Ising Model on an Affine Plane (Lattice 2022)

Evan Owen (speaker, Boston University)
Richard Brower (Boston University)

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Overview

- Lattice Radial Quantization
- Critical Ising Model on S^2
- Ising Model on an Affine Plane

Lattice Radial Quantization

Lattice Radial Quantization

- We wish to use the lattice to study field theories at or near conformal fixed points
- On a periodic square lattice, wraparound effects are always relevant
- Weyl transform from flat Euclidean manifold to a “cylinder”

$$\mathbb{R}^d \to S^{d-1} \times \mathbb{R}$$

$$ds^2_{\text{flat}} = r^2[(d\log r)^2 + d\Omega^2_{d-1}] \to ds^2_{\text{cyl.}} = dt^2 + d\Omega^2_{d-1}$$

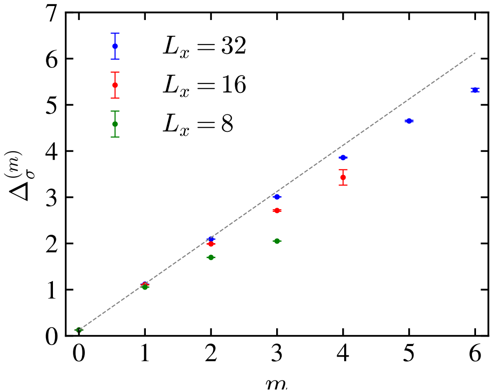
- Angular directions are periodic by definition
- Radial coordinate defined as $t = \log r$
- Power-law correlation functions decay exponentially in t
- Lattice volume grows exponentially with number of time-slices

Lattice Radial Quantization

Critical Ising model in 2d is relatively easy to simulate

$$\mathbb{R}^2 \to S^1 \times \mathbb{R}$$

Scaling exponents of σ operator and descendants can be extracted from two-point function



Descendant operators approach integer-spacing in the continuum limit

$$\sum_x \sigma(0)\sigma(t,x)\cos(2\pi m x/L_x) \propto e^{-c\Delta_\sigma^{(m)} t} \qquad \Delta_\sigma^{(m)} \to \Delta_\sigma + m$$

Lattice Radial Quantization

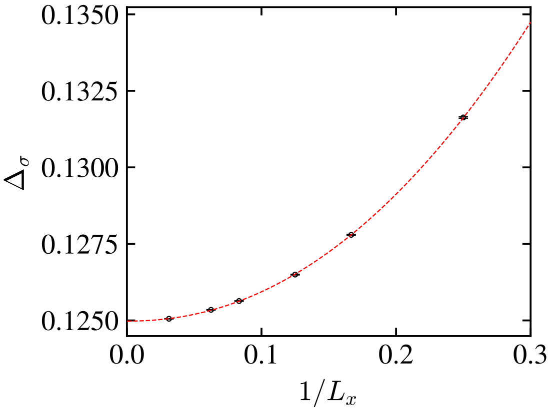
Continuum limit extrapolation of σ operator scaling exponent

$$\Delta_\sigma(L_x) = \Delta_\sigma(\infty) + \frac{b}{L_x^\gamma}$$

$$\Delta_\sigma(\infty) = 0.1249781(62)$$

$$b = 0.1245(22)$$

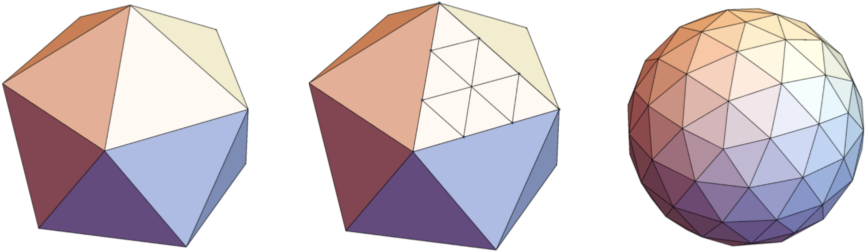
$$\gamma = 2.114(11)$$



Critical Ising Model on S^2

Critical Ising Model on S^2

- In $d > 2$, angular part of manifold cannot be discretized uniformly
- In 3 dimensions, $\mathbb{R}^3 \to S^2 \times \mathbb{R}$ can be discretized by tessellating an icosahedron [1]



- Produces a non-uniform simplicial complex
- Higher dimensions can be discretized in a similar manner

Critical Ising Model on S^2

- Test case is scalar field theory on S^2 with discretized action

$$S = \frac{1}{2}\sum_{\langle xy\rangle} \frac{V_{xy}}{\ell_{xy}^2} (\phi_x - \phi_y)^2 + \sum_x \sqrt{g_x} \left(\frac{1}{2} m^2 \phi_x^2 + \lambda \phi_x^4 \right)$$

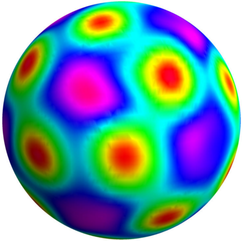
- Free scalar theory ($\lambda = 0$) on a simplicial complex can be solved exactly with the finite element method (FEM)
- Geometric factors V_{xy}, ℓ_{xy}, $\sqrt{g_x}$ are determined by lattice geometry via discrete exterior calculus (DEC)

Critical Ising Model on S^2

- Interacting theory ($\lambda \neq 0$) has UV divergences due to quantum fluctuations from loops
- Due to non-uniform cutoff, the quantum theory does not become spherical as $a \to 0$ so conformal symmetry is lost
- At small λ, a local perturbative mass counterterm renormalizes the theory [2]

$$S \to S + \frac{1}{2}\sum_x \sqrt{g_x}\delta m_x^2 \phi_x^2$$

- Applied correctly, counterterm restores conformal symmetry in the continuum limit



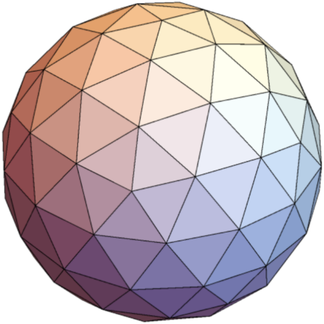
Critical Ising Model on S^2

- Can be used to accurately determine CFT parameters Δ_σ, $\Delta_{\sigma'}$, Δ_ϵ, $\Delta_{\epsilon'}$, etc. [3]
- Scaling exponents for σ' and ϵ' operators are required as inputs for the conformal bootstrap program [6]
- Simulations and data analysis are ongoing for critical ϕ^4 theory in 3 dimensions on $S^2 \times \mathbb{R}$
- We are also planning to pursue finite element formulations with gauge theories and fermions, preliminary work has been done [4]

Ising Model on an Affine Plane

Ising Model on an Affine Plane

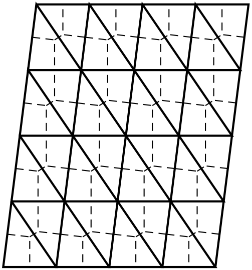
- Motivation: Instead of ϕ^4 theory with a counterterm, can we simulate an Ising spin model on S^2?
- As $a \to 0$, tangent planes of the discretized sphere S^2 become locally uniform, with smoothly varying triangles
- Point defects at 12 "exceptional" points



Ising Model on an Affine Plane

- Critical Ising model can be solved exactly in 2d via free massless fermion [7]
- Wolff [8] relates Ising model to free Majorana fermion on an equilateral triangular lattice with periodic boundaries via a loop expansion
- We generalize this to an affine-transformed triangular lattice to relate Ising couplings (K_1, K_2, K_3) to lattice geometry (ℓ_1, ℓ_2, ℓ_3)

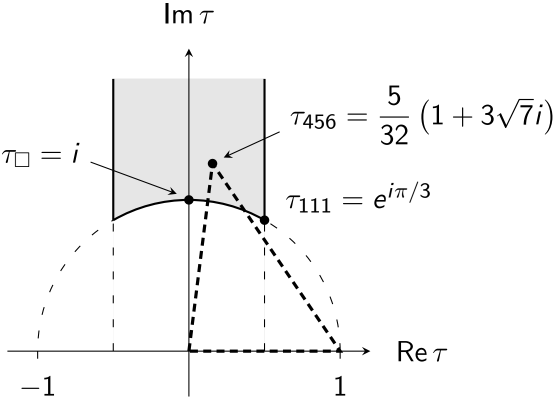
$$S = -\sum_{x,i} K_i \sigma_x \sigma_{x+i} \qquad \sinh(2K_i) = \frac{\ell_i^*}{\ell_i}$$



- This lattice is the tangent plane of our discretized sphere

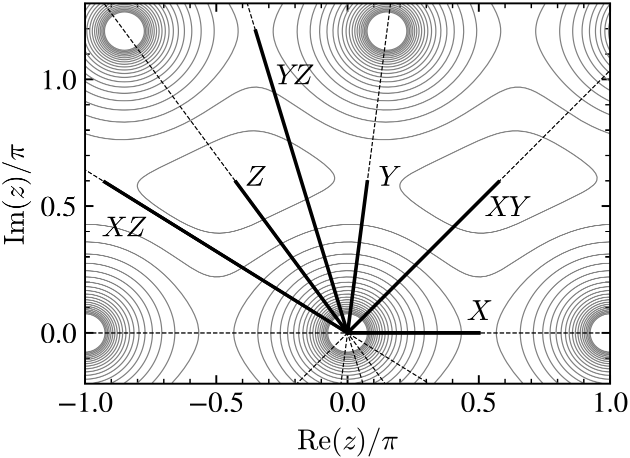
Ising Model on an Affine Plane

- This allows us to simulate the 2d Ising model on a torus with an arbitrary modular parameter τ (related to ℓ_i's)



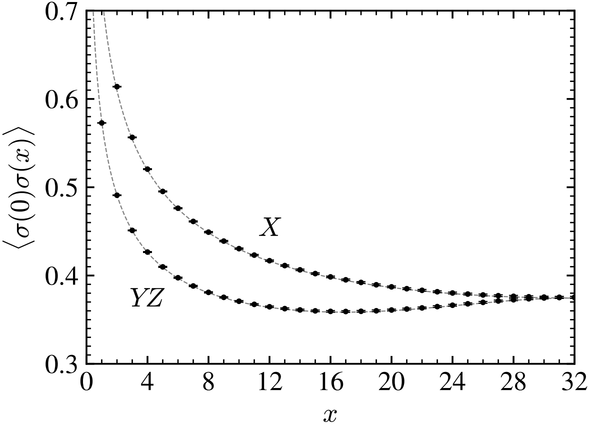
Ising Model on an Affine Plane

- Continuum spin-spin correlation function is known analytically for arbitrary τ [5], shown for $\ell_i \propto \{4, 5, 6\}$



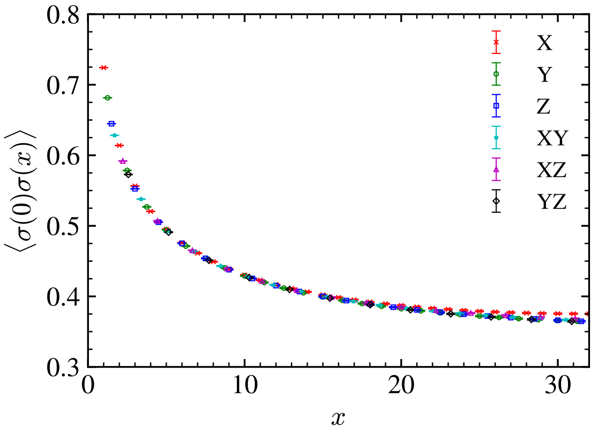
Ising Model on an Affine Plane

Comparison of lattice simulation to analytic continuum result, horizontal axis is in lattice steps



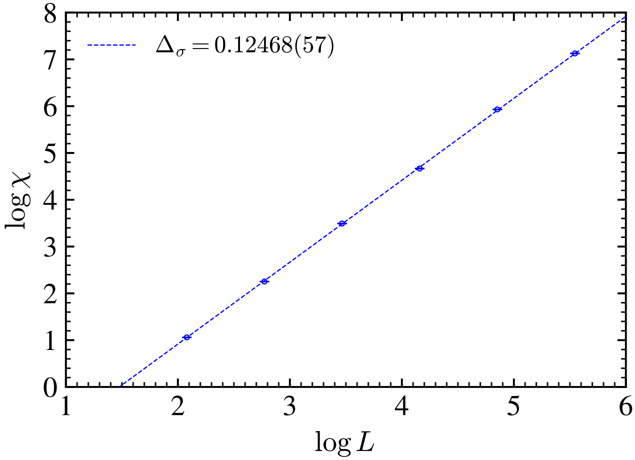
Ising Model on an Affine Plane

- Horizontal axis scaled according to $\{4, 5, 6\}$ lattice geometry
- Emergent geometry from operator scaling matches input geometry



Ising Model on an Affine Plane

- Conventional finite-size scaling analysis on $\{4, 5, 6\}$ lattice



Ising Model on an Affine Plane

Future directions:

- We expect that critical couplings applied locally on discretized S^2 lattice will restore conformal symmetry in the continuum limit, preliminary results support this
- Can potentially be applied to other 2-dimensional manifolds embedded in $\mathbb{R}^3$

Wrap-Up

- Lattice radial quantization is effective for studying field theories at or near conformal fixed points
- We are working on several approaches for studying strongly-coupled field theories on curved manifolds using finite element methods
- We have developed a framework for performing simulations of the 2d critical Ising model on a torus with arbitrary modular parameter with potential application to simulations on arbitrary curved manifolds

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Thank you!

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