline{\psi}, U\right] = \sum_{f=1}^{N_f} \sum_{x,y} \psi^f(x) Q^f(x,y) \psi(y) \tag{3}$$

with the WD-Operator

$$Q^f(x,y) \equiv \mathbb{1} - \kappa_f H(x,y). \tag{4}$$

- Here $\kappa_f = \left(2am_f + 8\right)^{-1}$ is the so called *hopping parameter* and

$$H(x,y) = \underbrace{T^+_{xy} + T^-_{xy}}_{\equiv T} + \sum_{i=1}^{3} \left[\underbrace{S^+_{xy,i} + S^-_{xy,i}}_{\equiv S}\right]$$

is the Hopping-Matrix

-

$$\det Q_f(x,y) = \underbrace{\det\left(\mathbb{1} - \kappa\, T\right)}_{\det Q_{\text{stat}}} \underbrace{\det\left(\mathbb{1} - \frac{\kappa\, S}{\mathbb{1} - \kappa\, T}\right)}_{\det Q_{\text{kin}}} \tag{5}$$

Reminder..

$$\boxed{-S_{\text{eff}} = \ln \int [\mathrm{d}U_i] \left[\det Q^f\, e^{-S_g}\right]}$$

Gluon Action

- Character-Expansion:

$$-S^g_{\text{eff}} = \sum_{\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle} \ln \left[1 + \lambda_1 \left(L^\star_{\vec{\mathbf{x}}} L_{\vec{\mathbf{y}}} + L_{\vec{\mathbf{x}}} L^\star_{\vec{\mathbf{y}}}\right)\right]. \tag{6}$$

- The expansion parameter is here $a_f \equiv u$ and the effective coupling is $\lambda_1 \equiv u^{N_\tau} + \mathcal{O}\left(u^{N_\tau + 4}\right)$.

Reminder..

Final Partition Function

$$\boxed{\mathcal{Z}_{\text{eff}} = \int [\mathrm{d}U_0]\, e^{-S_{\text{eff}}}}$$

- Final expression[1] for the partition function (cf. Eq. (1)) :

$$\mathcal{Z}_{\text{eff}} = \int \mathrm{d}L \underbrace{\prod_{\langle \vec{\mathbf{x}},\vec{\mathbf{y}}\rangle} \log\left[1 + \lambda_1 \left(L^\star_{\vec{\mathbf{x}}} L_{\vec{\mathbf{y}}} + L_{\vec{\mathbf{x}}} L^\star_{\vec{\mathbf{y}}}\right)\right]}_{\text{pure gauge}} \tag{7}$$

$$\underbrace{\times \prod_{\vec{\mathbf{x}}} \left(1 + h_1 L_{\vec{\mathbf{x}}} + h_1^2 L^\dagger_{\vec{\mathbf{x}}} + h_1^3\right)^2 \left(1 + \bar{h}_1 L^\dagger_{\vec{\mathbf{x}}} + \bar{h}_1^2 L_{\vec{\mathbf{x}}} + \bar{h}_1^3\right)^2}_{\text{static}}$$

$$\underbrace{\times \prod_{\langle \vec{\mathbf{x}},\vec{\mathbf{y}}\rangle} \left(\mathbb{1} - h_2 \operatorname{Tr} \frac{h_1 W_{\vec{\mathbf{x}}}}{\mathbb{1} + h_1 W_{\vec{\mathbf{x}}}} \operatorname{Tr} \frac{h_1 W_{\vec{\mathbf{y}}}}{\mathbb{1} + h_1 W_{\vec{\mathbf{y}}}}\right) \left(\mathbb{1} - h_2 \operatorname{Tr} \frac{\bar{h}_1 W^\dagger_{\vec{\mathbf{x}}}}{\mathbb{1} + \bar{h}_1 W^\dagger_{\vec{\mathbf{x}}}} \operatorname{Tr} \frac{\bar{h}_1 W^\dagger_{\vec{\mathbf{y}}}}{\mathbb{1} + \bar{h}_1 W^\dagger_{\vec{\mathbf{y}}}}\right)}_{\text{kinetic}}$$

- Effective quark coupling

$$\begin{aligned} h_1(\mu, N_\tau) &\equiv (2\kappa)^{N_\tau} e^{N_\tau a\mu} \\ \bar{h}_1(\mu, N_\tau) &\equiv (2\kappa)^{N_\tau} e^{N_\tau a\mu} . \end{aligned} \tag{8}$$

and the nearest neighbour coupling constant $h_2(\kappa, N_\tau) \equiv \frac{\kappa^2 N_\tau}{N_c}$.

[1] Owe Philipsen and Jonas Scheunert. “QCD in the heavy dense regime for general N_c: on the existence of quarkyonic matter”. In: *JHEP* 11 (2019), p. 022. DOI: 10.1007/JHEP11(2019)022. arXiv: 1908.03136 [hep-lat].

Implementing Mean-Field into the Theory

- Ansatz[2]:

$$L = \overline{L} + \delta L,$$
$$L^\star = \overline{L} + \delta L^\star$$

- Expand action in linear order:

$$S_{\text{eff}}[L] \approx S\left[\overline{L}\right] + \frac{\partial S_{eff}}{\partial L}|_{\overline{L}} \delta L + \frac{\partial S_{eff}}{\partial L}|_{\overline{L}} \delta L^\star + \dots$$

- Then we get:

$$\mathcal{Z} \approx f\left(\overline{L}\right) \int \mathrm{d}L \exp\left\{ \frac{\partial S_{\text{eff}}}{\partial \overline{L}} L + \frac{\partial S_{\text{eff}}}{\partial \overline{L}^\star} L^\star \right\} \tag{9}$$

[2] "Mean field and Monte Carlo studies of SU(N) chiral models in three dimensions". In: *Nuclear Physics B* 200.1 (1982), pp. 211–231. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(82)90065-7. URL: https://www.sciencedirect.com/science/article/pii/0550321382900657.

Example: Single Site (s.s) Pure Gauge Action

- Pure gauge action (cf Eq. (6)) for the nearest neighbour (NN) interactions:

$$S_{\mathrm{p.g.,NN}} = \sum_{\vec{\mathbf{x}}} S\left(L_{\vec{\mathbf{x}}}, L^{\dagger}_{\vec{\mathbf{x}}}\right) = -\sum_{\vec{\mathbf{x}}} \sum_{i=1}^{d} \log\left[1 + \lambda_1 \left(L_{\vec{\mathbf{x}}} L^{\dagger}_{\vec{\mathbf{x}}+i} + L^{\dagger}_{\vec{\mathbf{x}}} L_{\vec{\mathbf{x}}+i}\right)\right] \tag{10}$$

- Insert the MF Ansatz for the Polyakov-Loop $L_{\vec{\mathbf{x}},i} = \overline{L}_{\vec{\mathbf{x}}} + \delta L_{\vec{\mathbf{x}},i}$ and $L^{\star}_{\vec{\mathbf{x}},i} = \overline{L}_{\vec{\mathbf{x}}} + \delta L^{\star}_{\vec{\mathbf{x}},i}$ in Eq. (10). Finally with Eq. (9), we also can calculate s.s. free energy

$$\mathcal{F}_{ss} = -\log\left[\int \mathrm{d}L \exp\left\{\frac{2d\lambda_1 \overline{L}^{\star}}{1 + 2\lambda_1 \left|\overline{L}\right|^2} L + \frac{2d\lambda_1 \overline{L}}{1 + 2\lambda_1 \left|\overline{L}\right|^2} L^{\star}\right\}\right]. \tag{11}$$

Solving $SU(3)$ Integrals Numerically

- In this work the formula

$$\mathcal{I}(n,m) = \sum_{j=\max\left(0,\frac{n-m}{3}\right)}^{\lfloor \frac{n}{3} \rfloor} \frac{T(n-m)\,2n!\,m!\,\binom{3\left(n-j-\frac{n-m}{3}+1\right)}{n-3j}\binom{2j-\frac{n-m}{3}}{j}}{\left(n-j-\frac{n-m}{3}+1\right)!\left(n-j-\frac{n-m}{3}+2\right)!\left(2j-\frac{n-m}{3}\right)!} \tag{12}$$

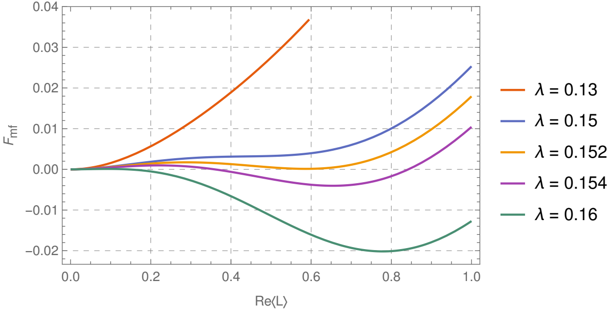
  was used for calculating the Polyakov-Loop integrals[3].
- A few examples (for more, see Ref.[4] (App. B)):

$$\int \mathrm{d}L\, L^3 = 1$$
$$\int \mathrm{d}L\, L^6 = 5$$
$$\int \mathrm{d}L\, L^2 \left(L^\dagger\right)^5 = 11$$

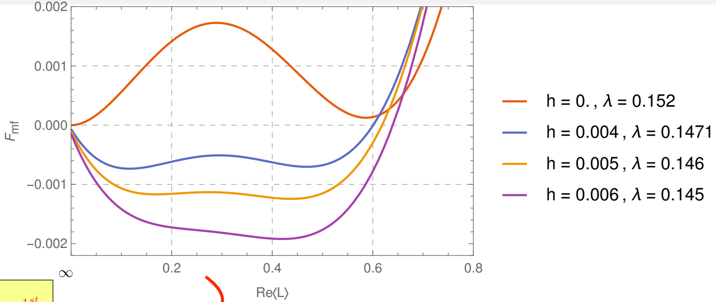
[3] Christof Gattringer. "Flux representation of an effective Polyakov loop model for QCD thermodynamics". In: *Nucl. Phys. B* 850 (2011), pp. 242–252. DOI: 10.1016/j.nuclphysb.2011.04.018. arXiv: 1104.2503 [hep-lat].

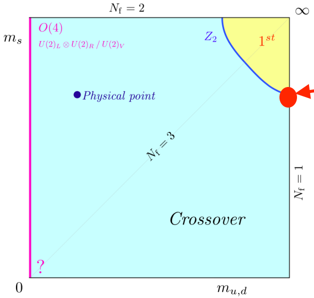
[4] S. Uhlmann, R. Meinel, and A. Wipf. "Ward identities for invariant group integrals". In: *J. Phys. A* 40 (2007), pp. 4367–4390. DOI: 10.1088/1751-8113/40/16/008. arXiv: hep-th/0611170.

Deconfinement Transition: Pure Gauge



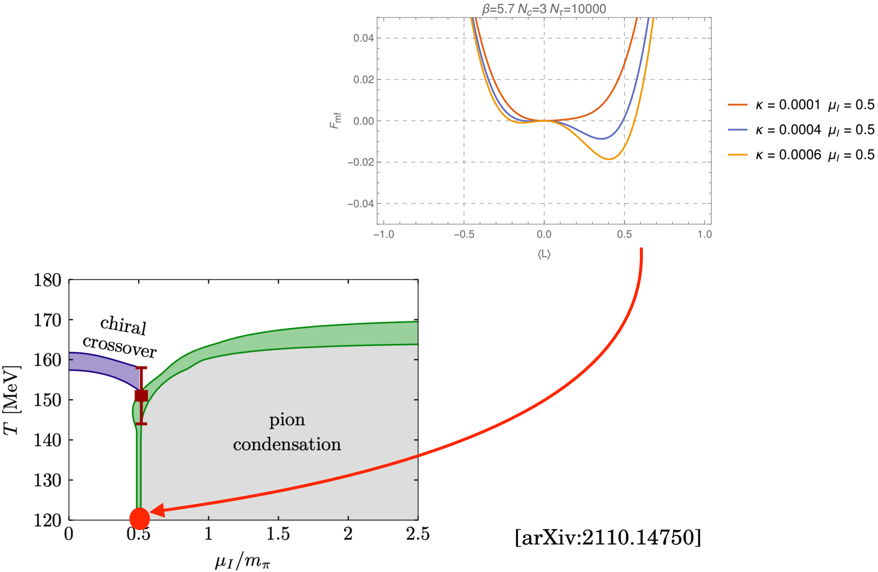
Deconfinement Transition: Static Determinant and Pure Gauge for $N_f = 1$





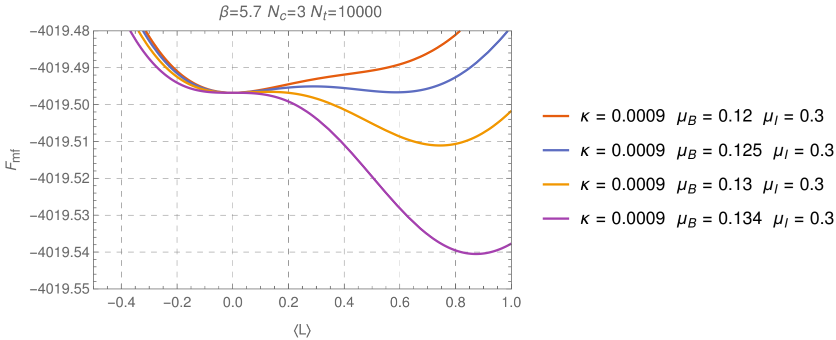
[Philipsen et al. PoS LAT 21]

Adding Isospin for $T \to 0$

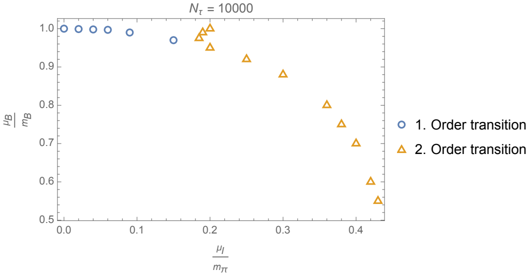


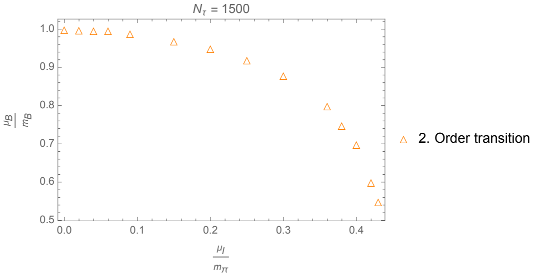
[arXiv:2110.14750]

Adding Isospin And Baryonic Potential for $T \to 0$



(μ_I, μ_B) Phase Diagram: Different Temperatures





Summary

- MFA produces qualitative results
- Deconfinement transition
- (μ_I, μ_B)-phase diagram:
  - 1. Case $\mu_B \neq 0$, $\mu_I = 0$ for $T \to 0$: Baryon Onset $\to$ 1st order transition .
  - 2. Case $\mu_B = 0$, $\mu_I \neq 0$ for $T \to 0$: Leads to 2nd order phase transition.
  - 3. Case $\mu_B \neq 0$ and $\mu_I \neq 0$ for $T \to 0$: 1st **and** 2nd order phase transition meat to in a tricritical endpoint.