Numerical studies on the finite-temperature CP restoration in 4D SU(N) gauge theory at $\theta = \pi$

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Gauge theory with a theta term

☆ θ term : topological nature of the gauge theory, nonperturbative

$$S_\theta = -i\theta Q = -\frac{i\theta}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) \qquad Z = \int dA\, e^{-S_g + i\theta Q}$$

- topological charge : $Q \in \mathbb{Z}$
- periodicity : $\theta \to \theta + 2\pi$
- CP $(\theta \to -\theta)$ exists not only at $\theta = 0$ but also $\theta = \pi$
- Possible phase structures at $\theta = \pi$ are constrained by 't Hooft anomaly matching.

Prediction by ’t Hooft anomaly matching

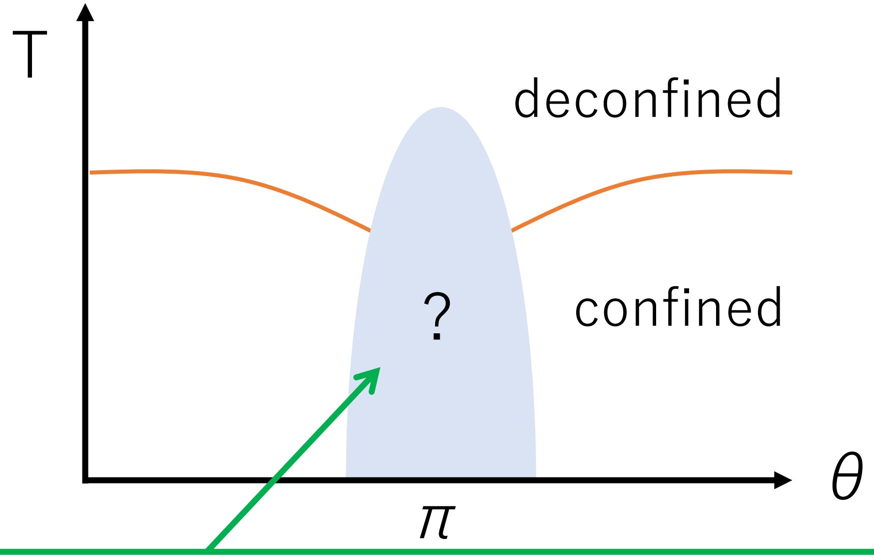
☆ ’t Hooft anomaly matching for 4D SU(N) gauge theory

→ constrain the phase structure at $\theta = \pi$

mixed ’t Hooft anomaly between
CP symmetry & Z_N 1-form center symmetry at $\theta = \pi$

[D. Gaiotto, A. Kapustin, Z. Komargodski, N. Seiberg (2017)]

- SSB of CP
- SSB of $Z_N^{(1)}$
- gapless (CFT)
- topological QFT



determined only for large N, but not for finite N (in particular, N=2)

Phase structure of 4D SU(2) gauge theory

Consider possible (θ , T) phase diagrams for N=2

- two critical temperatures at $\theta = \pi$:

  (1) CP is broken at low temperature $T < T_{CP}$

  indication of CP breaking at T=0 by subvolume method
  [R. Kitano, R. Matsudo, N. Yamada, M. Yamazaki (2021)]

  (2) Z_2 is broken at high temperature $T > T_{dec}$ (deconfinement)

  [D. J. Gross, R. D. Pisarski, L. G. Yaffe (1981)]

  [N. Weiss (1981)]

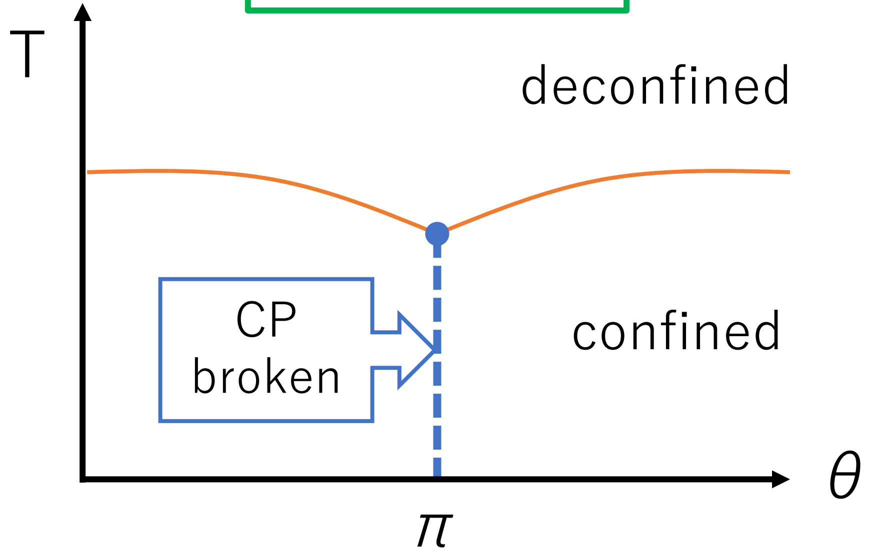
- constraint by the anomaly matching :

  "CP cannot be restored in the confined phase" → $T_{CP} \geqq T_{dec}$

T_{dec} VS T_{CP}

☆ examples of possible (θ , T) phase diagram

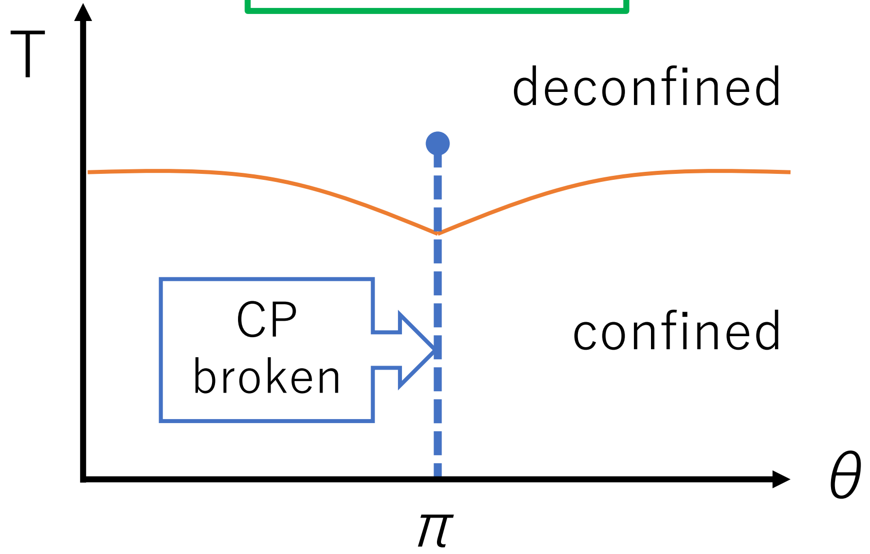
$T_{CP} = T_{dec}$



large N (holography)

[F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]

$T_{CP} > T_{dec}$



soft SUSY breaking of SU(2) SYM

[S. Chen, K. Fukushima, H. Nishimura, Y. Tanizaki (2020)]

Which diagram is realized for N=2 ?

Short summary

- Direct lattice simulation at $\theta = \pi$ is hard due to the sign problem.
- The crucial point of our work :
  CP breaking/restoration can be probed by the tail of topological charge distribution at $\theta = 0$!
- We find a sudden change of the tail
  by simulating the theory at imaginary θ (no sign problem).

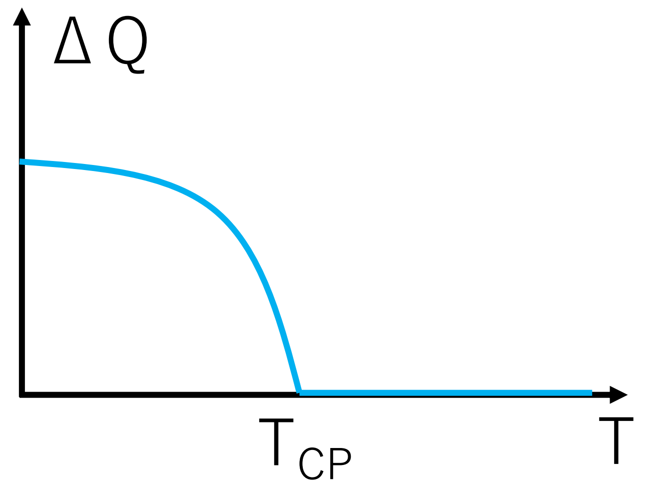
  → Our results suggest $T_{CP} > T_{dec}$ for SU(2).

Identifying CP restoration

- Q is a CP odd operator

  → If CP is spontaneously broken at $\theta = \pi$,
  $\langle Q \rangle$ is discontinuous there.

$$\Delta Q = |\langle Q \rangle_{\theta=\pi-\epsilon} - \langle Q \rangle_{\theta=\pi+\epsilon}| \begin{cases} > 0 & : \text{CP broken} \\ = 0 & : \text{CP restored} \end{cases}$$



- T_{CP} can be regarded as a temperature where ΔQ vanishes.

  → Can we probe it without simulations at $\theta = \pi$?

$\langle Q \rangle$ and the topological charge distribution

- We consider the topological charge distribution at $\theta = 0$.

$$\rho(q) = \frac{1}{Z_0}\int dU\,\delta(q-Q)e^{-S_g} = \frac{1}{Z_0}\int \frac{d\theta}{2\pi}\,e^{-i\theta q}Z_\theta$$

= Fourier transform of the partition function

$$Z_\theta = \int dU\,e^{-S_g+i\theta Q} = Z_0\int dq\,e^{i\theta q}\rho(q)$$

- θ dependence of $\langle Q \rangle$ is completely determined by ρ (q)

$$\langle Q\rangle = -i\frac{\partial}{\partial\theta}\log Z_\theta = \frac{\int dq\,qe^{i\theta q}\rho(q)}{\int dq\,e^{i\theta q}\rho(q)}$$

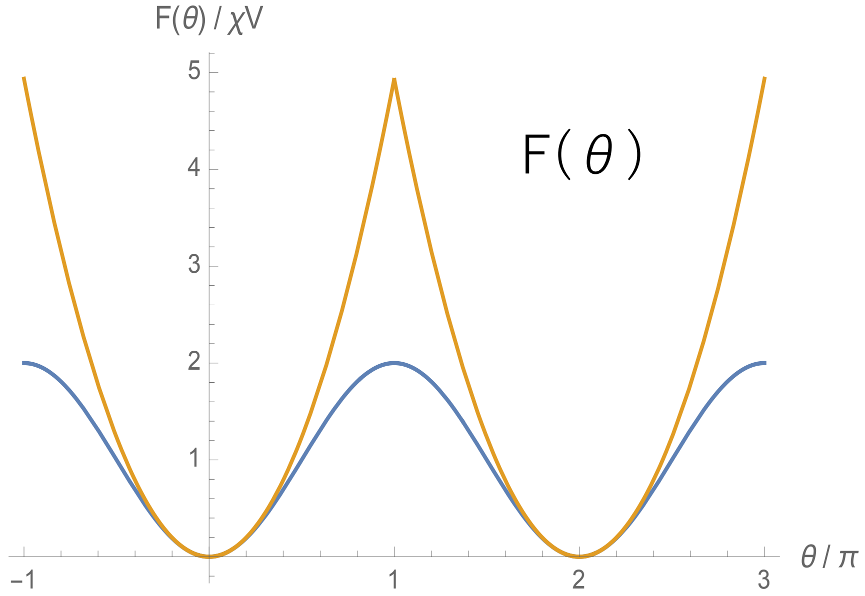
- If ΔQ=0 or not depends on ρ (q) → ρ (q) changes suddenly at T_{CP}

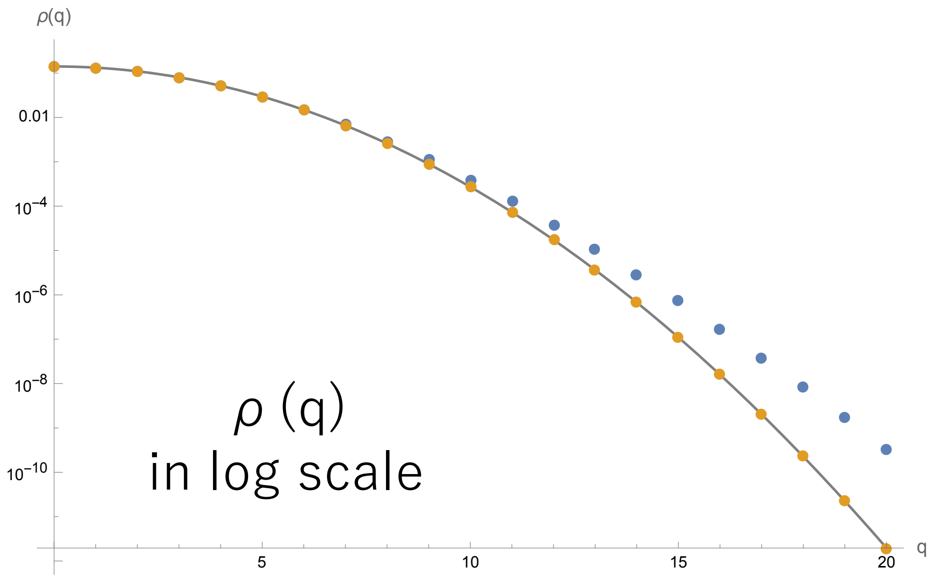
Comparison of simplified models

- Consider two different types of free energy $F(\theta) = -\log Z_\theta$

model	$F(\theta)$	CP at $\theta = \pi$
instanton gas (high T)	$\chi_0 V(1-\cos\theta)$	restored $F'(\pi+\epsilon) = F'(\pi-\epsilon)$
large N (low T)	$\frac{1}{2}\chi_0 V \min_n(\theta - 2\pi n)^2$	broken $F'(\pi+\epsilon) \neq F'(\pi-\epsilon)$

[E. Witten (1979)]





← different asymptotic behavior

Imaginary θ as a probe

- ΔQ (CP at $\theta=\pi$) seems to be related to the asymptotic behavior of the distribution ρ(q).

$$\rho(q) \sim \begin{cases} \exp\left(-q\log\frac{2q}{\chi_0 V}\right) & : \text{instanton} \quad \Longleftrightarrow \quad \Delta Q = 0 \\ \exp\left(-\frac{q^2}{2\chi_0 V}\right) & : \text{large N} \quad \Longleftrightarrow \quad \Delta Q \neq 0 \end{cases}$$

- We can observe the tail of $\rho(q)$ through $\langle Q\rangle$ at imaginary θ because of the $e^{-\tilde{\theta}q}$ factor.

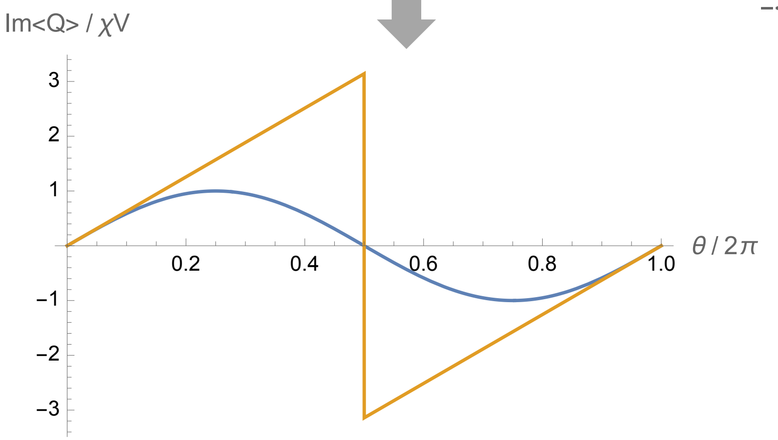
$$\langle Q\rangle = \frac{\int dq\, q e^{i\theta q}\rho(q)}{\int dq\, e^{i\theta q}\rho(q)} \quad \Rightarrow \quad \frac{\int dq\, q e^{-\tilde{\theta}q}\rho(q)}{\int dq\, e^{-\tilde{\theta}q}\rho(q)}$$

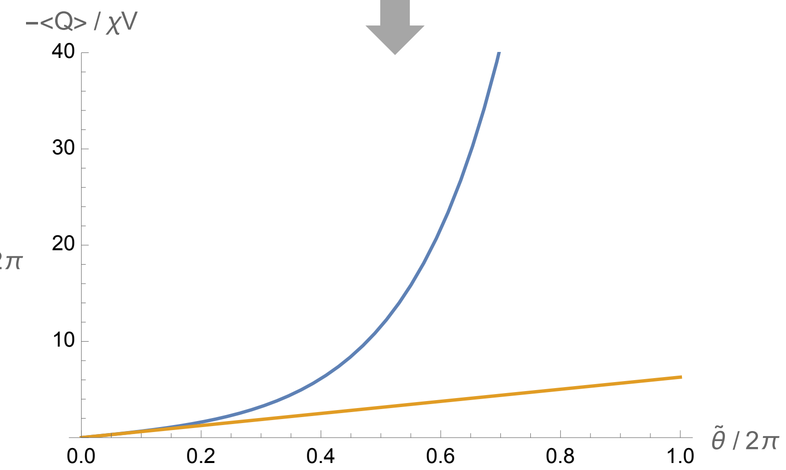
Comparison of simplified models

- $\langle Q \rangle$ at imaginary θ reflects the difference of ρ (q) clearly.

model	$\langle Q \rangle$ for $\theta \in \mathbb{R}$	$\langle Q \rangle$ for $\theta = i\tilde{\theta} \in i\mathbb{R}$	CP at $\theta = \pi$
instanton gas	$i\chi_0 V \sin\theta$	$-\chi_0 V \sinh\tilde{\theta}$	restored
large N	$i\chi_0 V\theta$	$-\chi_0 V\tilde{\theta}$	broken

common factor
$\chi_0 V = \langle Q^2 \rangle_{\theta=0}$





← exponential
vs
linear

We identify T_{CP} by simulating the theory with imaginary θ .

Lattice regularization

- gauge action : Wilson action

$$S_\beta = \frac{\beta}{2N} \sum_n \sum_{\mu \neq \nu} \mathrm{Tr}\, P_n^{\mu\nu} \qquad P_n^{\mu\nu} = U_{n,\mu} U_{n+\hat{\mu},\nu} U^{-1}_{n+\hat{\nu},\mu} U^{-1}_{n,\nu} \qquad \beta = \frac{4}{g^2}$$

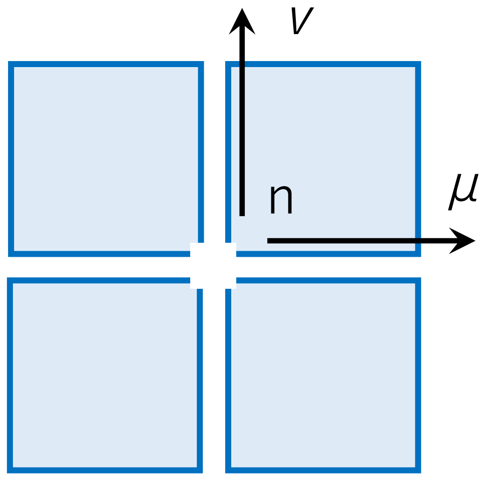
- topological charge :

  clover leaf definition [P. Di Vecchia, K. Fabricius, G. C. Rossi, G. Veneziano (1981)]

  + stout smearing [C. Morningstar, M. Peardon (2004)]

$$Q_{\mathrm{clov}} = -\frac{1}{32\pi^2} \sum_n \frac{1}{16} \sum_{\mu,\nu,\rho,\sigma=1}^{4} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left(\bar{P}_n^{\mu\nu} \bar{P}_n^{\rho\sigma} \right)$$

$$\bar{P}_n^{\mu\nu} = P_n^{\mu\nu} - P_n^{-\mu\nu} - P_n^{\mu-\nu} + P_n^{-\mu-\nu}$$

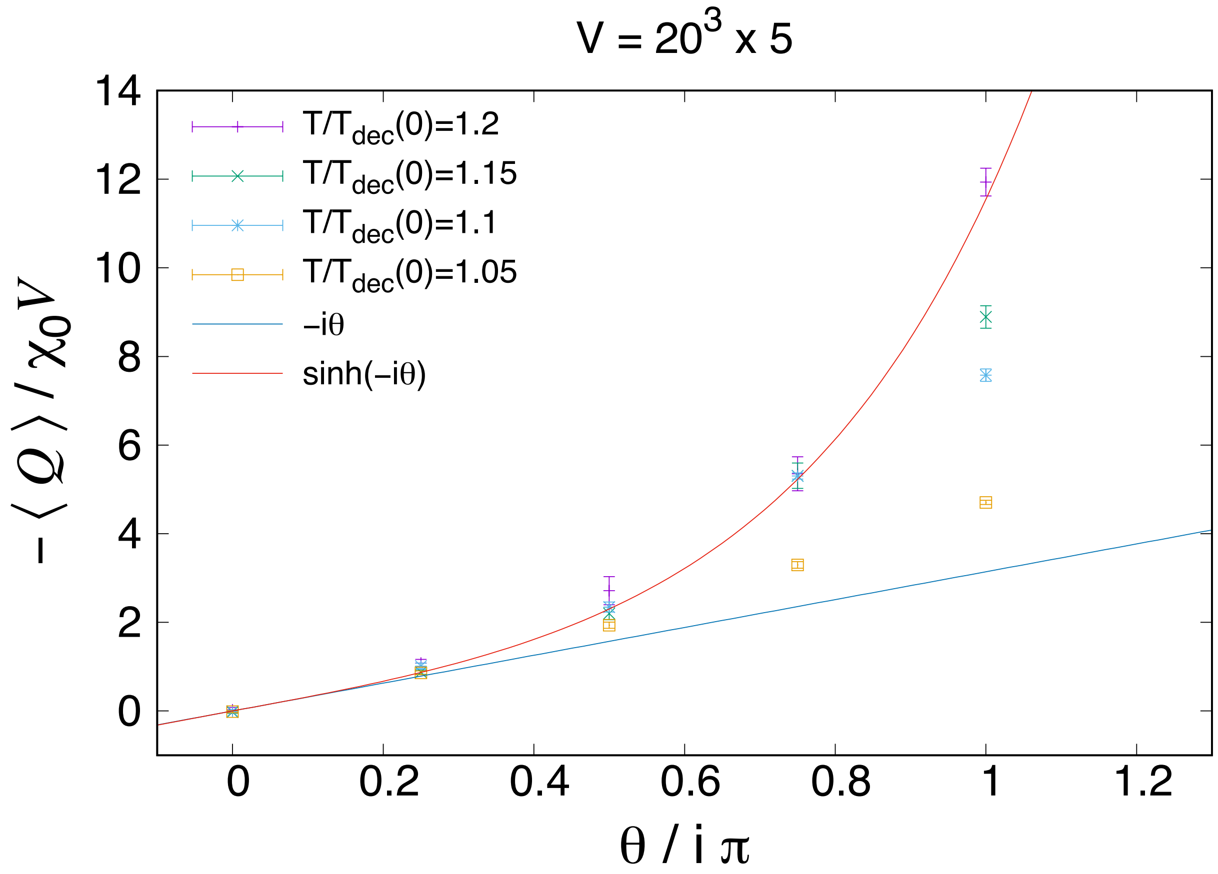


Result of HMC for imaginary θ

imaginary θ dependence of $\langle Q \rangle$ (normalized by $\chi_0 V = \langle Q^2 \rangle_{\theta=0}$)

- transition from large-N-low-T (linear) behavior to instanton gas model (sinh) around $0.9 < T/T_{\mathrm{dec}(\theta=0)} < 1.2$

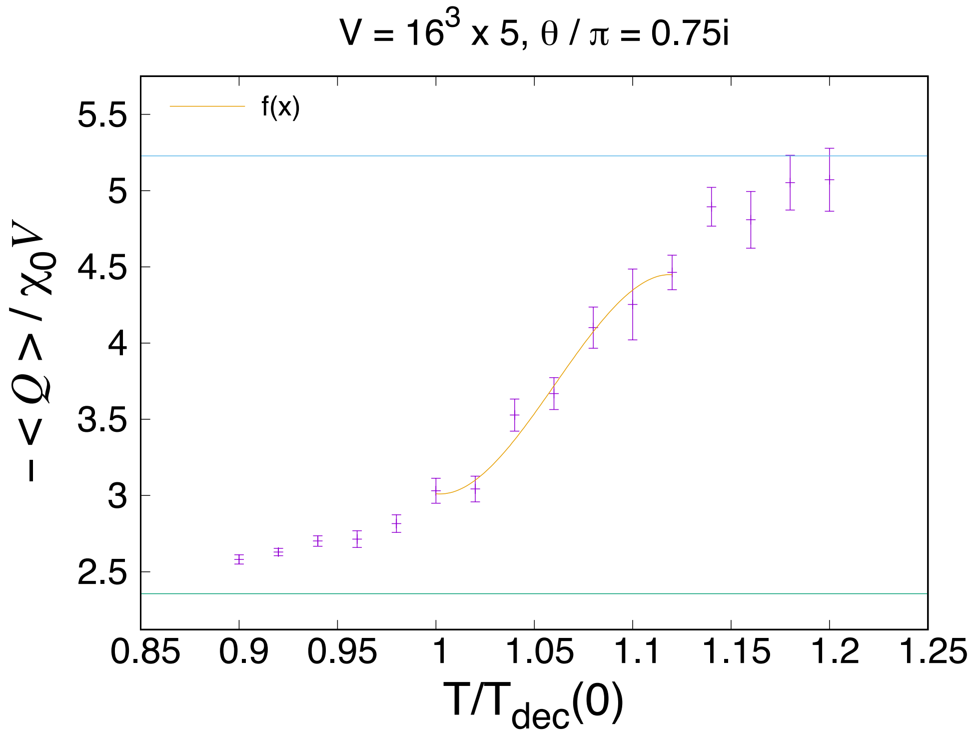
- T_{CP} can be identified by sharp jump of $\langle Q \rangle / \chi_0 V$ at sufficiently large imaginary θ.

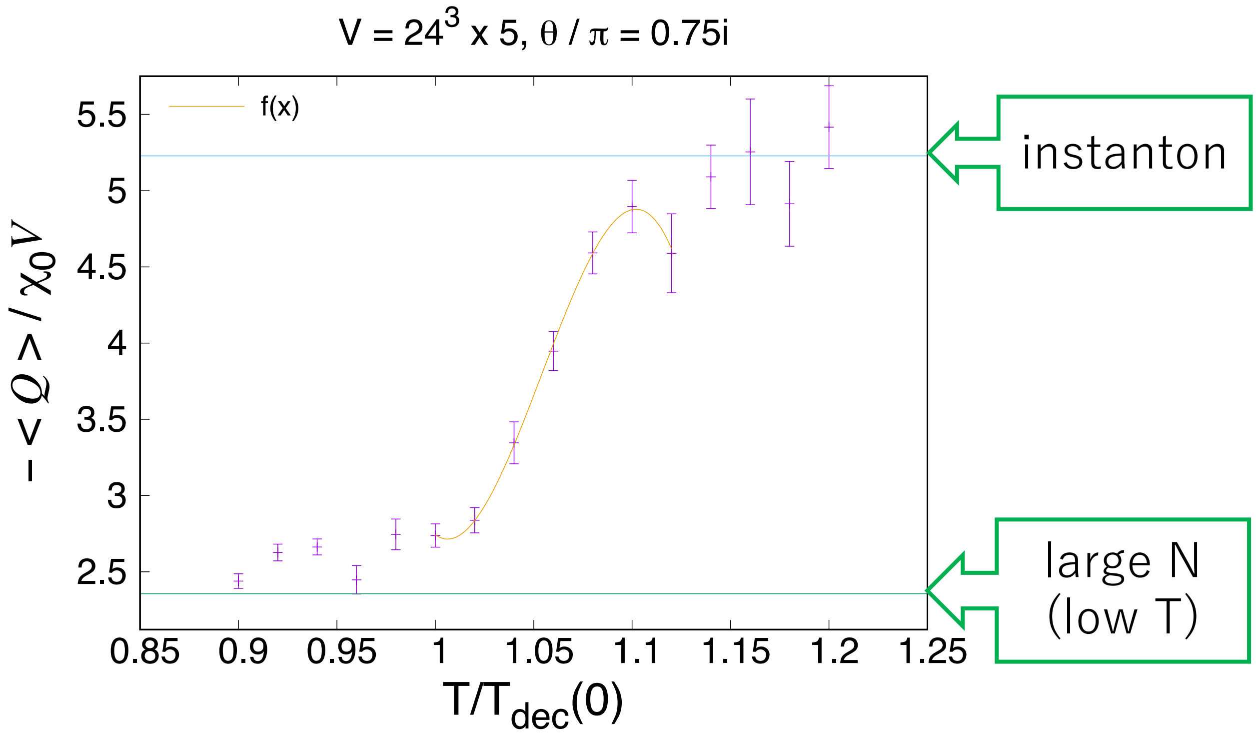


Determination of T_{CP}

temperature dependence of $\langle Q \rangle / \chi_0 V$ for the fixed θ

- fitting by $f(x) = ax^3 + bx^2 + cx + d$



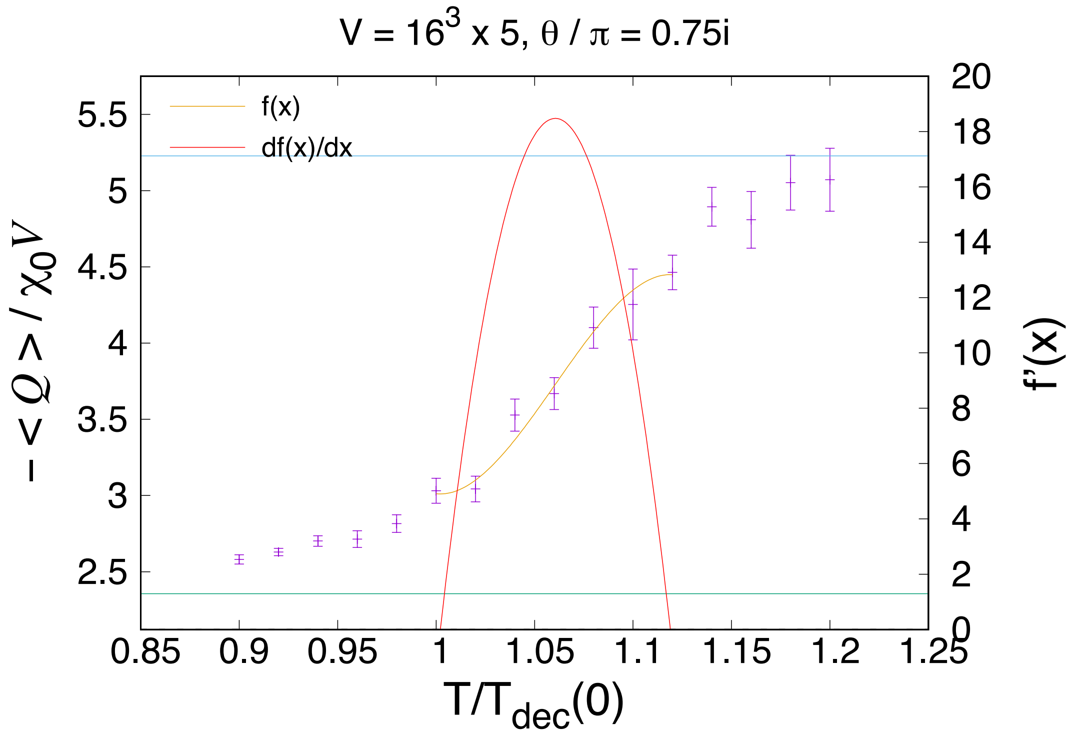


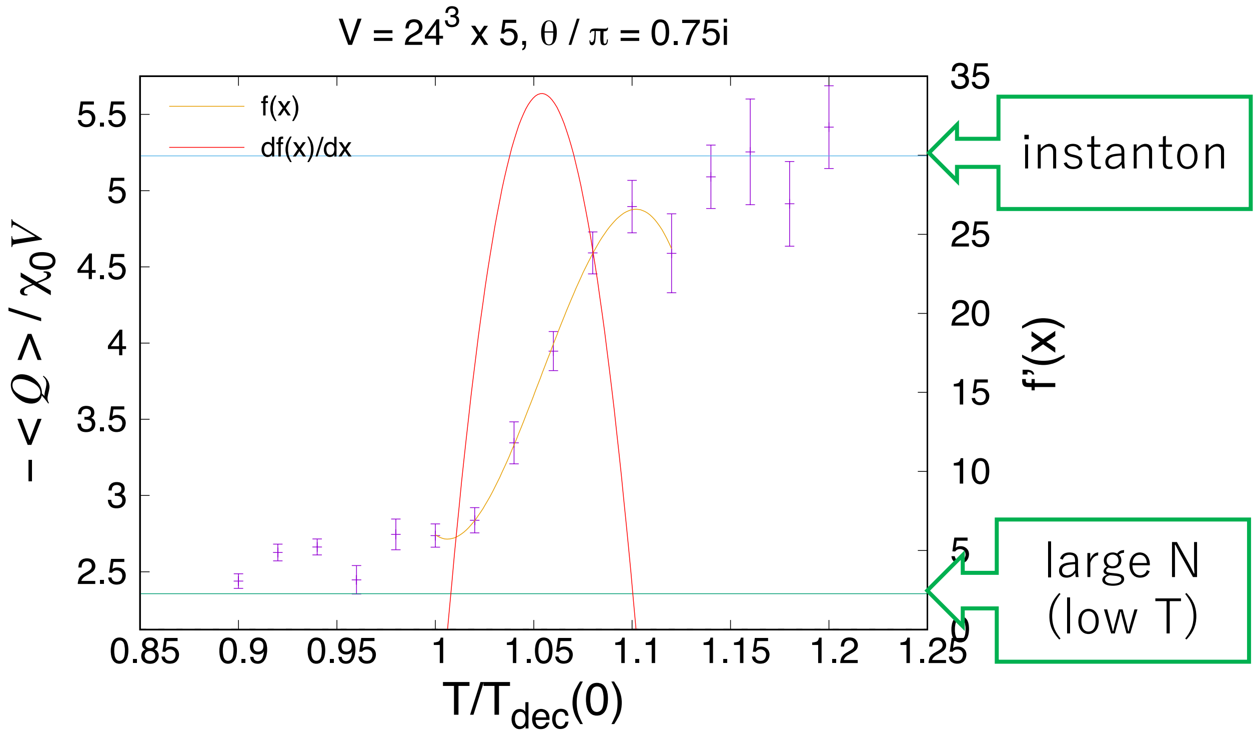
Determination of T_{CP}

<u>temperature dependence of $\langle Q \rangle / \chi_0 V$ for the fixed θ</u>

- fitting by $f(x) = ax^3 + bx^2 + cx + d$

  → T_{CP} is identified as a peak position of the derivative $f'(x)$.

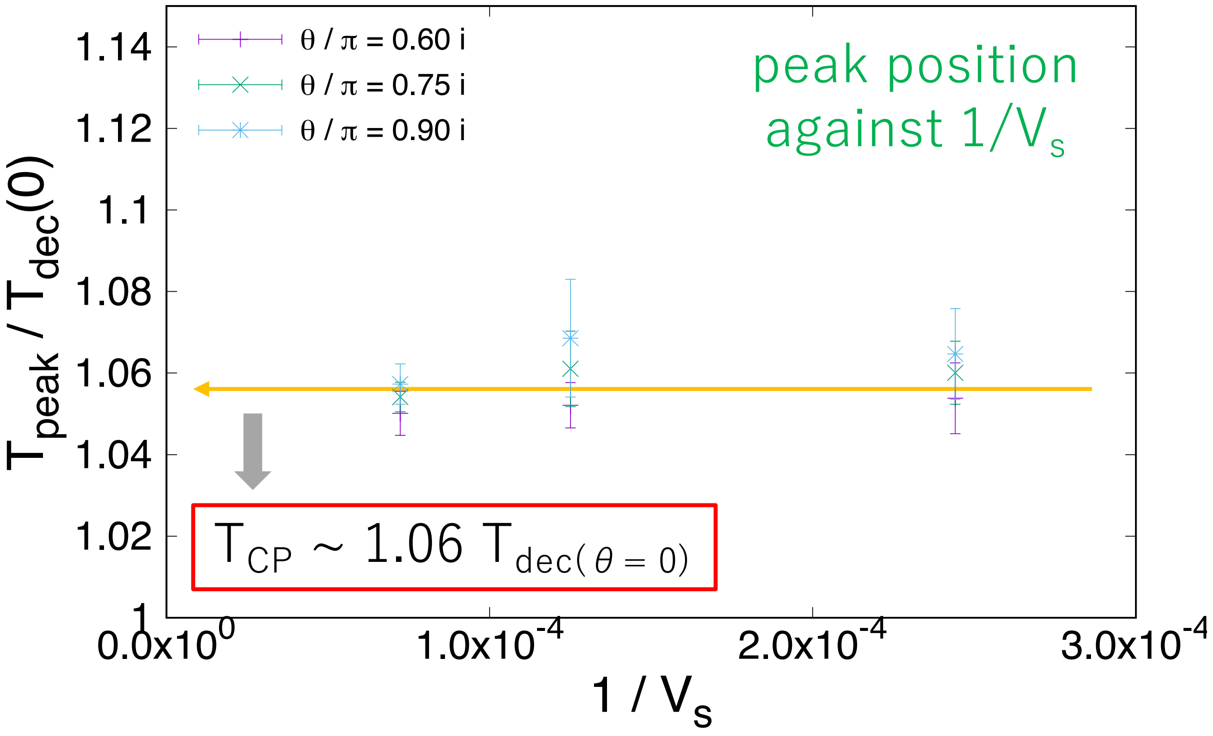


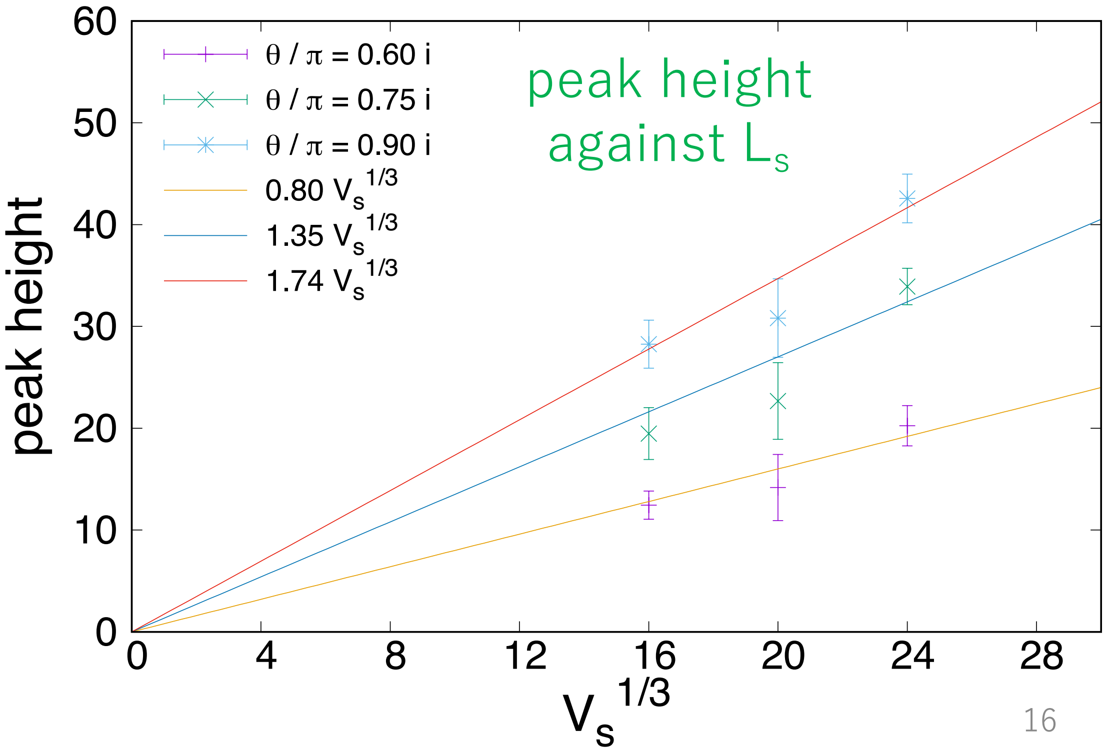


Finite volume effect

- The peak position T_{peak} should converge to the unique T_{CP} .

  (The finite volume effect is not significant already.)
- The peak height grows $\sim V_s^{1/3}$ → 2nd order transition or higher

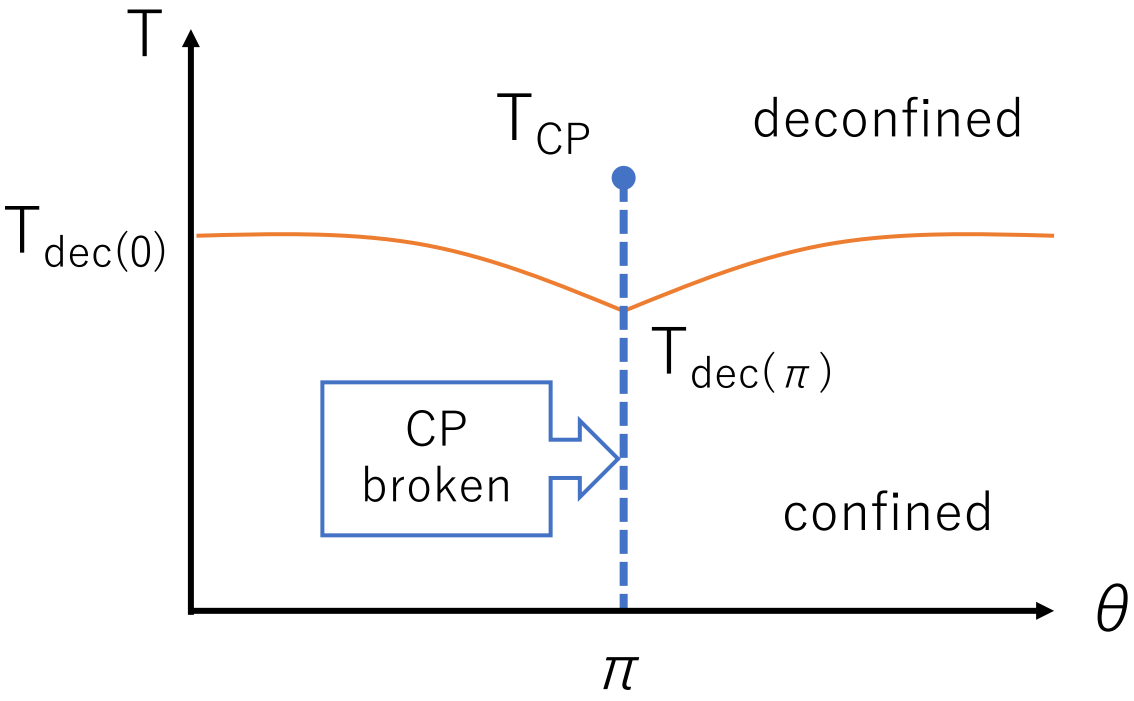




Conjectured phase diagram

- Our results indicates $T_{CP} \sim 1.06\ T_{dec(\theta=0)}$.
- If $T_{dec(\theta=\pi)}$ is lower than $T_{dec(\theta=0)}$ as expected in SU(3) case,

  $\rightarrow T_{CP} > T_{dec(\theta=0)} > T_{dec(\theta=\pi)}$



$$\frac{T_{\rm dec}(\theta)}{T_{\rm dec}(0)} \simeq 1 - R_2\,\theta^2$$

cf.) $R_2 \sim 0.018$ for SU(3)

[M. D'Elia, F. Negro (2013)]

[N. Otake, N. Yamada (2022)]

Summary

- The CP breaking/restoration at $\theta = \pi$ can be seen as a sudden change of the tail of topological charge distribution at $\theta = 0$.
- We can see it by $\langle Q\rangle/\chi_0 V$ at sufficiently large imaginary θ.
- We obtained $T_{CP} \sim 1.06\ T_{dec(\theta=0)}$.

  (Note that the sign problem is severest at $\theta = \pi$.)

- This is interesting from the viewpoint of the 't Hooft <u>anomaly matching condition</u> in 4D SU(N) gauge theory. $\rightarrow T_{CP} \geqq T_{dec}$
- Our results suggest $T_{CP} > T_{dec}$ for SU(2) unlike large N result ($T_{CP} = T_{dec}$).
- A similar study for SU(3) is ongoing.

  ($T_{CP} = T_{dec(\theta=\pi)} < T_{dec(\theta=0)}$ is expected for SU(3))

Thank you!

Stout smearing

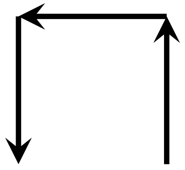
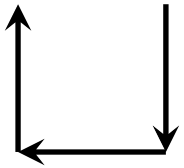
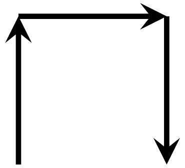
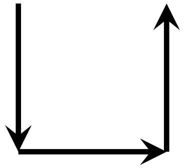
- The topological charge on the lattice is contaminated by UV fluctuation.

→ Recover the topological property by smoothing the gauge field

☆ stout smearing

[C. Morningstar, M. Peardon (2004)]

$$U^{(k+1)}_{n,\mu} = e^{iY_{n,\mu}} U^{(k)}_{n,\mu} \qquad\qquad iY_{n,\mu} = -\frac{1}{2}\left(J_{n,\mu} - \frac{1}{2}\mathrm{Tr}\left[J_{n,\mu}\right]\right)$$

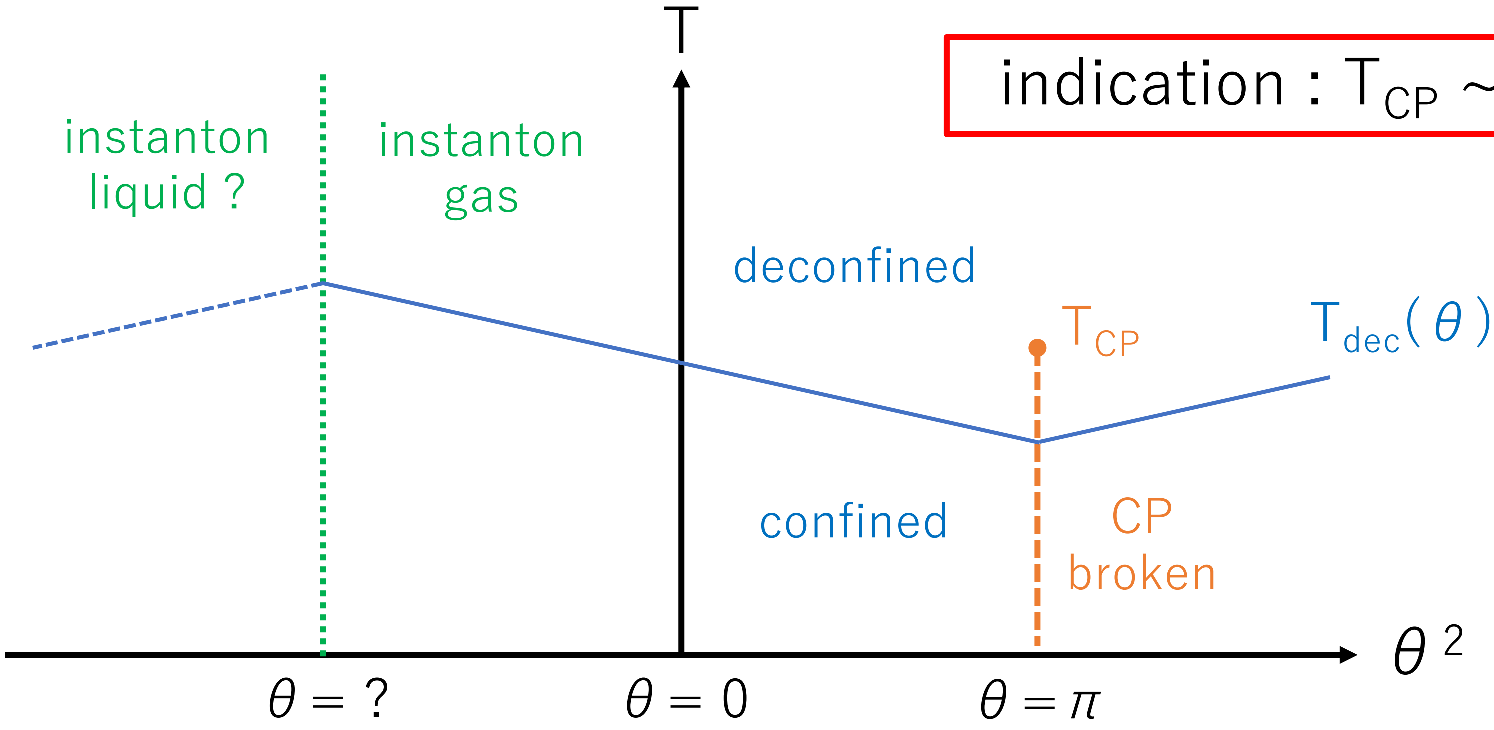
$J_{n,\mu} = \sum_{\nu(\neq\mu)} \rho_{\mu\nu} \Big[U_{n,\mu} \Big($  $+$  $\Big) - \Big($  $+$  $\Big) U^{-1}_{n,\mu} \Big]$

$\rho_{\mu\nu}$: weight of smearing

Conjectured phase diagram

- (θ^2, T) phase diagram including both real & imaginary θ

indication : $T_{CP} \sim 1.06 T_{dec}(0) \geqq T_{dec}(\pi)$



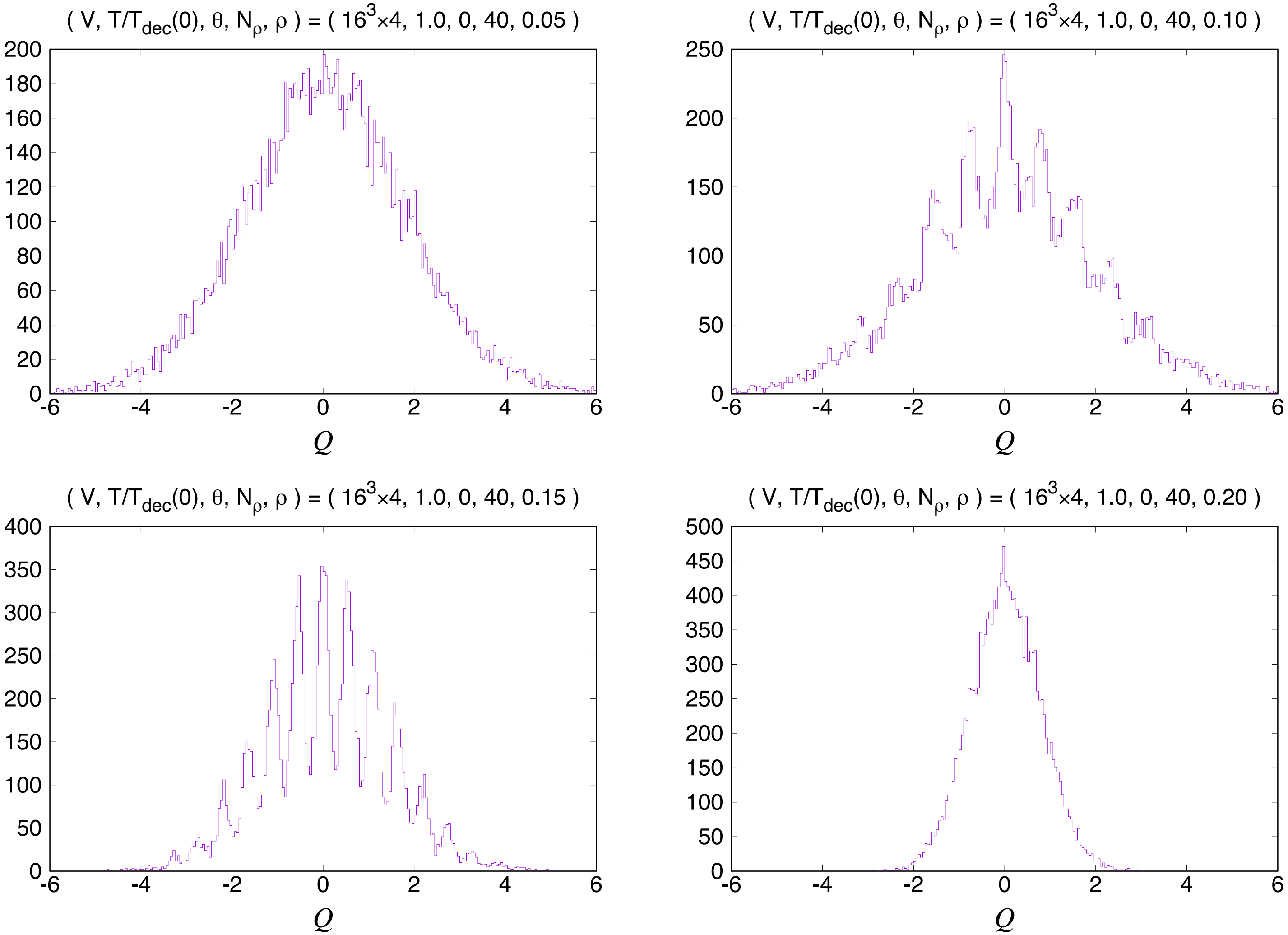
$$\frac{T_{\rm dec}(\theta)}{T_{\rm dec}(0)} \simeq 1 - R_2\,\theta^2$$

$R_2 > 0$?

$R_2 \sim 0.018$ for SU(3)
[M. D'Elia, F. Negro (2013)]

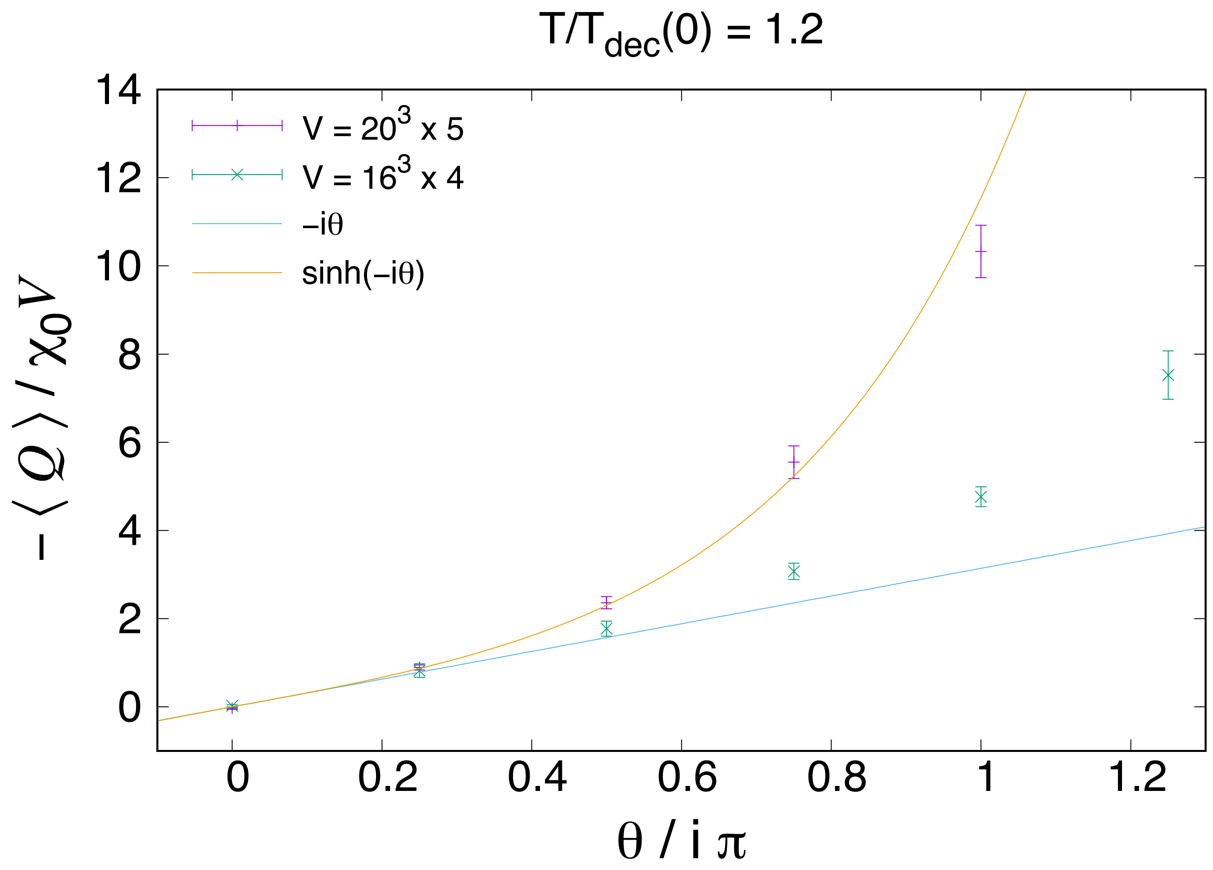
Effect of the smearing

- distribution of Q for various ρ



Finite spacing effect

- Increase β with fixing the physical volume and temperature
- $T/T_{dec}(0) = 1.2$
- $V = 16^3 \times 4,\ 20^3 \times 5$
- The results approaches instanton gas behavior on the finer lattice.



Naive method to treat the complex action

(1) Reweighting method

- Treat the phase of Boltzmann weight as a part of the observable and use the absolute value as a probability

$$\langle \mathcal{O} \rangle_S = \frac{\langle \mathcal{O}\, e^{-i\mathrm{Im}S} \rangle_{\mathrm{Re}S}}{\langle e^{-i\mathrm{Im}S} \rangle_{\mathrm{Re}S}}$$

- oscillation of phase → numerator and denominator ~ $e^{-O(N_{\mathrm{dof}})}$
  → It is hard to evaluate "0/0" due to the statistical error.
- valid only when $\mathrm{Im}\, S$ is sufficiently small

Improvement of the reweighting

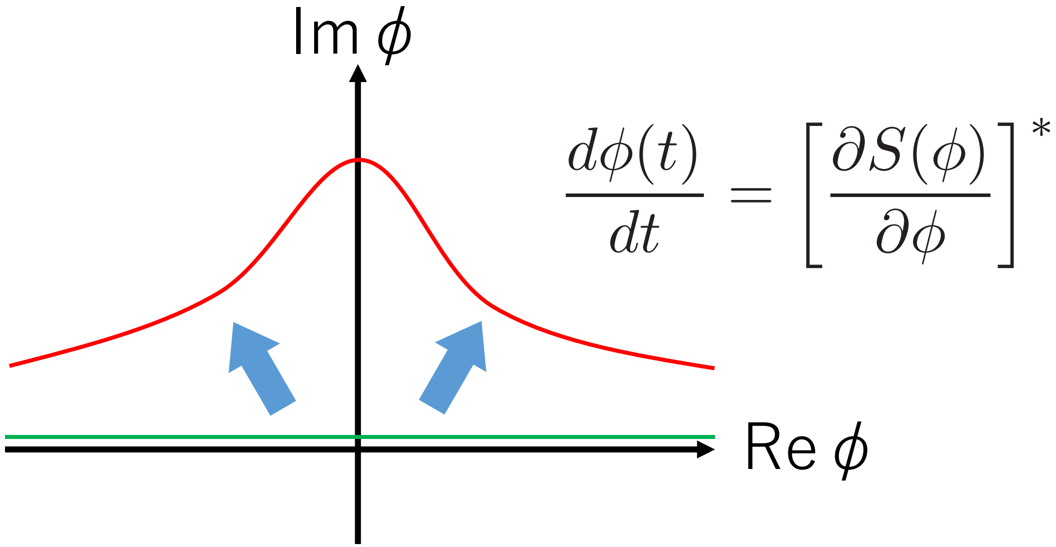
(2) Lefschetz thimble method (LTM) [E. Witten (2010)]

- Deform the integration contour from $\mathbb{R}$ to $\mathbb{C}$ by the flow equation.

  → oscillation of Boltzmann weight is suppressed

- Since the calculation cast grows $O(N_{dof}^3)$, this method is limited to lower dimensions.

※ Recently, a new technique to improve both ergodicity and calculation cost is proposed.

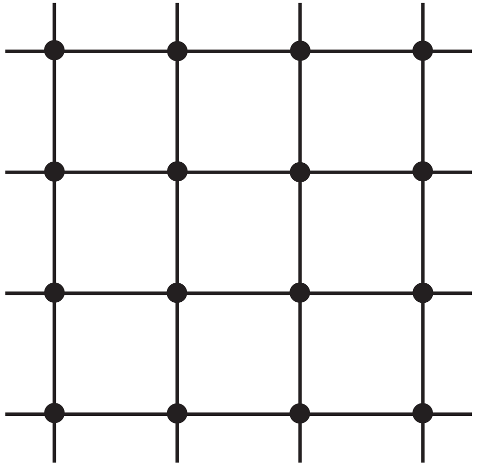


Deterministic approach

(3) Tensor renormalization group (TRG) [M. Levin, C. P. Nave (2007)]

- Evaluate the partition function represented by a tensor network.
- deterministic calculation
  $\rightarrow$ no statistical error
- calculation cost $\propto$ log(volume)
- Since the rank of the tensor becomes higher, the calculation cost grows rapidly in D > 2.

$$Z = \sum_{a,b,c,d,\cdots} T_{abcd} T_{defg} \cdots$$



Our approach

(4) Complex Langevin method (CLM) [G. Parisi (1983)] [J. R. Klauder (1983)]

- fictitious time evolution of dynamical variables by Langevin eq.

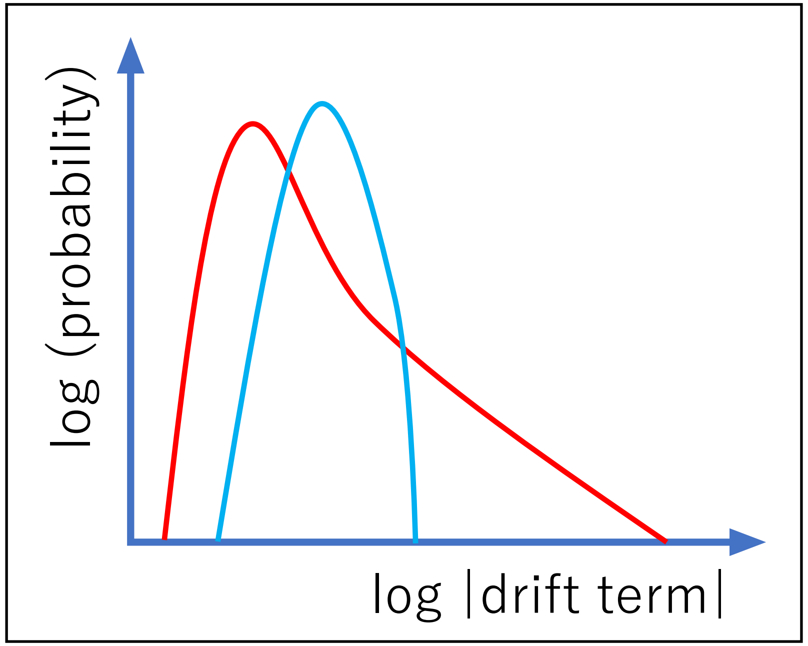
$$x \longmapsto z = x + iy$$

$$\frac{dz(t)}{dt} = -\frac{\partial S(t)}{\partial z} + \eta(t) \qquad Z = \int dx e^{-S}$$

drift term

Gaussian noise

- calculation cost $\propto$ (system size)
- condition for justifying the result



The distribution of the drift term should falls off exponentially or faster.

[K. Nagata, J. Nishimura, S. Shimasaki (2016)]