A New Way Of Resumming QCD At A Finite Chemical Potential

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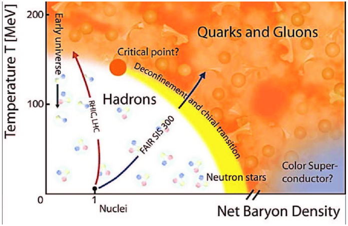
Outline of the talk

- Motivation and Introduction
- Taylor Series Expansion
- Exponential Resummation
- Bias in Exponential Resummation
- Cumulant Expansion
- An unbiased exponential formalism
- Conclusions and Outlook

Motivation and Introduction



- **In principle**, QCD can explain the above phase diagram **completely**
- Unfortunately, in reality, it still remains to be a **conjectured** one
- A **grand canonical** prescription is followed, forming the ensemble along with a grand canonical partition function $\mathcal{Z}(\mu, V, T)$
- The formulation is done by remaining in a **non-perturbative** regime
- The estimates of excess pressure and number density are considered here
- Initially, the setup of calculation is being briefly discussed

Computational Setup

- The present work has made use of **2+1-flavor HISQ** ensembles for three temperatures at T = 135 , 157 and 176 MeV
- The quark masses are tuned to their respective physical values.
- The calculations have been performed on a $32^3 \cdot 8$ lattice for all three T's.
- Within every gauge configuration, the scaled n-point correlation functions $\tilde{D}_n$ are calculated stochastically using random volume sources of $\mathcal{O}(500)$
- We have considered only upto 4 point correlation functions $(1 \leq n \leq 4)$
- The work is done with an ensemble of 20 K gauge field configurations
- The computations are mostly done with μ_B
- A few also done with μ_I as the sign problem is evaded

Taylor Series (TS) Expansion

- In form of a Taylor series (TS) in μ_B, the excess pressure is given by

$$\frac{\Delta P_N^E(T,\mu_B)}{T^4} = \frac{1}{VT^3}\ln\left[\frac{\mathcal{Z}(\mu)}{\mathcal{Z}(0)}\right] = \sum_{n=1}^{N}\frac{\mathcal{X}_{2n}}{(2n)!}\left(\frac{\mu_B}{T}\right)^{2n}$$
$$\text{QNS}:\mathcal{X}_{2n} = \frac{\partial^{2n}}{\partial(\mu_B/T)^{2n}}\left[\frac{\Delta P}{T^4}\right]\Bigg|_{\mu_B=0} \tag{1}$$

- The number density in a Taylor form, is given by

$$\frac{\mathcal{N}}{T^3} = \frac{\partial}{\partial(\mu_B/T)}\left[\frac{\Delta P}{T^4}\right] = \sum_{n=1}^{N}\frac{\mathcal{X}_{2n}}{(2n-1)!}\left(\frac{\mu_B}{T}\right)^{2n-1} \tag{2}$$

- There is a **slow convergence rate** and **non-monotonic** behaviour
- It is therefore essential to calculate TS to **sufficiently high orders** in μ_B
- Calculation of high-order TC are very tedious, computationally expensive
- **Is there any possible way around ?**
- The immediate solution is an **all-ordered resummation** maybe

A brief on Resummation methods

- Primarily, two methods of resummation are briefly enlightened here
- The **Padé resummation** provides important Padé approximants
- They are useful in approximating the behaviour of a function near a given value of the argument
- Which is done by a rational function of an order, equal to that of the TS being approximated
- They may work in certain domains where Taylor approximations fail to converge

▶▶▶ For a **more detailed** explanation, please refer to the TALKS ON

"**Isentropic equation of state in (2+1) flavor QCD**" by **Jishnu Goswami**

"**Multi-point Padé for the study of phase transitions**" by **Francesco Di Renzo**

- The present work is focused on a second method of resummation
- Which is the **exponential resummation** method

Exponential Resummation (ER)

- The resummed estimate to **all orders** in μ_B for $D_n (1 \leq n \leq N)$ is given by

$$\frac{\Delta P_N^R(T, \mu_B)}{T^4} = \frac{1}{VT^3} \ln \left\langle \exp\left(\sum_{n=1}^{N} \overline{D}_n \hat{\mu}_B^n\right)\right\rangle, \quad \overline{D}_n = \frac{1}{N_R}\sum_{r=1}^{N_R} \tilde{D}_n^{(r)} \qquad (3)$$

  where $\langle \cdot \rangle$ is the expectation value over all possible gauge field configurations in an ensemble **generated at $\mu_B = 0$** with N_R random vectors per configuration [S. Mondal, S. Mukherjee, P. Hegde, Phys. Rev. Lett **128**, 022001 (2022)]

- D_n are n-point correlation functions which are **real** for **even** n, **imaginary** for **odd** n, given by

$$\tilde{D}_n = \frac{D_n}{n!} = \frac{1}{n!} \frac{\partial^n}{\partial \hat{\mu}_B^n} \ln \det M\left(T, \hat{\mu}_B\right)\Bigg|_{\mu_B=0} \qquad (\hat{\mu}_B \equiv \mu_B/T). \qquad (4)$$

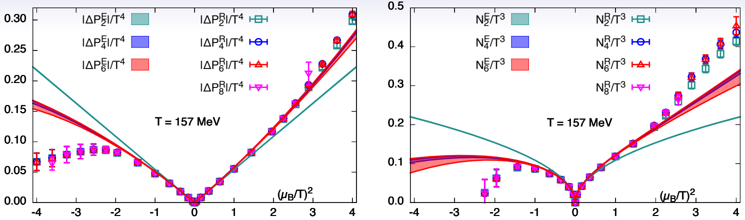
- Hence, $\Delta P_N^R(T, \hat{\mu}_B) = \Delta P_N^E(T, \hat{\mu}_B) + \sum_{n>N}^{\infty} \langle \overline{D}_1^i \overline{D}_2^j \ldots \overline{D}_N^k \rangle \, \hat{\mu}_B^n$, where $1 \cdot i + 2 \cdot j + \cdots + N \cdot k = n$.

A major but hidden problem: Bias

- The highly useful ER suffers from the emergence of **biased estimates**
- That is purely by virtue of the way the series is constructed
- In principle, they manifest when the derivative estimate from **at least one** random vector, is raised **at least** to **quadratic** integral powers

$$\left(\overline{D}_n\right)^m = \left[\frac{1}{N_R}\sum_{r=1}^{N_R} D_n^{(r)}\right]^m = \left[\left(\frac{1}{N_R}\right)^m \sum_{r_1=1}^{N_R} \dots \sum_{r_m=1}^{N_R} D_n^{(r_1)} \dots D_n^{(r_m)}\right]$$
$$\approx \text{Biased estimate} + \sum_{r_1 \neq \dots \neq r_m}^{N_R} \dots \sum^{N_R} D_n^{(r_1)} \dots D_n^{(r_m)} \tag{5}$$

- These effects can prove to be very drastic in the long run involving
  - **Large values** of μ
  - **Higher orders** of μ in series expansion
  - **Higher order μ derivatives** of free energy
- It is therefore high time we try to identify and minimise their emergence and subsequent effects in calculations



Plots of pressure (left) and number density (right) [S. Mondal, S. Mukherjee, P. Hegde, Phys. Rev. Lett **128**, 022001 (2022)]

- The different **unbiased powers** of D_n are used for constructing TC in QNS
- There is **no such scope** to introduce unbiased powers within the given formulation of ER with **transcendental functions** being present
- The above pressure and number density plots clearly indicate the significant difference between the two approaches for large orders and values of μ
- This is clearly attributable to biased and unbiased estimates
- Hence, one is motivated to **truncate** the ER series in such a manner so that the truncated series **reproduces** QNS upto $\mathcal{O}(\mu_B^N)$

Cumulant Expansion (CE): Formalism

- Considering $X = \sum_{n=1}^{N} \frac{\mu^n}{n!} \overline{D}_n$, the cumulant expansion (CE) of ER series in eqn. (3) yields (barring the $1/VT^3$ factor)

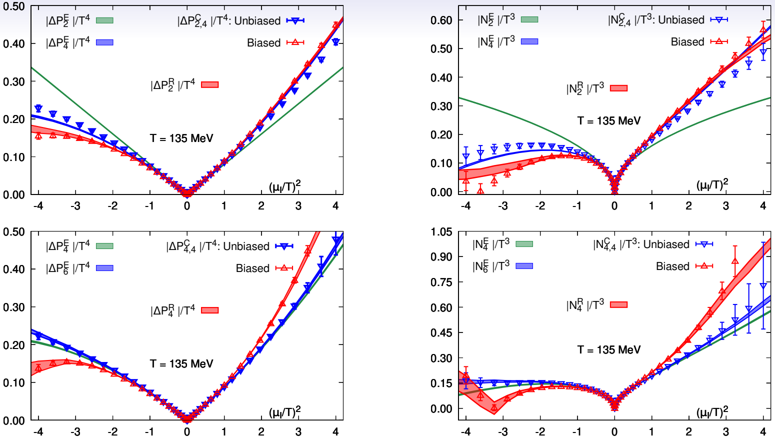
$$\ln \left\langle e^X \right\rangle = \sum_{n=1}^{M} \frac{\kappa_n}{n!} + \mathcal{O}(\kappa_{M+1}) \tag{6}$$

where κ_n is the nth cumulant, N represents the highest derivative order and M is the total number of cumulants

- $|\Delta P^C_{N,M}|/T^4$ and $\mathcal{N}^C_{N,M}/T^3$ are calculated with $M = 4$ with $N = 2, 4$
- The first 4 cumulants in X are represented as follows

$$\begin{aligned}
\kappa_1 &= \langle X \rangle \,, \\
\kappa_2 &= \left\langle X^2 \right\rangle - \langle X \rangle^2 \,, \\
\kappa_3 &= \left\langle X^3 \right\rangle - 3 \left\langle X^2 \right\rangle \langle X \rangle + 2 \langle X \rangle^3 \\
\kappa_4 &= \left\langle X^4 \right\rangle - 4 \left\langle X^3 \right\rangle \langle X \rangle + 12 \left\langle X^2 \right\rangle \langle X \rangle^2 - 6 \langle X \rangle^4 - 3 \left\langle X^2 \right\rangle^2
\end{aligned} \tag{7}$$

[S. Mitra, P. Hegde, and C. Schmidt, (2022), arXiv:**2205.08517**]



- The **higher-order fluctuations** are truly captured by the **unbiased cumulant** estimates, which used to get **suppressed** by ER
- A good agreement is observed between
  - biased cumulant and ER (△ and **red bands**)
  - unbiased cumulant and QNS (▽ and **blue bands**)

Cumulant Expansion: Debacle

- Despite the **all-important** introduction of unbiasedness in the calculations, the cumulant expansion does **deprive** us of some things which are as follows

  1. The reweighting factor
  2. The partition function $\mathcal{Z}$
  3. The phasefactor
  4. The singularities of partition function $\mathcal{Z}$ in the complex $\hat{\mu}_B$ plane

- The obvious search is therefore to **retrieve back** everything lost, but **preserving unbiasedness**

- **Is that achievable** ?

A possible unbiased ER formula: idea

- The idea is to search for an **unbiased counterpart** of ER which would reproduce **unbiased** powers **exactly** upto $\mathcal{O}(\mu^N)$
- In this new formalism, **all** mathematical manipulations are done **with the** the sample of random volume sources **available within every** gauge configuration constituting the configuration ensemble
- In μ **basis**, the new formalism resembles the following shape

$$P^{\mu}_{ub} = \frac{1}{VT^3}\ \ln\ \mathcal{Z}^{\mu}_{ub}, \qquad \mathcal{Z}^{\mu}_{ub} = \left\langle e^{A(\mu)} \right\rangle, \qquad A(\mu) = \sum_{n=1}^{N} \mu^n \frac{\mathcal{C}_n}{n!} \tag{8}$$

where the $\mathcal{C}_n$(different μ coefficients) for n = 1, 2, 3, 4 are given as follows:

Different μ coefficients

$$\mathcal{C}_1 = \overline{D_1},$$

$$\mathcal{C}_2 = \overline{D_2} + \left(\overline{D_1^2} - (\overline{D_1})^2\right),$$

$$\mathcal{C}_3 = \overline{D_3} + 3\left(\overline{D_2 D_1} - (\overline{D_2})\,(\overline{D_1})\right) + \left(\overline{D_1^3} - 3\,(\overline{D_1^2})\,(\overline{D_1}) + 2\,(\overline{D_1})^3\right),$$

$$\begin{aligned}\mathcal{C}_4 = {} & \overline{D_4} + 3\left(\overline{D_2^2} - (\overline{D_2})^2\right) + 4\left(\overline{D_3 D_1} - (\overline{D_3})\,(\overline{D_1})\right) + 6\left(\overline{D_2 D_1^2} - (\overline{D_2})\,(\overline{D_1^2})\right) \\ & - 12\left((\overline{D_2 D_1})\,(\overline{D_1}) - (\overline{D_2})\,(\overline{D_1})^2\right) + \\ & \left(\overline{D_1^4} - 4\,(\overline{D_1^3})\,(\overline{D_1}) + 12\,(\overline{D_1^2})\,(\overline{D_1})^2 - 6\,(\overline{D_1})^4 - 3\,(\overline{D_1^2})^2\right),\end{aligned}$$

- The **simplicity** of this basis is highlighted by the fact that
  the **degree** of the unbiased QNS expansion, being reproduced by this method
  is **exactly identical** to
  the **degree** of the polynomial $A(\mu)$, being exponentiated

Cumulant basis: The second one

- In **cumulant basis**, a new variable W is defined, where

  $W = \sum_{n=1}^{N} \frac{\mu^n}{n!} D_n \neq X$, we have

$$P_{ub}^W = \frac{1}{VT^3} \ln \mathcal{Z}_{ub}^W, \quad \mathcal{Z}_{ub}^W = \left\langle e^{Y(W)} \right\rangle, \quad Y(W) = \sum_{n=1}^{M} \frac{\mathcal{L}_n(W)}{n!} \tag{9}$$

- which would reproduce exactly the first M cumulants in UCE

$$\ln \left\langle e^Y \right\rangle = \sum_{n=1}^{M} \frac{\kappa_n^{ub}}{n!} + \mathcal{O}(\kappa_{M+1}) \tag{10}$$

- **N cumulants** in cumulant basis is **equivalent** to having unbiased powers to $\mathcal{O}(\mu^{\mathbf{N}})$ for $\mu = \mu_{(B,Q,S)}$ and $\mathcal{O}(\mu^{\mathbf{2N}})$ for $\mu = \mu_I$

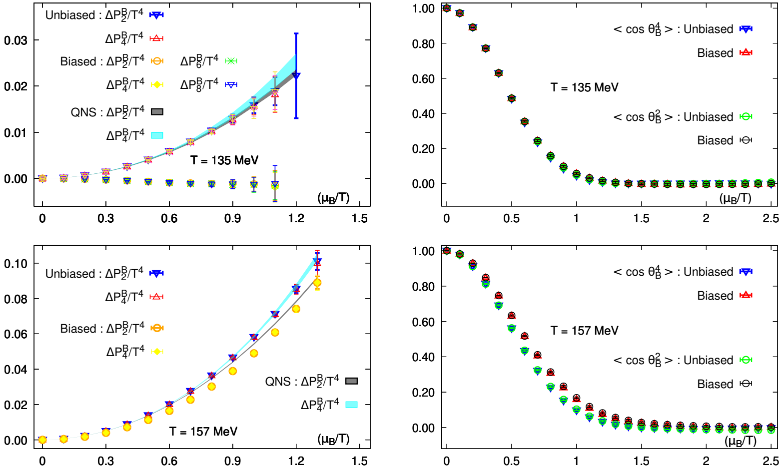
- The first four $\mathcal{L}_n$ in eqn. (9) and κ_n^{ub} in eqn. (10) are explained as follows

$$
\begin{aligned}
\mathcal{L}_1 &= (\overline{W}) \\
\mathcal{L}_2 &= \left[\left(\overline{W^2}\right) - \left(\overline{W}\right)^2\right] \\
\mathcal{L}_3 &= \left[\left(\overline{W^3}\right) - 3\left(\overline{W^2}\right)\left(\overline{W}\right) + 2\left(\overline{W}\right)^3\right] \\
\mathcal{L}_4 &= \Big[\left(\overline{W^4}\right) - 4\left(\overline{W^3}\right)\left(\overline{W}\right) + 12\left(\overline{W^2}\right)\left(\overline{W}\right)^2 \\
&\quad - 6\left(\overline{W}\right)^4 - 3\left(\overline{W^2}\right)^2\Big]
\end{aligned}
\tag{11}
$$

- The κ_n^{ub} are unbiased cumulants resembling eqn. (7) with following transformation : $X^n \Rightarrow U_n[X] \quad \text{for} \quad n = 1, 2, 3, 4$
- $U_n[X]$ is the unbiased nth power of X, (X $= \sum_{n=1}^{N} \frac{\mu^n}{n!}\overline{D}_n$) given by

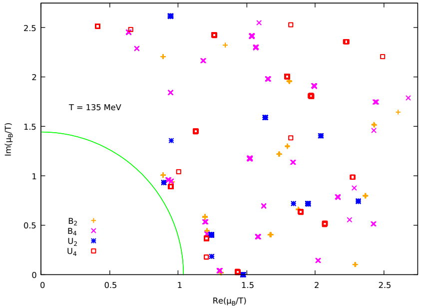
$$
U_n[D_m] = \frac{n!}{\prod_{k=0}^{n-1}(N_R - k)!} \sum_{r_1 \neq}^{N_R} \dots \sum_{\dots \neq r_n}^{N_R} D_m^{(r_1)} \dots D_m^{(r_n)}
$$

- A **faster** rate of convergence with **more** higher-order terms ensure that cumulant basis is the preferred basis to work with



Pressure (left) and phasefactor (right) plots for T = 135 and 157 MeV

Roots of $\mathcal{Z}$



Roots of $\mathcal{Z}_2$ and $\mathcal{Z}_4$ in complex μ_B plane at 135 MeV

- The unbiased formalism **ensures** a newly defined partition function $\mathcal{Z}$
- It is therefore **possible** to search for roots of $\mathcal{Z}$ in the complex $\hat{\mu}_B$ plane
- The **green** outline represents a naive **lower bound** of the roots appearing in the complex $\hat{\mu}_B$ plane

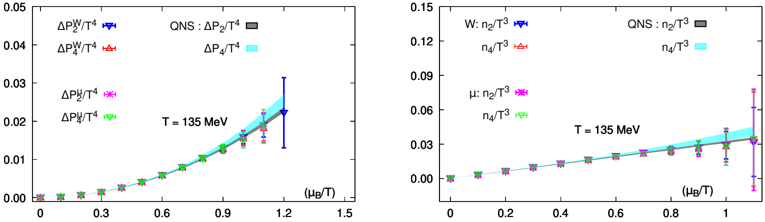
Conclusions and Future Outlook

- A cumulant expansion has been established which duly serves as a bridge between a strict Taylor expansion (QNS) and old exponential resummation
- It has been possible to regulate the degree of unbiasedness at the level of individual cumulants and also at different powers of μ
- The unbiased (partially, in principle) exponential resummation, is guaranteed to provide exact unbiased results upto $\mathcal{O}(\mu^N)$
- Along with a newly defined reweighting factor and $\mathcal{Z}$, it has been possible to re-obtain phasefactor and roots of $\mathcal{Z}$ in the complex μ_B plane
- Most significantly, it gives an **all-ordered unbiased exponentially resummed** series in the limit of a truly **infinite cumulant series**

In future, look for signs of **QCD critical point** by including **higher-order** derivatives

BACKUP SLIDES

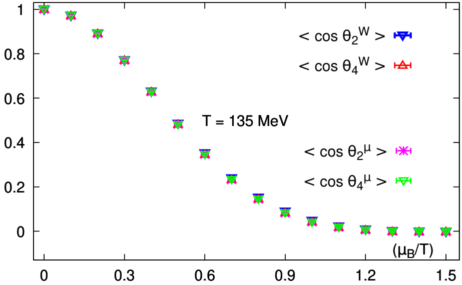
Plots of results from cumulant and μ bases



Pressure and number density plots in cumulant and μ bases for N = 2,4

- Cumulant basis provides **extra** higher order contribution terms over μ basis
- Fortunately, within the set of extra terms, we have **terms** and **counter terms** possibly
- Which **nullifies** the individual fluctuations among one another
- **Faster** convergence in cumulant basis over μ basis
- Making agreement with QNS, so good
- Hallmark of a **genuine series expansion**, where successive higher order contributions are less than the leading order

Phasefactor in cumulant and μ bases



Phasefactor plot at T = 135 MeV (cumulant and μ bases)

- Now, **can** calculate phasefactor
- Quite similar results for phasefactor from both the bases
- Plummeting to zero almost at the **same** value of μ_B/T

Comparison between cumulant and μ bases

- The work on unbiased formalism is primarily done in the cumulant basis, as it provides a faster rate of convergence and a genuine series expansion as compared to μ basis
- The difference due to bias proved to be very acute and qualitatively radical at least in the case of 135 MeV results while working with $\hat{\mu}_B$
- The formalism obviate the bias to some extent, managing to agree along QNS results with negligible higher-order contributions