Equation of state of a hot-and-dense QGP: lattice simulations at $\mu_B > 0$ vs extrapolations

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Introduction

Most results on $\mu_B > 0$ rely on extrapolation from $\mu_B = 0$ or $\mu_B^2 \leq 0$
Usually rely on some kind of expansion, e.g. Taylor series in μ_B
Goal: Quantitatively check the reliability of some extrapolation schemes

Two steps

- Direct results for $\mu_B > 0$ (reweighting with no overlap problem)
- Reimplement extrapolation schemes with the same setup

Lattice setup

- 2stout $N_\tau = 8$ (tree-level improvement on p takes it close to cont.)
- Aspect ratio $LT = 2$

Formulas are schematic
$\hat{\mu} \equiv \mu/T \qquad \hat{n} \equiv n/T^3 \qquad \hat{p} \equiv p/T^4$
$\det M_l(m_l, \hat{\mu}_l = \hat{\mu}_B/3, U)^{1/2} \det M_s(m_s, \hat{\mu}_s = 0, U)^{1/4} \to \det M(\hat{\mu}_B)$

Reweighting from $\mu_B = 0$

Simulate at $\mu_B = 0$. Use:

$$\Delta\hat{p} \equiv \hat{p}(\hat{\mu}_B) - \hat{p}(0) = \frac{1}{(LT)^3} \ln \left\langle \frac{\det M(\hat{\mu}_B)}{\det M(0)} \right\rangle_{\mu=0}$$

Two exponential problems:

- sign problem: fluctuating phases of $\frac{\det M(\hat{\mu}_B)}{\det M(0)}$
  $\rightarrow$ bad signal-to-noise ratio
- overlap problem: heavy tail of the distribution of $\frac{\det M(\hat{\mu}_B)}{\det M(0)}$
  $\rightarrow$ unknown systematic error **(not reliable)**

The numerical evidence suggests that the main bottleneck is the overlap problem, and not the sign problem.

Giordano, Kapas, Katz, Nogradi, Pasztor; PRD 102 (2020) 3, 034503 [2003.04355]

Phase reweighting

Corresponds to finite isospin: $\mu_d = -\mu_u$ (no symm. breaking term)
Phases: $\frac{\det M(\hat{\mu}_B)}{|\det M(\hat{\mu}_B)|} = e^{i\theta}$ compact $\rightarrow$ no tails (no overlap prob.) **(reliable)**

Method 1

$$\hat{n}_I = \left\langle \frac{\partial}{\partial \mu_I} \ln |\det M| \right\rangle_{PQ}$$

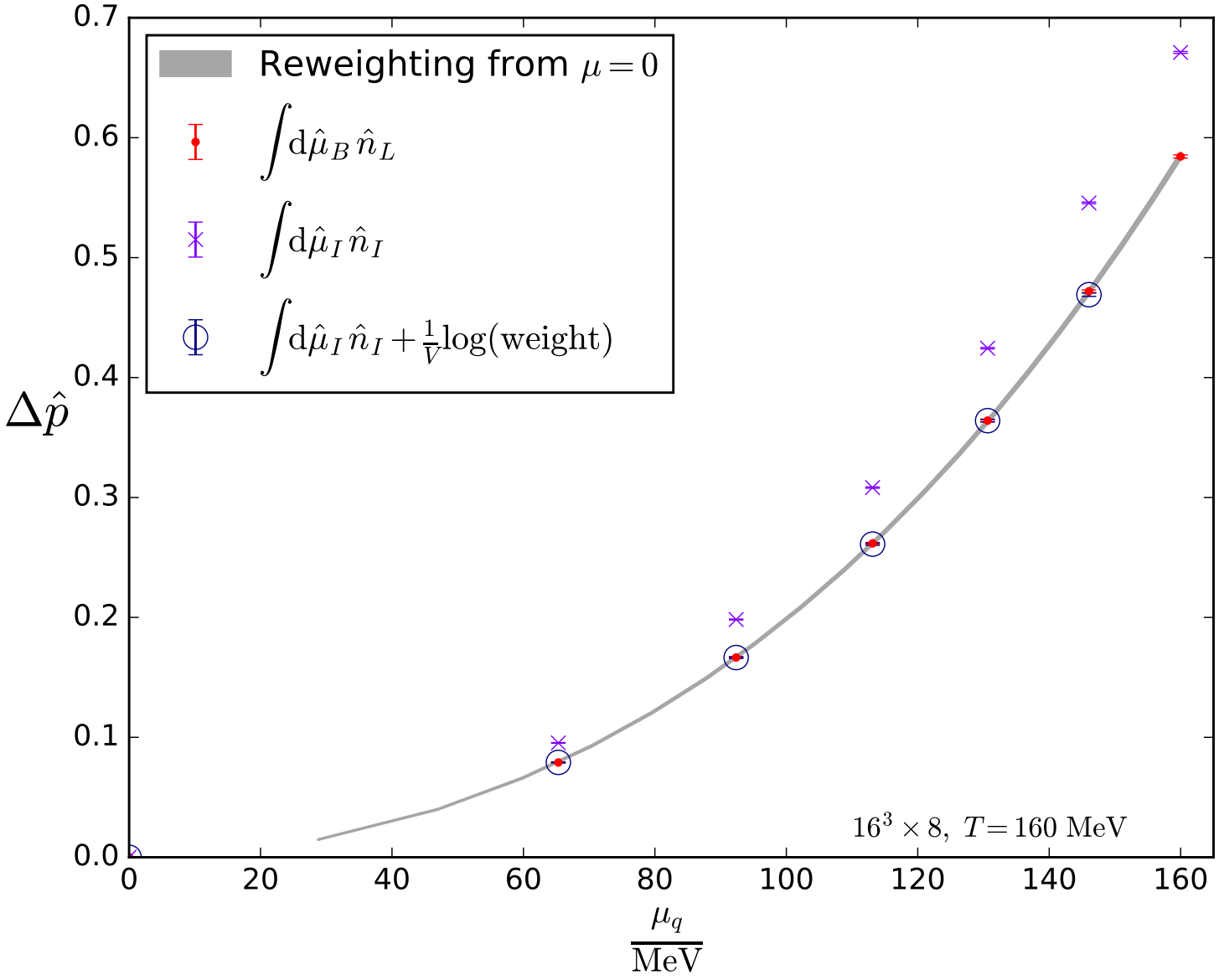
$$\Delta \hat{p} = \int_0^{\mu} \hat{n}_I \, d\hat{\mu}_I + \frac{1}{(LT)^3} \ln \left\langle e^{i\theta} \right\rangle_{PQ}$$

Method 2

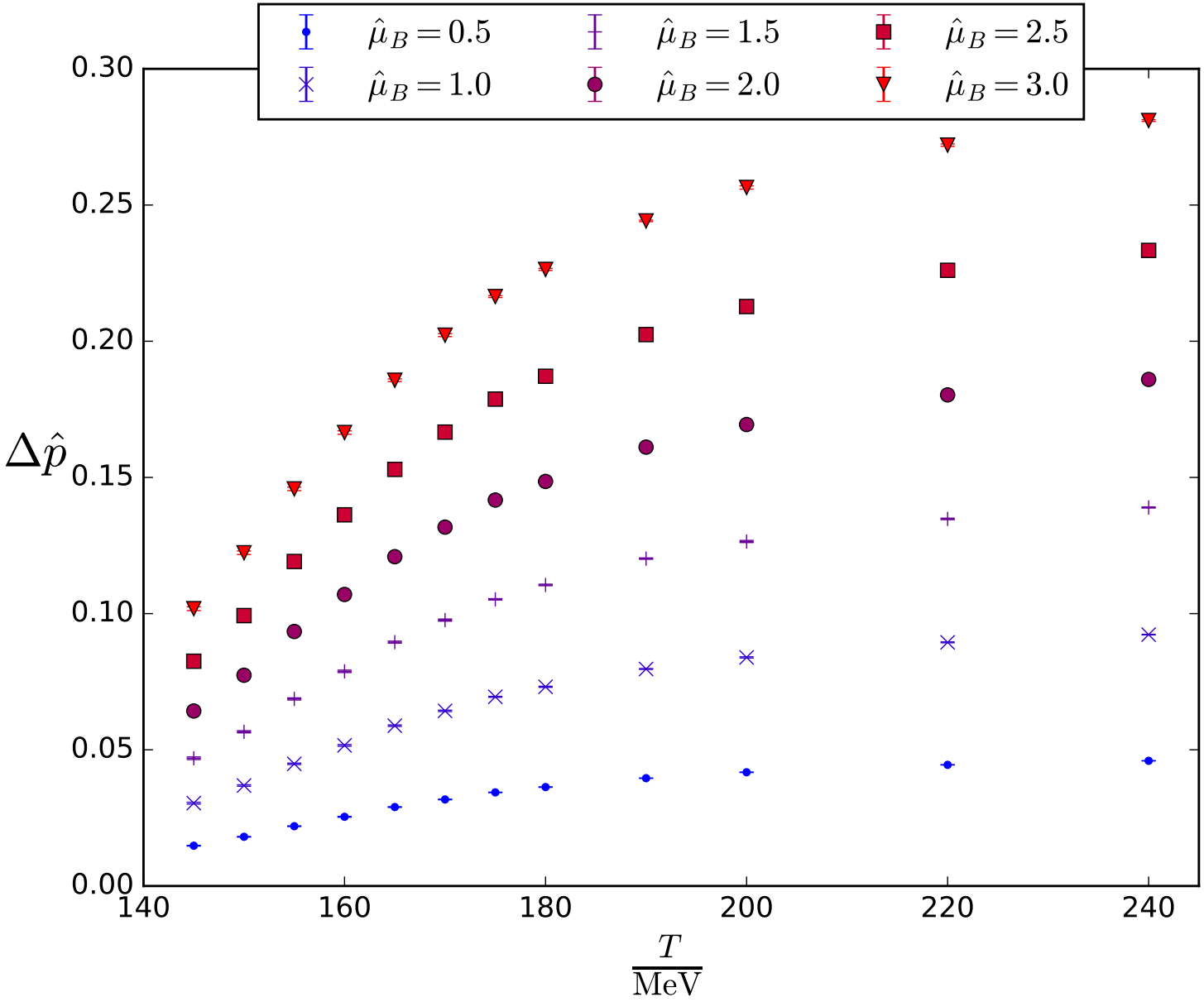
$$\hat{n}_L \equiv \frac{d\hat{p}}{d\hat{\mu}_B} = \frac{1}{(LT)^3 \left\langle e^{i\theta} \right\rangle_{PQ}} \left\langle e^{i\theta} \frac{\partial}{\partial \hat{\mu}_B} \ln \det \mathrm{M} \right\rangle_{PQ}$$

$$\Delta \hat{p} = \int_0^{\mu_B} \hat{n}_L(\hat{\mu}'_B) \, d\hat{\mu}'_B$$

Different reweighting methods on our ensembles



Results for the pressure



The Taylor expansion

$$\Delta\hat{p} = p_2(T)\hat{\mu}_B^2 + p_4(T)\hat{\mu}_B^4 + \dots$$

Two ways to calculate them:

1. From $\mu_B = 0$ simulations. Need polynomials of $\mathcal{D}_n \equiv \left(\frac{\partial^n}{\partial\hat{\mu}_B^n} \ln\det M\right)_{\hat{\mu}_B=0}$
2. Fit to simulations at imaginary chemical potential

Method 1 inherits an overlap problem from reweighting from $\mu_B = 0$ (BUT: on these ensembles reweighting from $\mu_B = 0$ works)
Method 2 has systematic errors from the fit function (BUT: we can compare with Method 1)

Exponential resummation

A truncation of reweighting from $\mu = 0$, where one approximates:

$$\mathcal{D}_n \equiv \left(\frac{\partial^n}{\partial \hat{\mu}_B^n} \ln \det M \right)_{\hat{\mu}_B = 0}$$

$$\frac{\det M(\hat{\mu}_B)}{\det M(0)} \simeq \exp\left(\sum_{n=1}^{N} \frac{1}{n!} \mathcal{D}_n \hat{\mu}_B^n \right)$$

$$\Delta \hat{p} \simeq \frac{1}{(LT)^3} \ln \left\langle \exp\left(\sum_{n=1}^{N} \frac{1}{n!} \mathcal{D}_n \hat{\mu}_B^n \right) \right\rangle_{\mu=0}$$

Note:

- The exponential overlap problem of reweighting comes back (but: for our ensembles, reweighting from $\mu_B = 0$ actually works!)
- No unbiased stochastic estimator known for the exponential (but: we calculate the $\mathcal{D}_n$ exactly, without stochastic estimators!)

Mondal, Mukherjee, Hegde; PRL 128 (2022) 2, 022001 [2106.03165]

Resummations based on shifting sigmoid functions

Defining implicit equation
Instead of fix T, we work at a fixed value for some observable:

$$F\left(T, \hat{\mu}_B\right) = F\left(T'(T, \hat{\mu}_B), 0\right)$$
$$T' = T\left(1 + \#\hat{\mu}_B^2 + \#\hat{\mu}_B^4 + \dots\right)$$

F is some observable that looks like a sigmoid function in T.

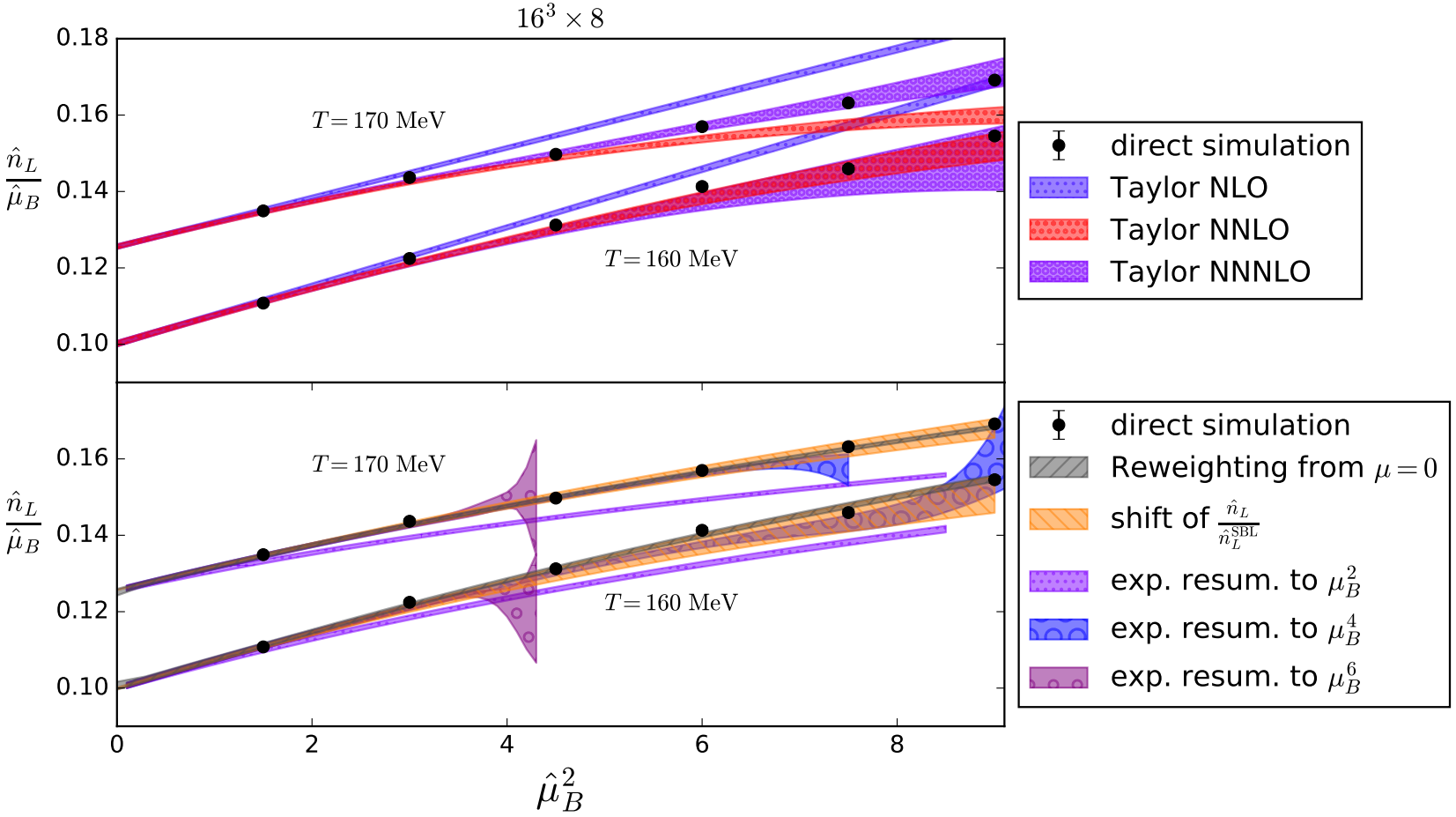
Two schemes

$$F = \frac{\hat{n}_L}{\hat{\mu}_B} \qquad \rightarrow \qquad T' = T\left(1 + \kappa_2\hat{\mu}_B^2 + \kappa_4\hat{\mu}_B^4 + \dots\right)$$
$$F = \frac{\hat{n}_L}{\hat{n}_L^{SBL}} \qquad \rightarrow \qquad T' = T\left(1 + \lambda_2\hat{\mu}_B^2 + \lambda_4\hat{\mu}_B^4 + \dots\right)$$

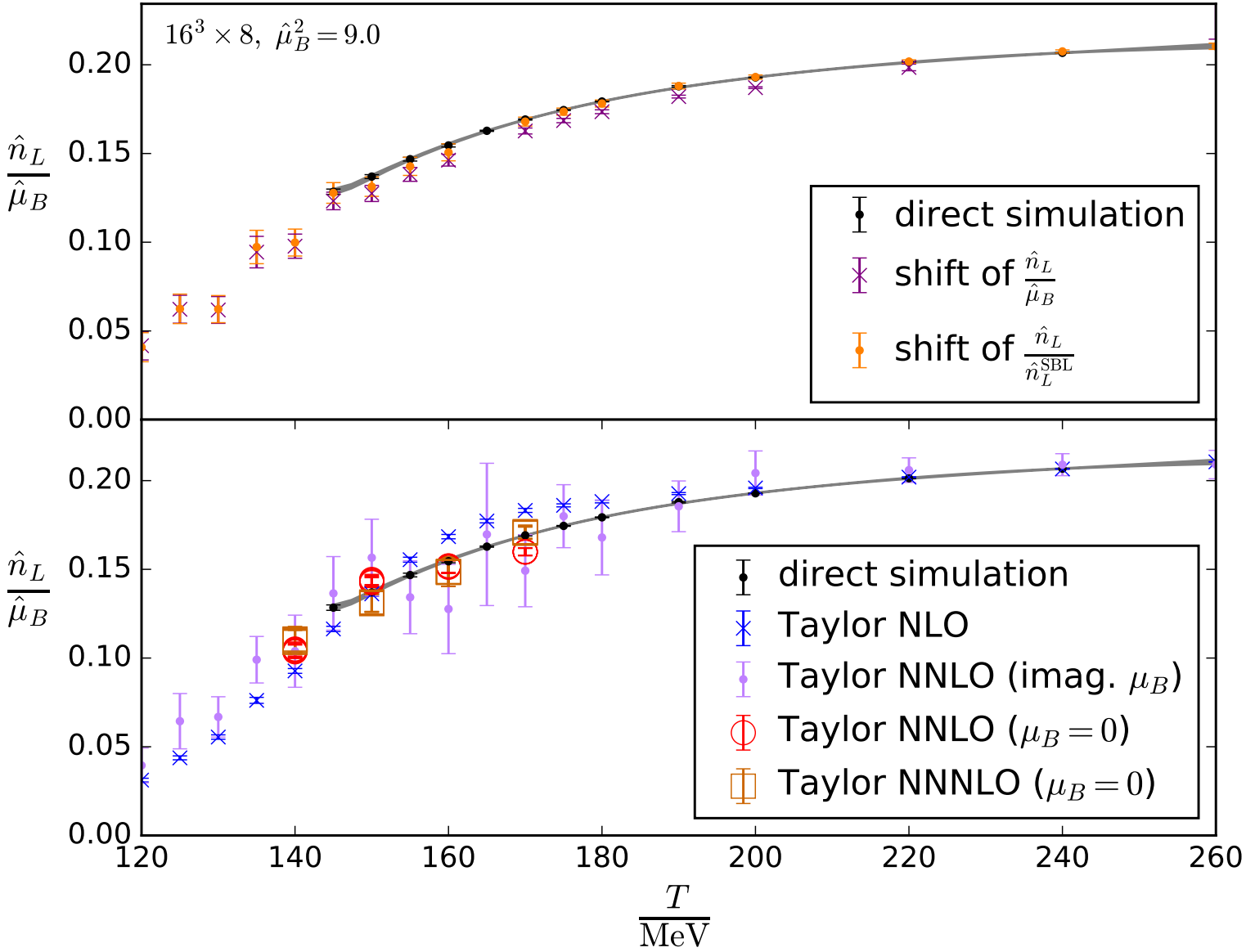
Both can be calculated with $\mathrm{Im}\,\mu_B$ simulations (Jana Guenther's talk)

W-B: PRL126 (2021) 23, 232001 [2102.06660 [hep-lat]]
W-B: PRD 105 (2022) 11, 114504 [2202.05574 [hep-lat]]

Chemical potential scan



Temperature scan



Summary

- Reliable results for $\mu_B > 0$ from phase reweighting
- We compared the direct results with extrapolations for 2stout $16^3 \times 8$ lattices in the range 145 MeV $\leq T \leq$ 240 MeV and $1 \leq \hat{\mu}_B \leq 3$.
- The Taylor expansion appears to converge, but to cover the entire range one needs at least $\mathcal{O}(\mu_B^8)$ in the pressure
- The shifting sigmoid method with the Stefan-Boltzmann correction converges faster, with order λ_4 covering the entire range (this corresponds to $\mathcal{O}\left(\mu_B^6\right)$ in the Taylor)
- The shifting method without the Stefan-Boltzmann correction to order κ_4 works in the crossover region but slightly underestimates the density at high temperatures
- The exponential resummation method shows pathological convergence properties

Reduced matrix formalism

- Hasenfratz, Toussaint '91: for the staggered operator we have:

$$\det M(\hat{\mu}) = e^{3V\hat{\mu}} \prod_{i=1}^{6V} \left(\xi_i - e^{-\hat{\mu}}\right)$$

- the ξ_i are the eigenvalues of the so-called **reduced matrix** P

$$\begin{aligned} P &= -\left(\prod_{i=0}^{N_t-1} P_i\right) L\,, & P_i &= \begin{pmatrix} B_i & 1 \\ 1 & 0 \end{pmatrix}, \\ B_i &= 2\eta_4 (D^{(3)} + am)\big|_{t=i}\,, & L &= \begin{pmatrix} U_4 & 0 \\ 0 & U_4 \end{pmatrix}\Bigg|_{t=N_t-1}, \end{aligned} \tag{1}$$

- Temporal gauge
- The ξ_i depend only on the gauge fields, and not μ
- Given the ξ_i we can calculate the reweighting factors from $\mu = 0$ as well as the $\mathcal{D}_n = \left(\frac{\partial^n}{\partial \hat{\mu}_B^n} \ln \det M\right)_{\hat{\mu}_B=0}$ exactly for each configuration