Quarks and Triality in a Finite Volume

Milad Ghanbarpour

Institut für Theoretische Physik
Justus-Liebig-Universität Gießen

August 9, 2022

Based On:
MG, L. von Smekal, arXiv:2206.11697

Confinement/Deconfinement

- pure gauge theory:
  - first-order transition ($\mathbb{Z}_3$-symmetry):
    Polyakov loop $\langle L\rangle \Longleftrightarrow$ infinitely heavy charge
  - 't Hooft string tension, static quark-anti-quark potential, etc.

Free Energy (Static Quarks)

- free energy difference

$$\mathrm{e}^{-\frac{1}{T}\Delta F_\mathrm{q}} \sim \langle L\rangle$$

- confinement/deconfinement:

$$\Delta F_\mathrm{q} = \begin{cases} < \infty, & \text{deconfined} \\ \infty, & \text{confined} \end{cases}$$

Free Energy (Dynamical Quarks)

- naively:

$$\Delta F_\mathrm{q} \overset{?}{=} -T \ln \frac{Z_{N=1}}{Z_{N=0}}$$

Naive Approach

- fugacity expansion:

$$Z(T, \mu = \mathrm{i}\theta T) = \sum_N \mathrm{e}^{\mathrm{i}N\theta} Z_N(T)$$

- canonical ensemble:

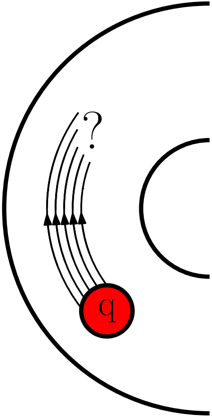
$$Z_N(T) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{-\mathrm{i}N\theta} Z(T, \mathrm{i}\theta T)$$

Problem

- Roberge-Weiss-Symmetry:

$$Z(\theta) = Z\left(\theta + \frac{2\pi}{3}\right)$$

- $Z_N = 0$ for all $N \neq 0 \mod 3$



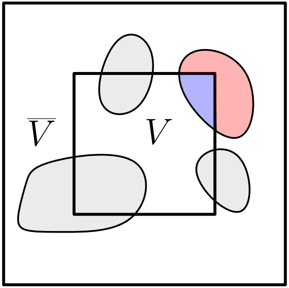
Goal/Objective

Construct ensemble $Z_{N=1,2 \bmod 3}$ of fractional baryon number, e.g.

$$N = 1 \mod 3 = \ldots, -11, -8, -5, -2, 1, 4, 7, 10, \ldots$$

Result

- construction for subvolume V of lattice.
- Flux-Tube Model (easy)
- full QCD (difficult)

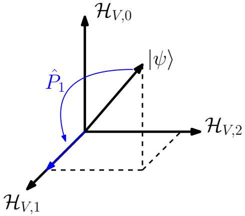


Here

Wilson Plaquette Action + Wilson Fermions (one flavor)

Full QCD (Naive)

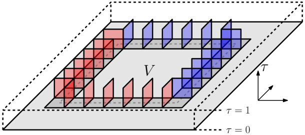
$$Z^{(1)} = \mathrm{tr}_{\mathcal{H}_{V,1}}\left(\mathrm{e}^{\frac{\mu}{T}\hat{N}}\mathrm{e}^{-\frac{1}{T}\hat{H}}\right) = \mathrm{tr}\left(\hat{P}_1\mathrm{e}^{\frac{\mu}{T}\hat{N}}\mathrm{e}^{-\frac{1}{T}\hat{H}}\right)$$



Path Integral (Naive)

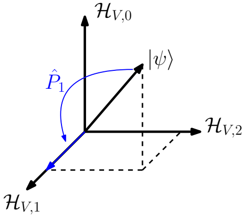
$$Z^{(1)} = \frac{1}{3}\sum_{z\in \mathrm{Z}_3} z^{-1}\int \mathcal{D}U\,\mathrm{e}^{S^z_{\text{Gauge}}}\det M$$

$$\mathrm{ReTr}\left(U_P\right) \to \begin{cases}\mathrm{ReTr}\left(zU_P\right)\\ \mathrm{ReTr}\left(z^{-1}U_P\right)\end{cases}$$



Full QCD (Naive)

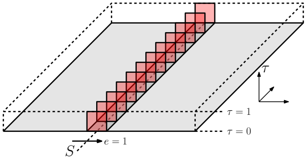
$$Z^{(1)} = \mathrm{tr}_{\mathcal{H}_{V,1}} \left(\mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right)$$
$$= \mathrm{tr} \left(\hat{P}_1 \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right)$$

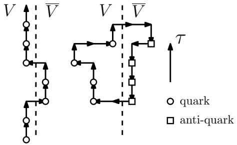


‘t Hooft’s Electric-Flux Ensemble

$$Z^{(1)} = \frac{1}{3} \sum_{z \in \mathrm{Z}_3} z^{-1} \int \mathcal{D}U \, \mathrm{e}^{S^z_{\mathrm{Gauge}}}$$

$$\mathrm{ReTr}\left(U_P\right) \to \mathrm{ReTr}\left(z^{-1} U_P\right)$$





Problem

$$[\hat{P}_1, \hat{H}] \neq 0$$

Solution

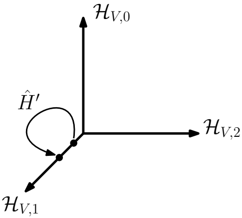
Modify $\hat{H}$ at surface of the subvolume:

$$\hat{H} \longrightarrow \hat{H}'$$

Requirements

$\hat{H}'$ should be

- self-adjoint
- gauge-invariant
- derived from $\hat{H}$ (only modify at surface)
- $[\hat{P}_{0,1,2}, \hat{H}'] = 0$



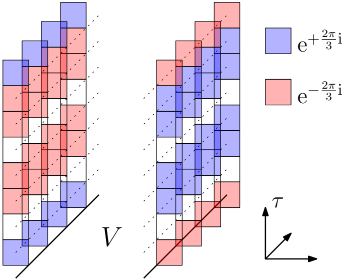
Modified Dynamics

$$Z^{(1)} = \mathrm{tr}\left(\hat{P}_1 \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}'}\right)$$

Path Integral

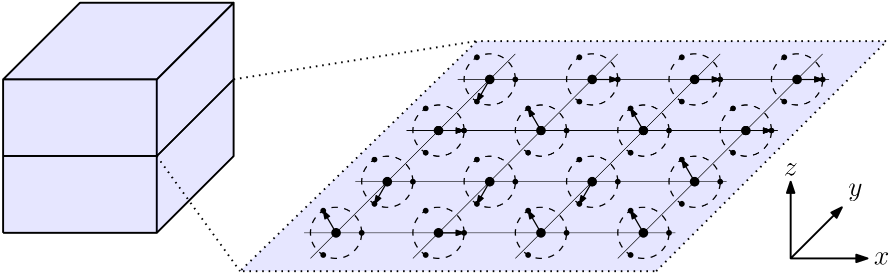
$$Z^{(1)} = \frac{1}{3^{L_4}} \sum_{\{z\}} \left[\prod_{\tau=0}^{L_4-1} z_\tau^{-1}\right] Z(\{z\})$$

$$Z(\{z\}) = \int \mathcal{D}U \, \mathrm{e}^{S_{\mathrm{Gauge}}(\{z\})} \det M$$



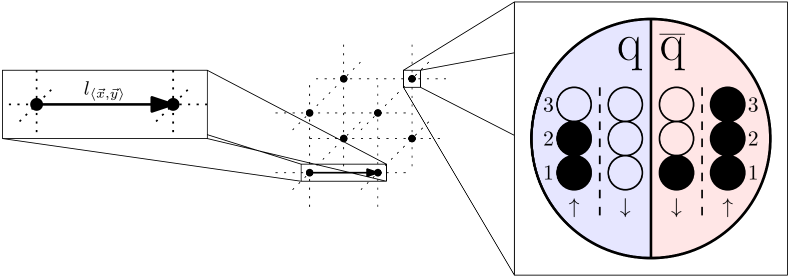
Effective Theory

$$Z_{\text{eff}}(T,\mu) = \frac{1}{3^{N_s}} \sum_{\{z_{\vec{x}}\}} \left(\prod_{\langle \vec{x},\vec{y} \rangle} \left[1 + 2\lambda \operatorname{Re} z_{\vec{x}} z_{\vec{y}}^{*} \right] \right) \prod_{\vec{x}} Q(z_{\vec{x}})$$



Flux-Tube Model[1]

- flux tube (electric fluxes): $l_{\langle\vec{x},\vec{y}\rangle} \in \{1, 0, -1\}$ $(l_{\langle\vec{y},\vec{x}\rangle} = -l_{\langle\vec{x},\vec{y}\rangle})$
- quark and anti-quarks: $\underbrace{n_{\vec{x},\uparrow}, n_{\vec{x},\downarrow}}_{\text{quarks}}, \underbrace{\overline{n}_{\vec{x},\uparrow}, \overline{n}_{\vec{x},\downarrow}}_{\text{anti}-\text{quarks}} \in \{0, 1, 2, 3\}$



---

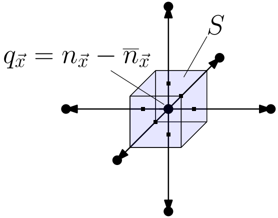
[1] A. Patel, Nucl. Phys. B 243 (1984) 411; C. Bernard et al., Phys. Rev. D 49 (1994) 6051; J. Condella and C. DeTar, Phys. Rev. D 61 (2000) 074023

Hamiltonian

$$H(\{l,n\}) = \sum_{\langle\vec{x},\vec{y}\rangle} \sigma a |l_{\langle\vec{x},\vec{y}\rangle}| + \sum_{\vec{x},s=\uparrow,\downarrow} m(n_{\vec{x},s} + \overline{n}_{\vec{x},s})$$

Local Gauss Law

$$q_{\vec{x}} = \underbrace{\sum_{\vec{y}\sim\vec{x}} l_{\langle\vec{x},\vec{y}\rangle}}_{=\phi_S=\phi_{\vec{x}}} \mod 3$$



Flux-Tube Model

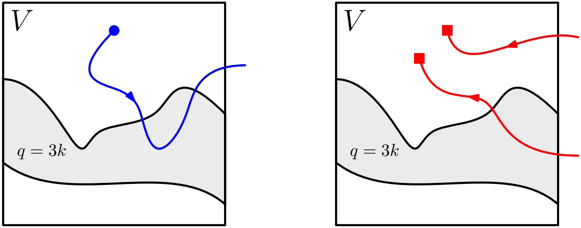
$$Z_{\text{flux}}(T,\mu) = \sum_{\{l,n\}} \mathrm{e}^{-\frac{1}{T}H(\{l,n\})+\frac{\mu}{T}\sum_{\vec{x}} q_{\vec{x}}} \prod_{\vec{x}} \delta(q_{\vec{x}} = \phi_{\vec{x}} \mod 3)$$

Ensemble with Fixed Quark Number

$$Z^{(1)}_{\text{flux}}(T,\mu) = \sum_{\{l,n\}} \ldots \delta(q_V = 1 \mod 3) \prod_{\vec{x}} \delta(q_{\vec{x}} = \phi_{\vec{x}} \mod 3)$$

- fix net quark number in partial volume V, e.g. $q_V = 1 \mod 3$:

$$q_V = \underbrace{\ldots, -11, -8, -5, -2,}_{\text{two additional anti-quarks}} \quad \underbrace{1, 4, 7, 10, \ldots}_{\text{one additional quark}}$$

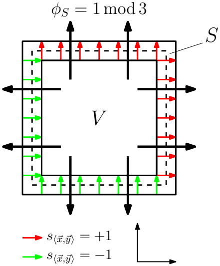


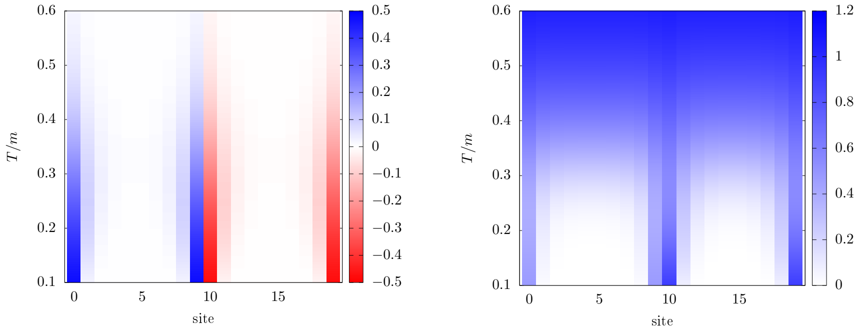
Ensemble with Fixed Quark Number

$$Z^{(1)}_{\text{flux}}(T,\mu) = \frac{1}{3}\sum_{k=0}^{2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} Z_{\text{eff}}(k)$$

Twisted Ensemble

$$Z_{\text{eff}}(k) = \sum_{\{z_i\}} \prod_{\langle\vec{x},\vec{y}\rangle} \left(1 + 2\lambda \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}s_{\langle\vec{x},\vec{y}\rangle}k} z_{\vec{x}} z^*_{\vec{y}}\right]\right) \times \prod_{\vec{x}} Q(z_{\vec{x}})$$





(a) $\mu/m = 0$ (mesonic) (b) $\mu/m = 1/2$ (baryonic)

Figure: $\langle q_i \rangle$ for $L = 20$, $\sigma a/m = 0.3$

Transfer-Matrix Construction[1]

Hilbert Space

$$\underset{\text{gauge links}}{f(U)} \quad \otimes \quad \dots (\hat{\xi}^{\overset{\text{spin}}{\sigma}}_{\underset{\text{quark/anti-quark}}{\mathrm{q}/\overline{\mathrm{q}}}})^{\dagger}_{\underset{\text{site}}{\vec{x}},\underset{\text{color}}{a}} \dots |0\rangle \in \mathcal{H}$$

Gauge Transformation

$$\begin{aligned} & \hat{\varrho}(\Omega)\,|\psi\rangle = |\psi\rangle \\ \Longleftrightarrow \quad & |\psi\rangle \in \mathcal{H}_{\text{phys.}} \end{aligned}$$

Transfer Operator

$$\begin{aligned} & \hat{T}(f(U) \otimes |\psi_F\rangle) \\ & = \int \mathcal{D}U'\, K(U,U')(f(U') \otimes |\psi_F\rangle) \end{aligned}$$

Partition Function

$$Z = \operatorname{tr}\left(\mathrm{e}^{\frac{\mu}{T}\hat{N}}\mathrm{e}^{-\frac{1}{T}\hat{H}}\hat{P}_0\right) \xrightarrow{\text{discretize}} Z = \operatorname{tr}\left(\mathrm{e}^{\frac{\mu}{T}\hat{N}}\hat{T}^{L_4}\hat{P}_0\right)$$

---

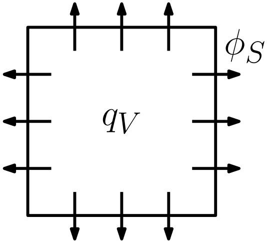
[1] C. Borgs and E. Seiler, Commun. Math. Phys. 91 (1983) 329-380; M. Lüscher, Commun. Math. Phys. 54 (1977) 283-292, F. Palumbo, Nucl. Phys. B 645 (2002) 309-320; V. K. Mitrjushkin, Nucl. Phys. B (Proc. Suppl.) 119 (2003) 326-328

Center Charge and Center Flux

$$\hat{E}^z_{\langle\vec{x},\vec{y}\rangle}\,|\psi\rangle = z^e\,|\psi\rangle$$
$$\hat{Q}^z_{\vec{x}}\,|\psi\rangle = z^q\,|\psi\rangle$$

Flux-Tube States

$$\mathcal{H}_{\text{phys.}} = \bigoplus_{\{q,e\}} \mathcal{H}_{\{q,e\}}$$
$$q_{\vec{x}} + \sum_{\vec{y}\sim\vec{x}} e_{\langle\vec{x},\vec{y}\rangle} = 0 \mod 3$$



$$q_V = -\phi_S \mod 3$$

Projection Operator

$$\hat{P}_{q_V}(e) = \hat{P}_{\phi_S}(-e)$$

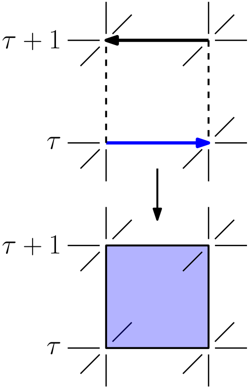
Asymmetric Transfer Operator

$$K(U,U') = S(U,U')T_G(U')T_F(U')$$

Modified Dynamics

$$\hat{T}' = \frac{1}{3}\sum_{z\in \mathrm{Z}_3} \hat{\phi}_S^z \hat{T} \hat{\phi}_S^{z^{-1}}$$

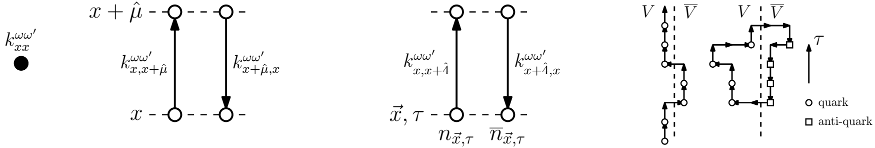
$$\hat{T}'\,\mathcal{H}_{q_V,e} \subseteq \mathcal{H}_{q_V,e}$$



$$Z^{(e)} = \frac{1}{3^{L_4}} \sum_{\{z_\tau\in \mathrm{Z}_3\}} \left[\prod_\tau z_\tau^{-e}\right] \mathrm{tr}\left(\mathrm{e}^{\frac{\mu}{T}\hat{N}}\hat{P}_0 \prod_\tau \left[\hat{\phi}_S^{z_\tau^{-1}}\hat{T}\right]\right)$$

Dualization[1]

$$Z = \sum_{\{k\}} \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, f(k, \overline{\psi}, \psi) \exp\left\{ a\mu \cdot \sum_{x} \left[n_x(k) - \overline{n}_x(k) \right] \right\}$$



Restriction

$$Z^{(e)} = \sum_{\{k\}} \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \ldots \times \left[\prod_{\tau} \delta(q_{V,\tau}(k) = e \mod 3) \right]$$

---

[1] C. Gattringer and C. Marchis, Nucl. Phys. B 916 (2017) 627-646; C. Marchis and C. Gattringer, Phys. Rev. D 97 (2018) 034508

Main Result

Construction of ensembles with fractional baryon number in spatial subvolume V. [arXiv:2206.11697]

Main Features

- Lattice QCD with Wilson fermions + Wilson plaquette action
- construction via dualization and transfer-matrix construction
- Fermion determinant is unchanged. Only plaquettes are twisted.
- Ensemble can be seen as a generalization of ‘t Hooft’s electric flux ensembles.

Outlook

- Is the construction helpful for understanding confinement/deconfinement with dynamical quarks?
- Understand percolation properties of flux tube states? Connection to ensembles with fractional baryon number?