Casimir effect for fermions on the lattice

[K.N., and K. Suzuki, arXiv:2207.14078]
[K.N., and K. Suzuki, arXiv:2204.12032]
[T. Ishikawa, K.N., and K. Suzuki, arXiv:2012.11398]
[T. Ishikawa, K.N., and K. Suzuki, arXiv:2005.10758]

Katsumasa Nakayama(DESY)

Kei Suzuki(JAEA)

2022/8/10@Bonn/Germany

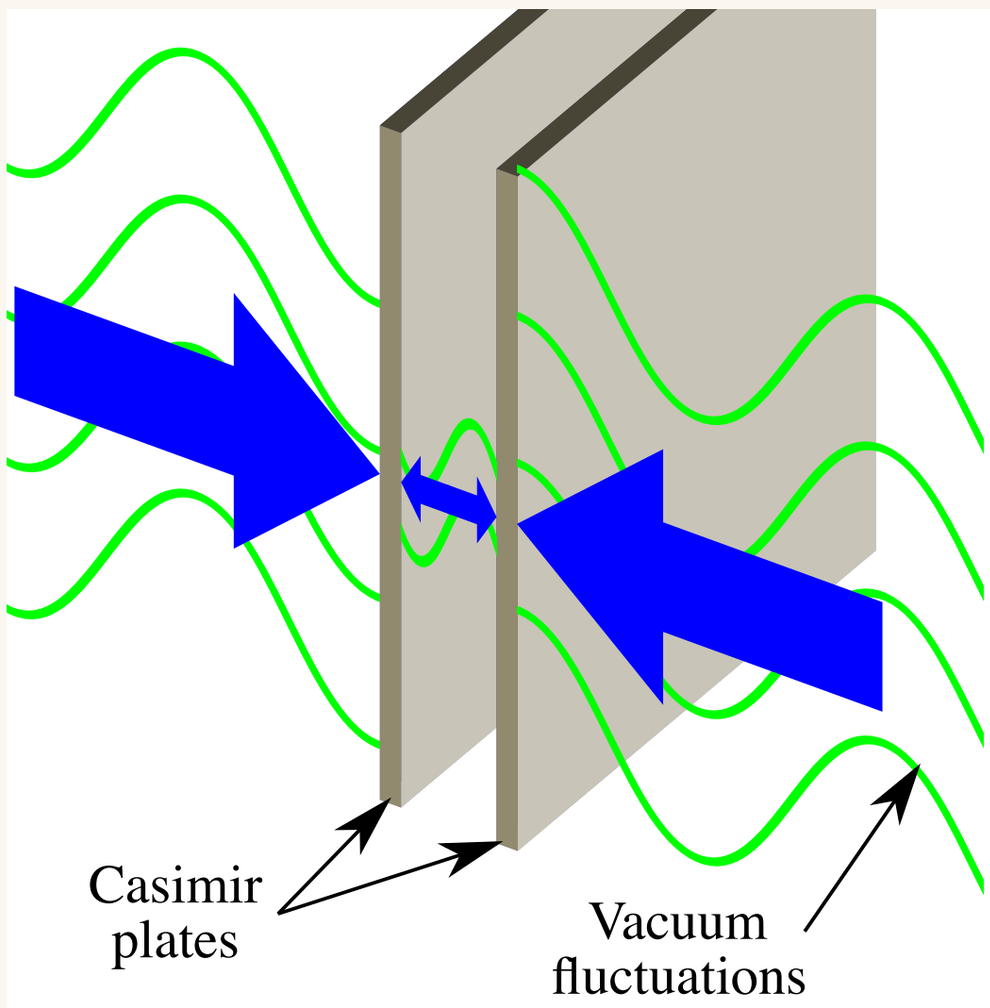
Photo: Berlin, Germany, 2019

◉ Casimir energy (theory and experiment)

$$E_{\rm Cas}(L) \equiv E_{\rm vac}(L) - E_{\rm vac}(\infty)$$

$$E_{\rm vac}(\infty) \propto \int d^3k \sqrt{\mathbf{k}^2 + m^2}$$

◇ Vacuum energy shift from the energy in infinite volume.



◇ It is originally suggested for photon field in 1948.

[H.B.G. Casimir, Proc. K. Ned. Acad. Wet. 51, 793 (1948)]

◇ An experiment confirmed in 1996.

[S. K. Lamoreaux, PRL78 (1997), 5]

[T. Ishikawa, K.N., and K. Suzuki, arXiv:2005.10758]

◉ Casimir energy and regularization

$$E_{\rm Cas}(L) \equiv E_{\rm vac}(L) - E_{\rm vac}(\infty)$$

◇ Vacuum energy shift from the energy in infinite volume.

◇ Regularization is needed.

$$E_{\rm vac}(\infty) \propto \int d^3k \sqrt{\mathbf{k}^2 + m^2} \qquad \to \infty$$

$$E_{\rm vac}(L) \propto \int d^2k \sum_{n=0}^{\infty} \sqrt{k_1^2 + k_2^2 + \left(\frac{2n\pi}{L}\right)^2 + m^2} \qquad \to \infty$$

◇ Zeta-function, Abel-Plana formula…etc.

→ We apply the lattice regularization.

◉ Contents

(1): Introduction using the free Wilson fermion.

[T. Ishikawa, K.N., and K. Suzuki, arXiv:2005.10758]

[T. Ishikawa, K.N., and K. Suzuki, arXiv:2012.11398]

(2): Casimir oscillation on DSM/WSM toy model.

[K.N., and K. Suzuki, arXiv:2207.14078]

(3): Casimir oscillation on realistic Dirac semimetals.

[K.N., and K. Suzuki, arXiv:2207.14078]

(4): Non-zero remnant Casimir of quadratic dispersion.

[K.N., and K. Suzuki, arXiv:2204.12032]

[T. Ishikawa, K.N., and K. Suzuki, arXiv:2005.10758]

◉ Lattice regularization for Casimir energy

→ For 1+1 dim., with continuum time direction,

◇ Wilson fermion with periodic boundary.

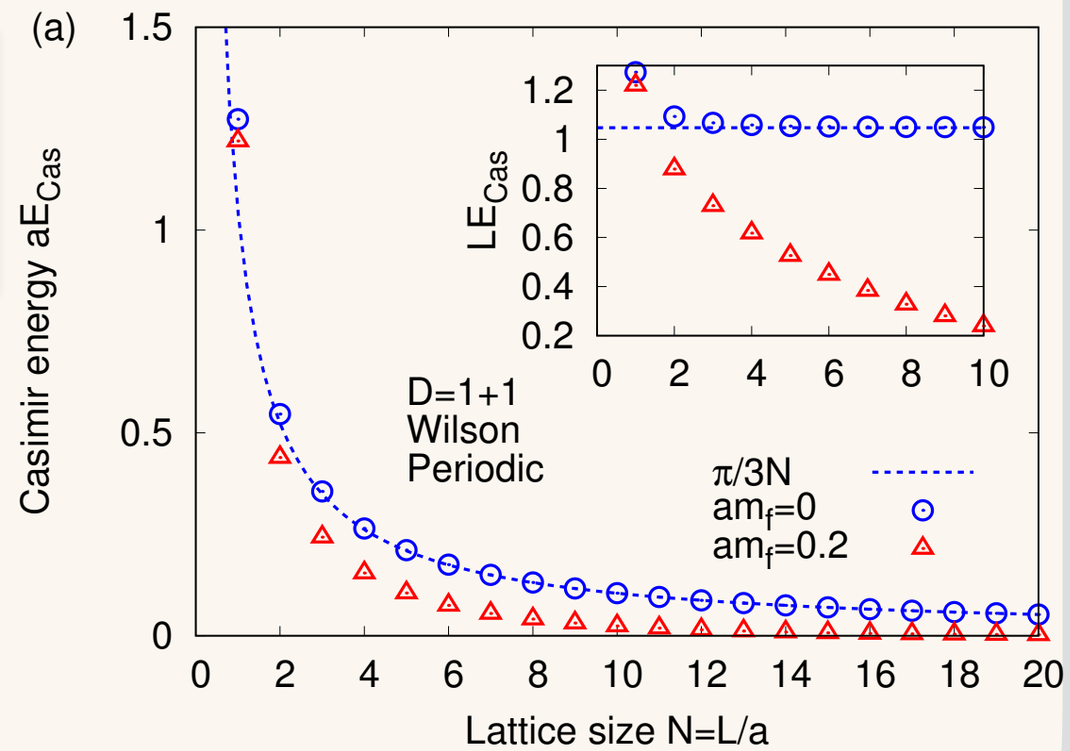
$$E_{\rm vac} \propto \int_{-\pi/a}^{\pi/a} dp \sqrt{a^2 D^\dagger D} = \int_{-\pi/a}^{\pi/a} dp \sqrt{2 - 2\cos ap}$$

$$E_{\rm Cas} \propto \left(\int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} - \sum_{-N/2}^{N/2} \right) \sqrt{2 - 2\cos ap}$$

(Analytically) ↓

$$aE_{\rm Cas} = \frac{4N}{\pi} - 2\cot\frac{\pi}{2N}$$

→ $$E_{\rm Cas} = \frac{\pi}{3L} + O(a^2)$$



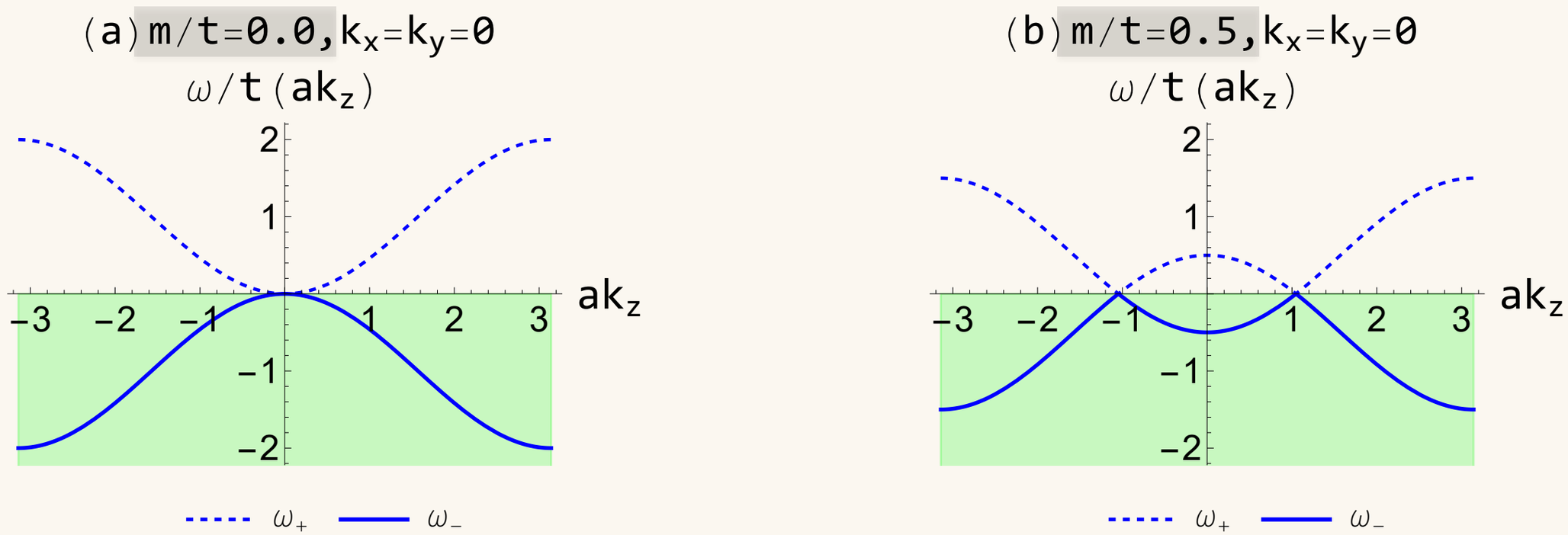
$(N = L/a)$

→ Lattice reg. reproduces Casimir energy in the continuum.

[K.Y. Yang, Y.M. Lu, and Y. Ran arXiv:1105.2353]

◉ Toy model of Dirac/Weyl semimetal

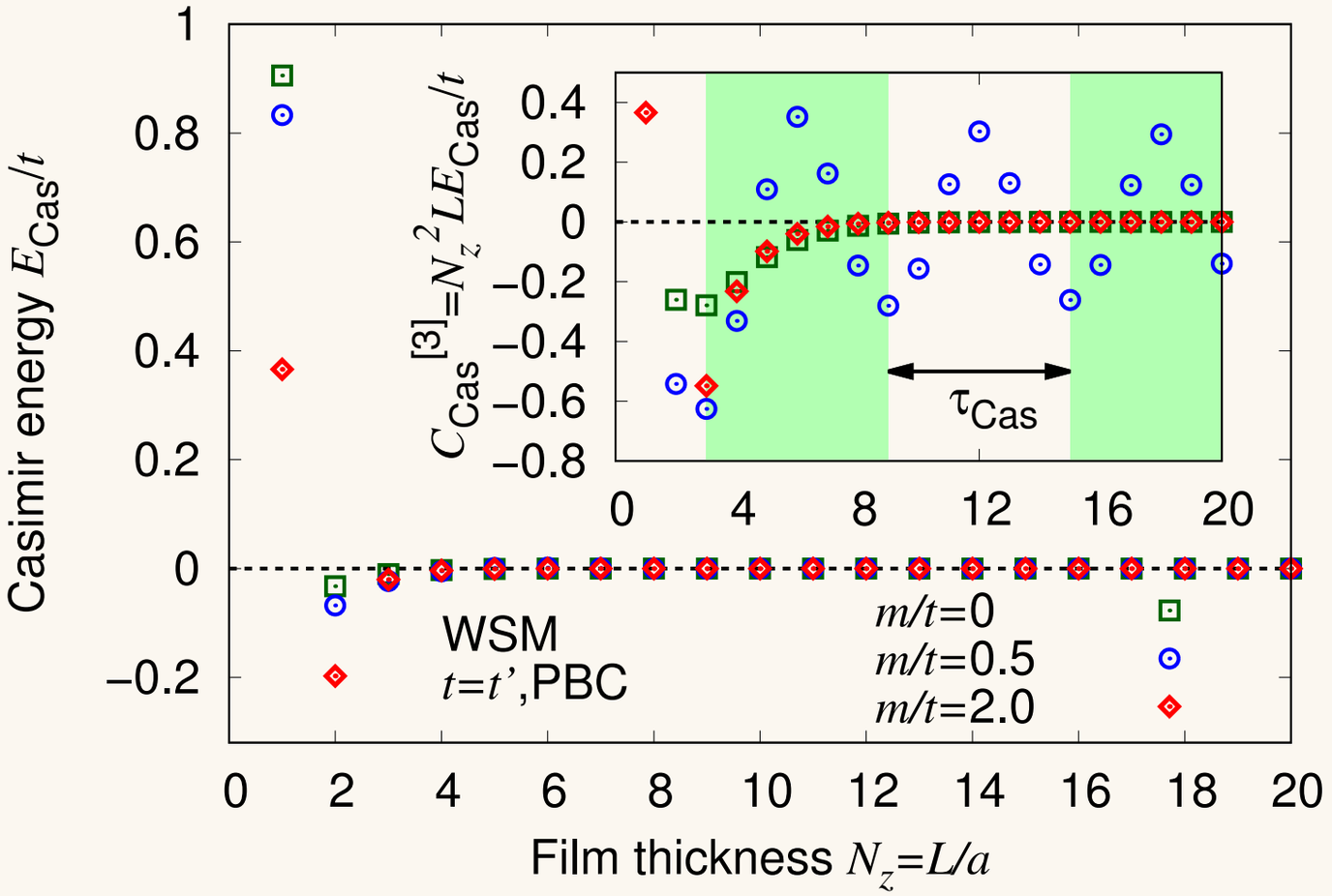
$$\omega_{\pm}^{\mathrm{WSM}} = \pm\sqrt{t^2 \sum_i^{x,y} \sin^2 a_i k_i + \left[m - t' \sum_i^{x,y,z} (1 - \cos a_i k_i)\right]^2}$$

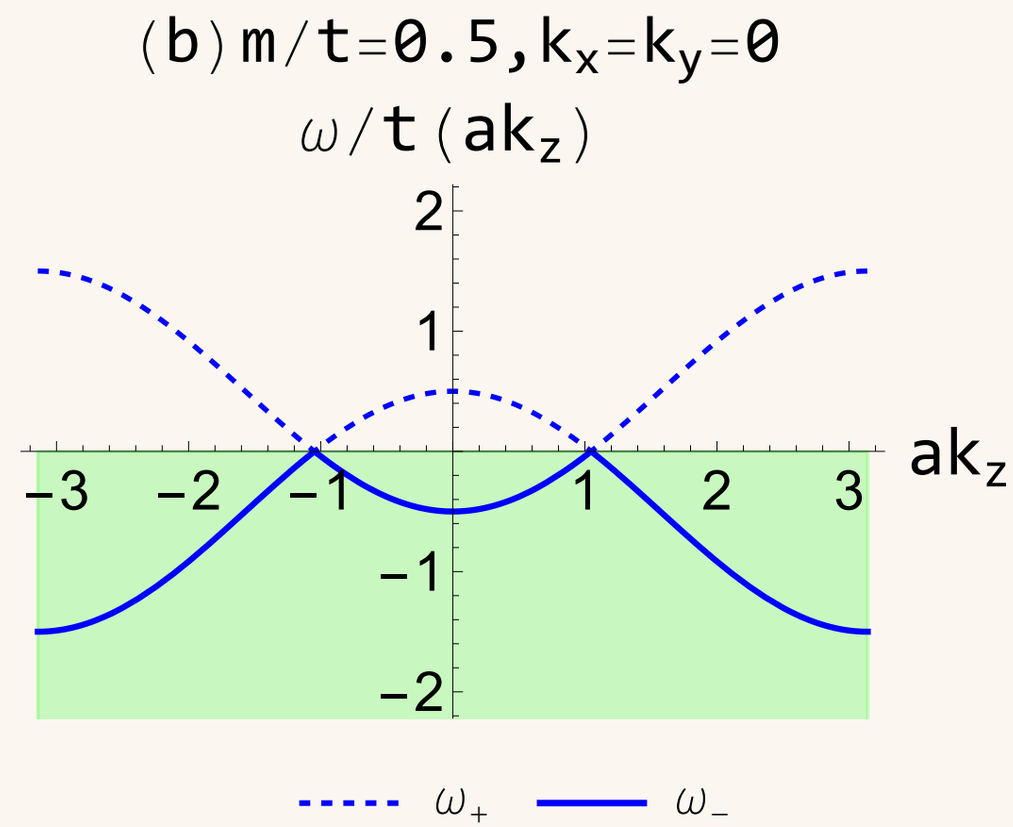


→ In (b), Dirac/Weyl points arise at $ak_z \neq 0$

[K.N., and K. Suzuki, arXiv:2207.14078]

◉ Oscillating Casimir effect





→ We find an oscillation with a period $\tau_{\rm Cas} = 6$.

→ Dirac/Weyl points are at $\frac{\pi}{3}$.

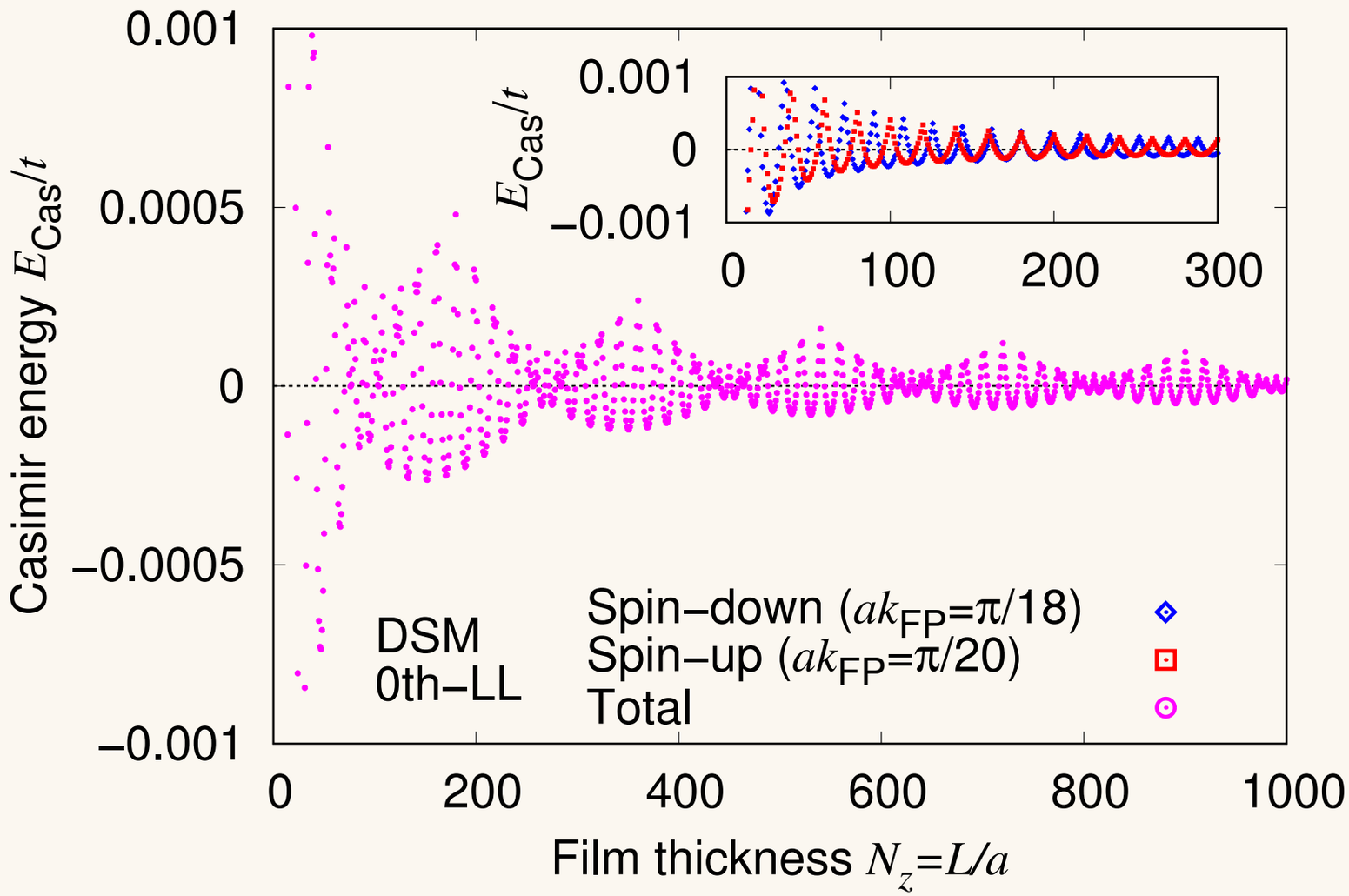
$$\tau_{\rm Cas} = \frac{2\pi}{a_z k_{\rm WP/DP}}$$

◉ Beat in spin-splitting Dirac semimetal

→ We consider Dirac semimetal with a magnetic field.

[D.H.M. Nguyen et al., arXiv:2105.06171]

$$\omega_{\uparrow}^{\mathrm{DSM-0LL}} = m - t'(1-\cos ak_z) - \pi t'\phi + \lambda_z g_{\uparrow}\phi,$$
$$\omega_{\downarrow}^{\mathrm{DSM-0LL}} = -m + t'(1-\cos ak_z) + \pi t'\phi - \lambda_z g_{\downarrow}\phi,$$



→ Similar but slightly different dispersions produce a beat.

[Z. Wang et al., arXiv:1202.5636]
[Z. Wang et al., arXiv:1305.6780]

◉ Model for realistic Dirac/Weyl semimetal

$$H^{\mathrm{DSM}}(\mathbf{k}) = \begin{pmatrix} \epsilon_0(\mathbf{k}) + M(\mathbf{k}) & A(k_x + ik_y) & D(k_x - ik_y) & B^*(\mathbf{k}) \\ A(k_x - ik_y) & \epsilon_0(\mathbf{k}) - M(\mathbf{k}) & B^*(\mathbf{k}) & 0 \\ D(k_x + ik_y) & B(\mathbf{k}) & \epsilon_0(\mathbf{k}) + M(\mathbf{k}) & -A(k_x - ik_y) \\ B(\mathbf{k}) & 0 & -A(k_x + ik_y) & \epsilon_0(\mathbf{k}) - M(\mathbf{k}) \end{pmatrix}$$

$$\epsilon_0(\mathbf{k}) = C_0 + C_1 k_z^2 + C_2(k_x^2 + k_y^2),$$

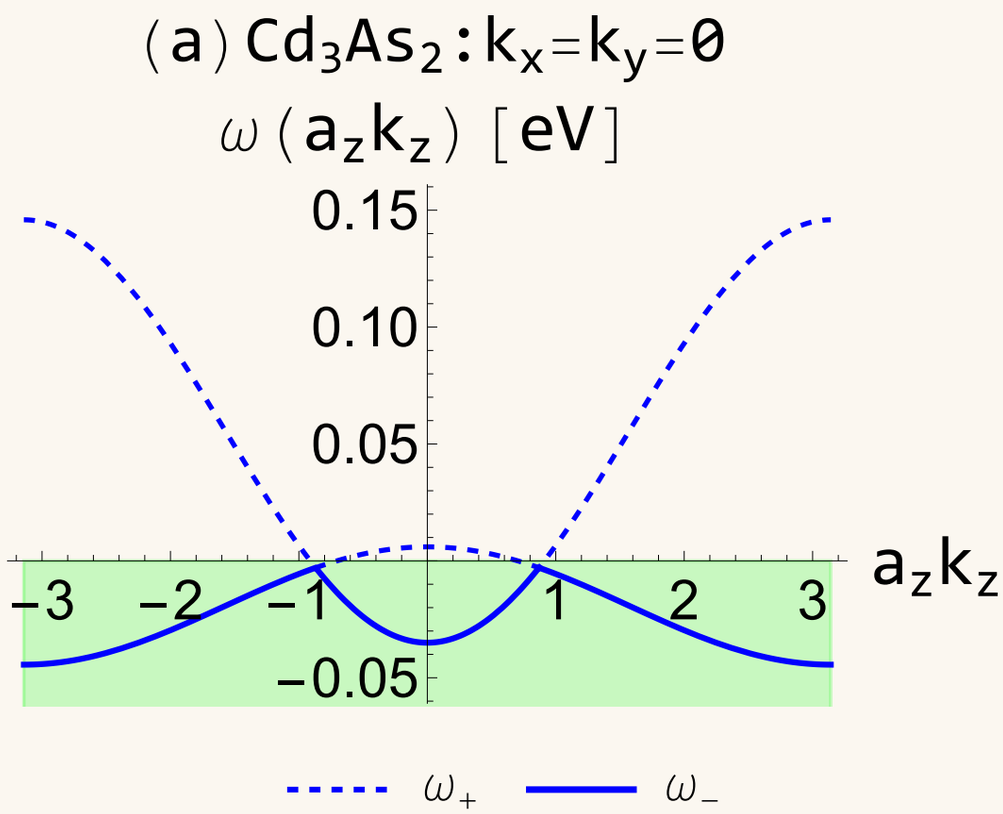
$$M(\mathbf{k}) = M_0 + M_1 k_z^2 + M_2(k_x^2 + k_y^2).$$

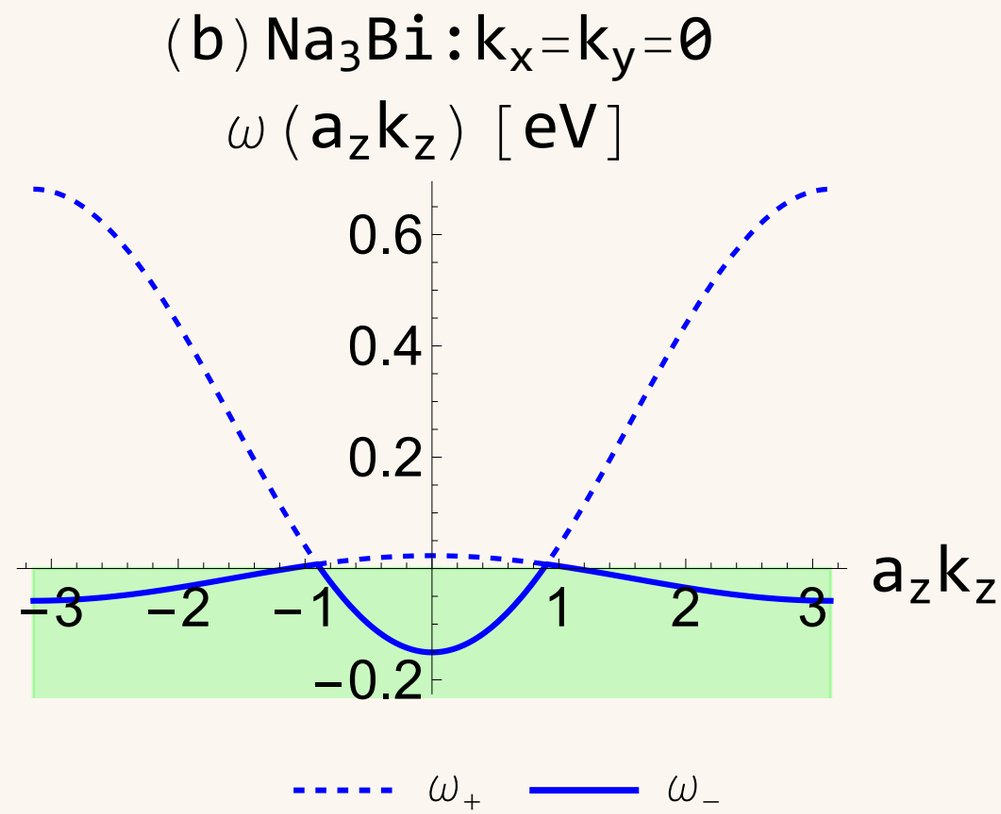
$$\omega_{\pm}^{\mathrm{DSM}} = \epsilon_0 \pm \sqrt{M^2 + A^2(k_x^2 + k_y^2) + B^2},$$

→ Parameters are consistent with the experimentally observed dispersions.

Parameters	Unstrained Cd_3As_2 [47]	Na_3Bi [11]
A (eVÅ)	0.889	2.4598
C_0 (eV)	−0.0145	−0.06382
C_1 (eVÅ^2)	10.59	8.7536
C_2 (eVÅ^2)	11.5	−8.4008
M_0 (eV)	−0.0205	−0.08686
M_1 (eVÅ^2)	18.77	10.6424
M_2 (eVÅ^2)	13.5	10.361
b_1 (eVÅ)	0	0
$a_x = a_y$ (Å)	12.67	5.448
a_z (Å)	25.48	9.655

◉ Dispersion relation of Cd3As2 and Na3Bi



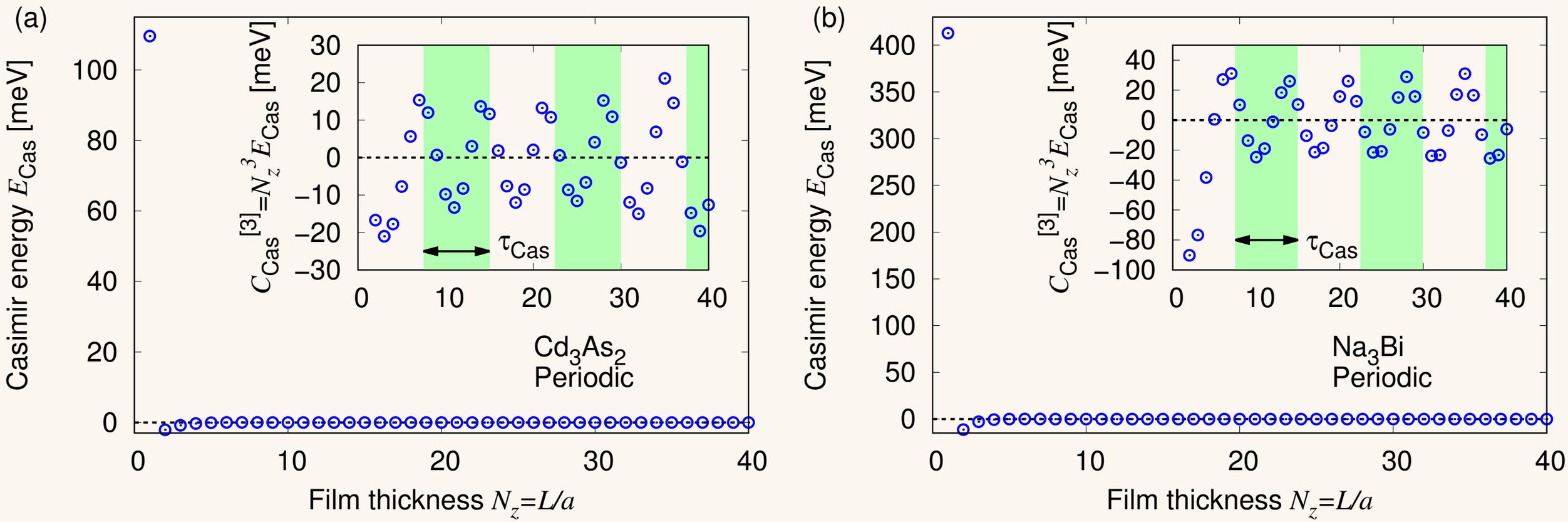


◇ Dirac points are at $\simeq \dfrac{\pi}{3.6}$

→ Periods are expected as $\tau_{\mathrm{Cas}} \simeq 7.2$

$$\tau_{\mathrm{Cas}} = \frac{2\pi}{a_z k_{\mathrm{WP/DP}}}$$

◉ Cd3As2 and Na3Bi



◇ We find oscillation periods, as expected.

→ The oscillation is a general behaviour in realistic DSMs.

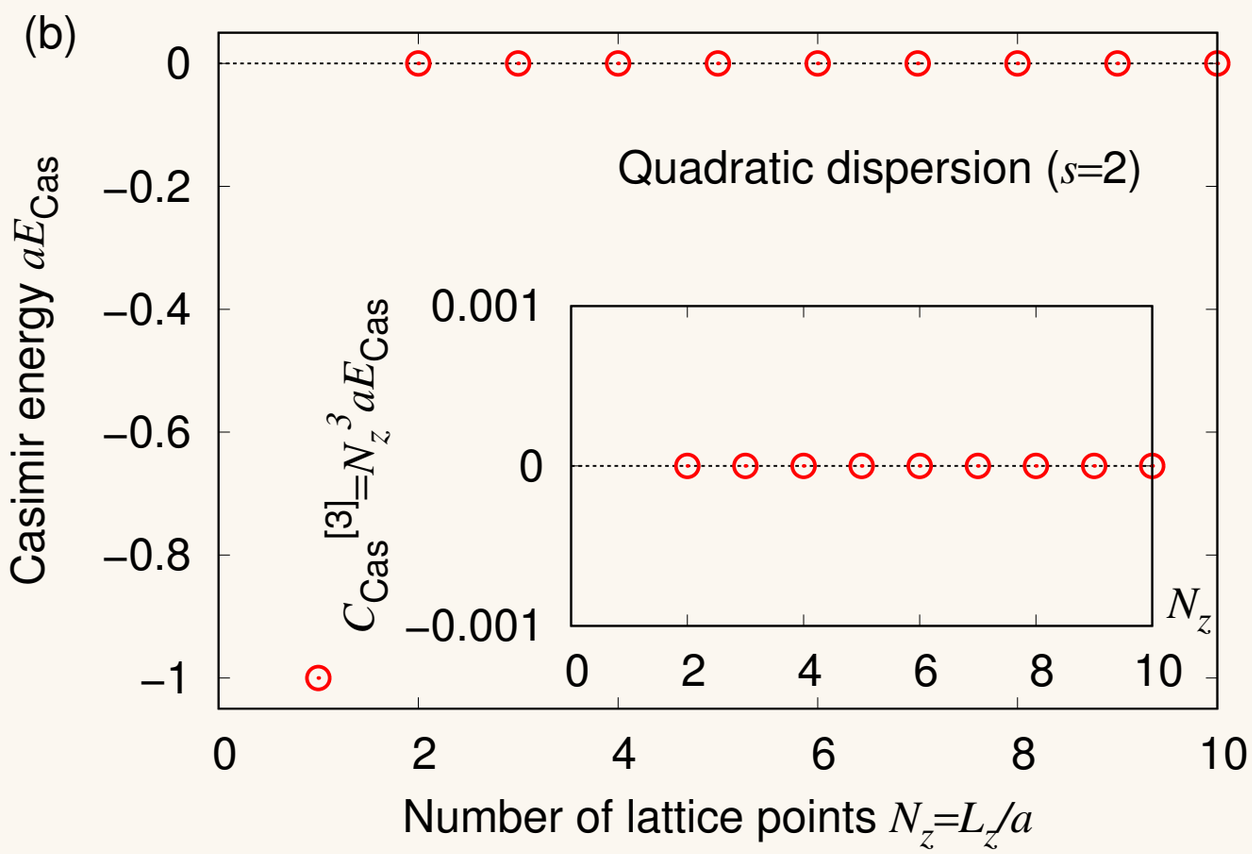
[K.N., and K. Suzuki, arXiv:2204.12032]

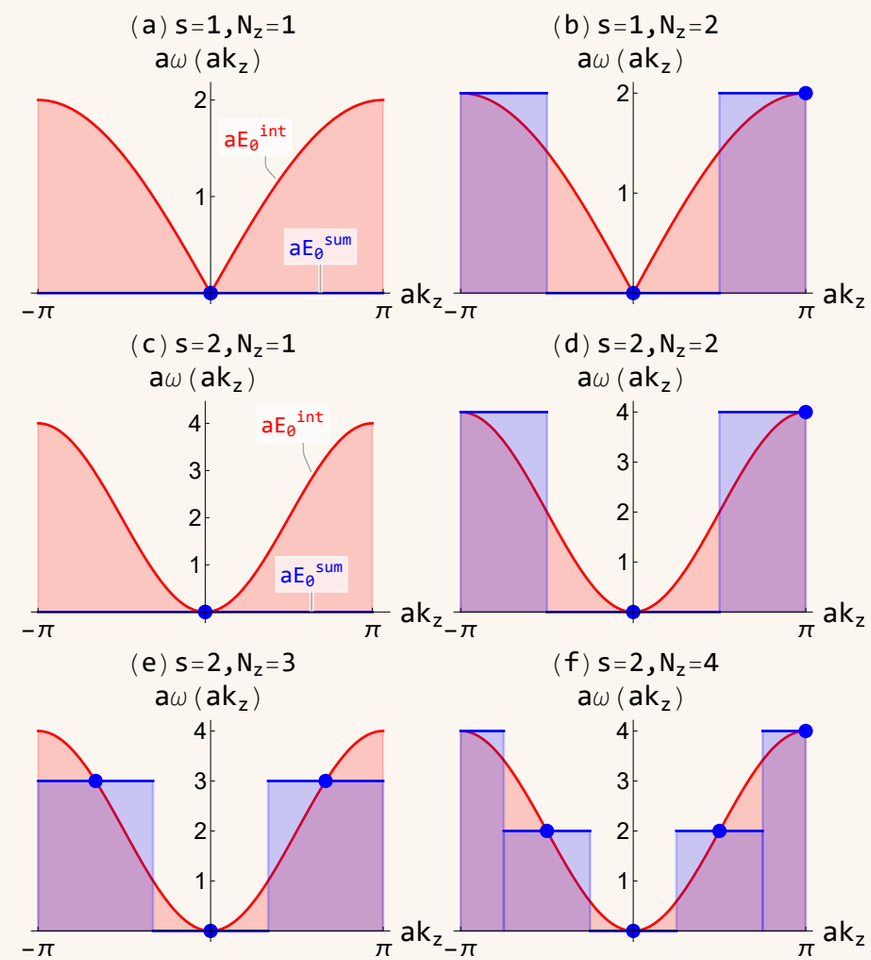
◉ Remnant Casimir effects on the lattice

◇ We also focus on quadratic dispersion.

→ In continuum, exactly zero (Zeta function regularization)

→ On the lattice, exactly zero except size N = 1.

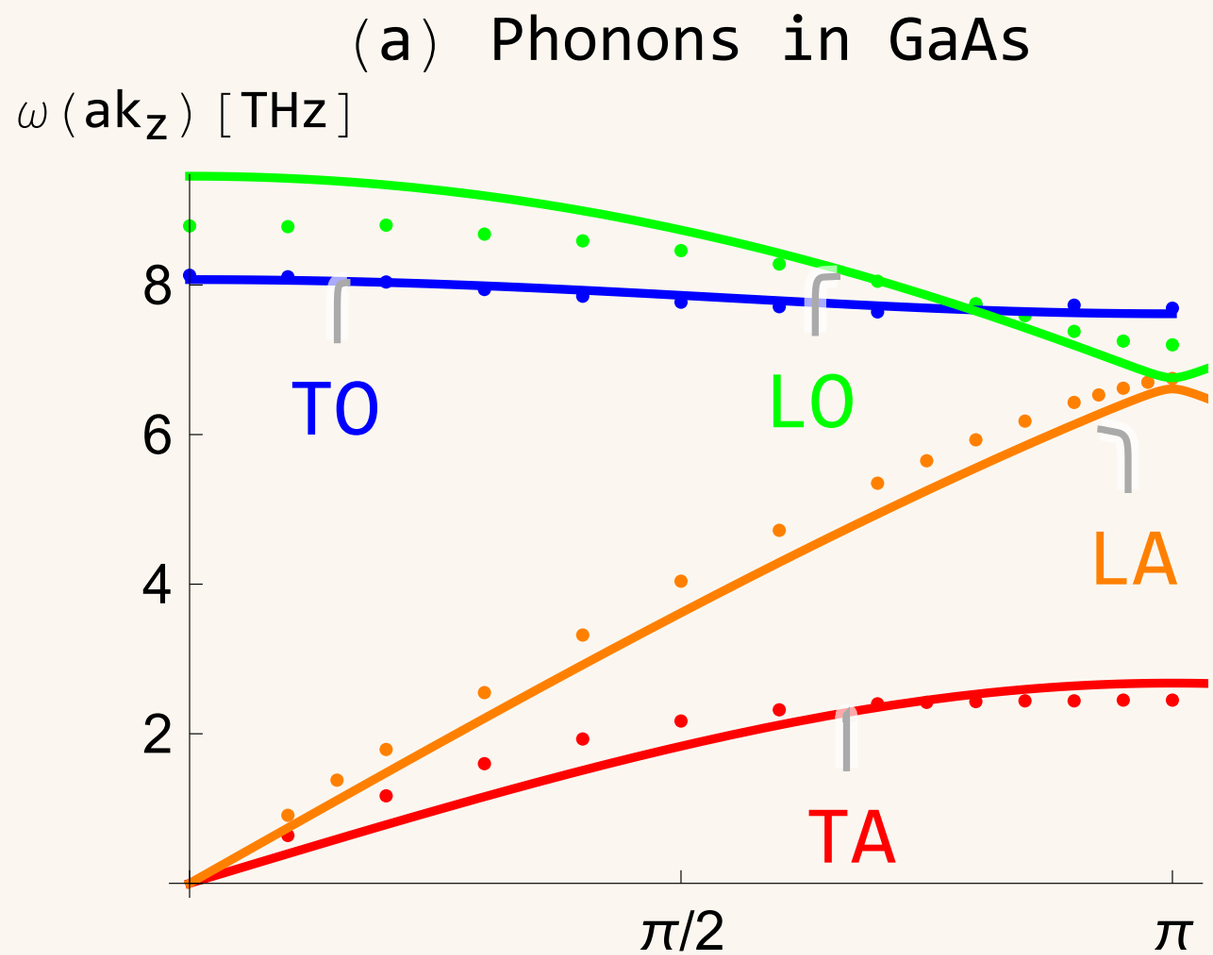


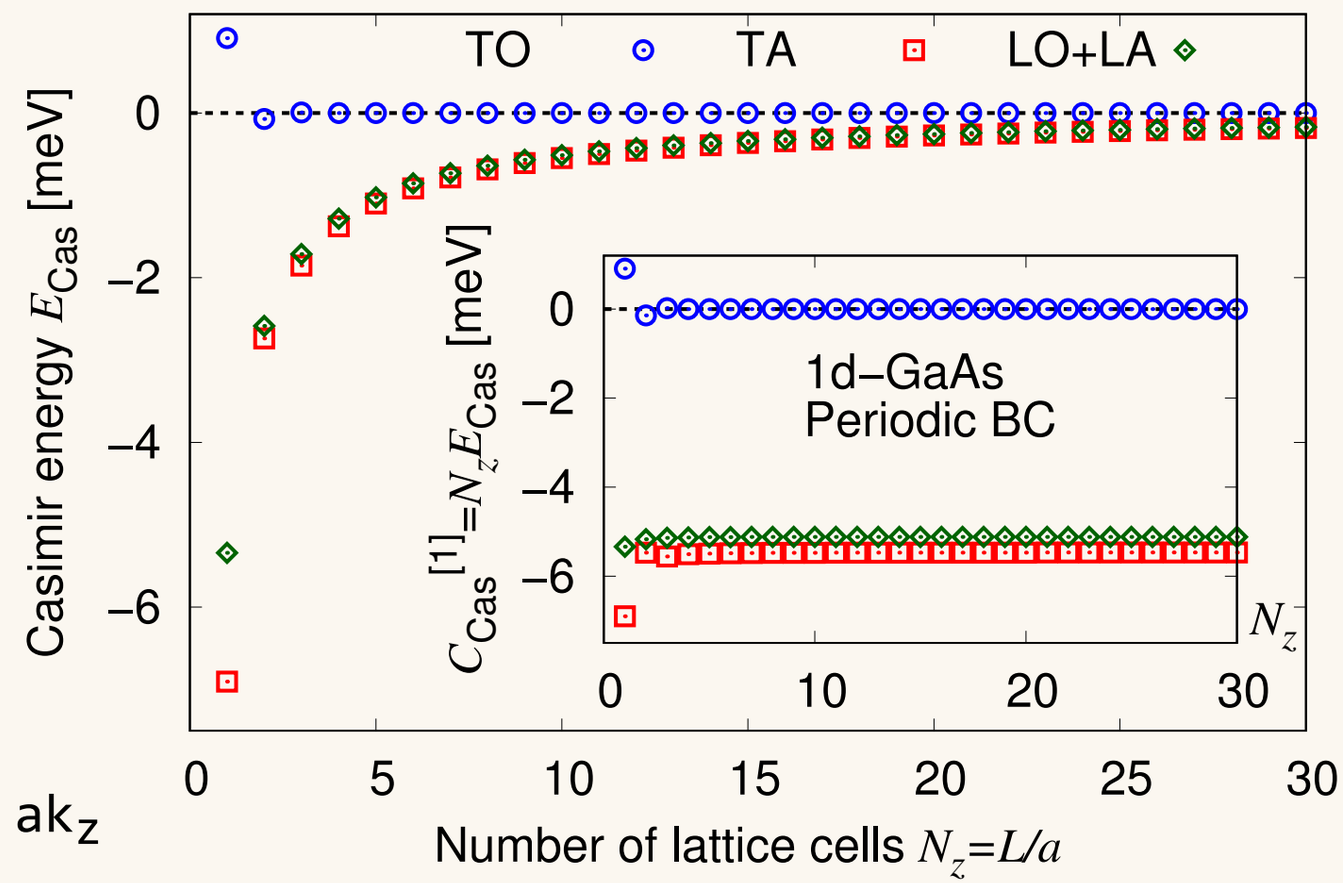


◉ Remnant Casimir effect on the lattice

◇ We also focus on quadratic dispersion.

→ Considering transverse optical phonon in GaAs nanowire





→ The remnant Casimir effect arises in realistic material.

◉ Summary

- We define and calculate Casimir energy with lattice regularization. [T. Ishikawa, K.N., and K. Suzuki, arXiv:2005.10758]

- We focus on the Dirac/Weyl semimetal and find an oscillating Casimir effect. [K.N., and K. Suzuki, arXiv:2207.14078]

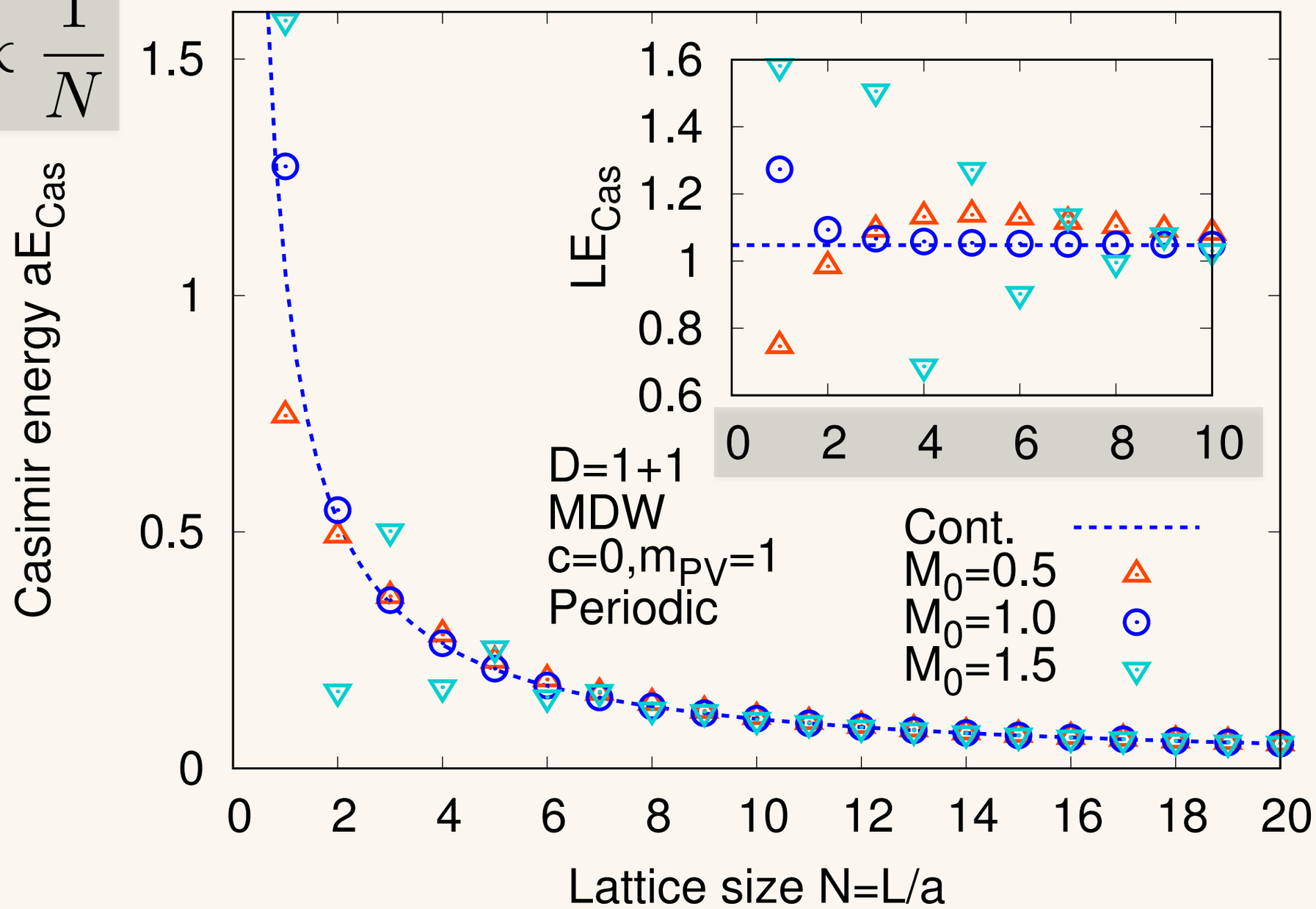
- Applying our approach to realistic materials Cd3As2 and Na3Bi. [K.N., and K. Suzuki, arXiv:2207.14078]

- The remnant Casimir effects for quadratic dispersion are also shown in phonons. [K.N., and K. Suzuki, arXiv:2204.12032]

◉ Numerical calc. of Domain-wall fermion

◇ Domain-wall fermion (Numerical)

$$aE_{\mathrm{Cas}} \propto \frac{1}{N}$$



→ Lattice "artifact" is small. $(10 \lesssim N)$

◉ Lattice regularization for Casimir energy

Lattice regularization

$$a\partial_\mu\psi(x) \to \frac{\psi(x+a)-\psi(x-a)}{2} = (\sinh a\partial_\mu)\psi(x)$$

$$\int_{-\infty}^{\infty} dk \to \int_{-\pi}^{\pi} dk \qquad \sum_{n=-\infty}^{\infty} \to \sum_{n=-N/2}^{n=N/2} \qquad (N = L/a)$$

◇ Wilson fermion

$$aD_W \equiv i\sum_k \gamma_k \sin ap_k + r\sum_k (1-\cos ap_k)$$

→ We confirm these reproduces Casimir energy.

◉ Lattice regularization for Casimir energy

→ For 1+1, with continuum time direction,

◇ Wilson fermion with periodic boundary.

$$E_{\rm vac} \propto \int_{-\pi/a}^{\pi/a} dp \sqrt{a^2 D^\dagger D} = \int_{-\pi/a}^{\pi/a} dp \sqrt{2 - 2\cos ap}$$

$$E_{\rm Cas} \propto \left(\int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} - \sum_{-N/2}^{N/2} \right) \sqrt{2 - 2\cos ap}$$

↓ (Analytical solution)

$$aE_{\rm Cas} = \frac{4N}{\pi} - 2\cot\frac{\pi}{2N}$$

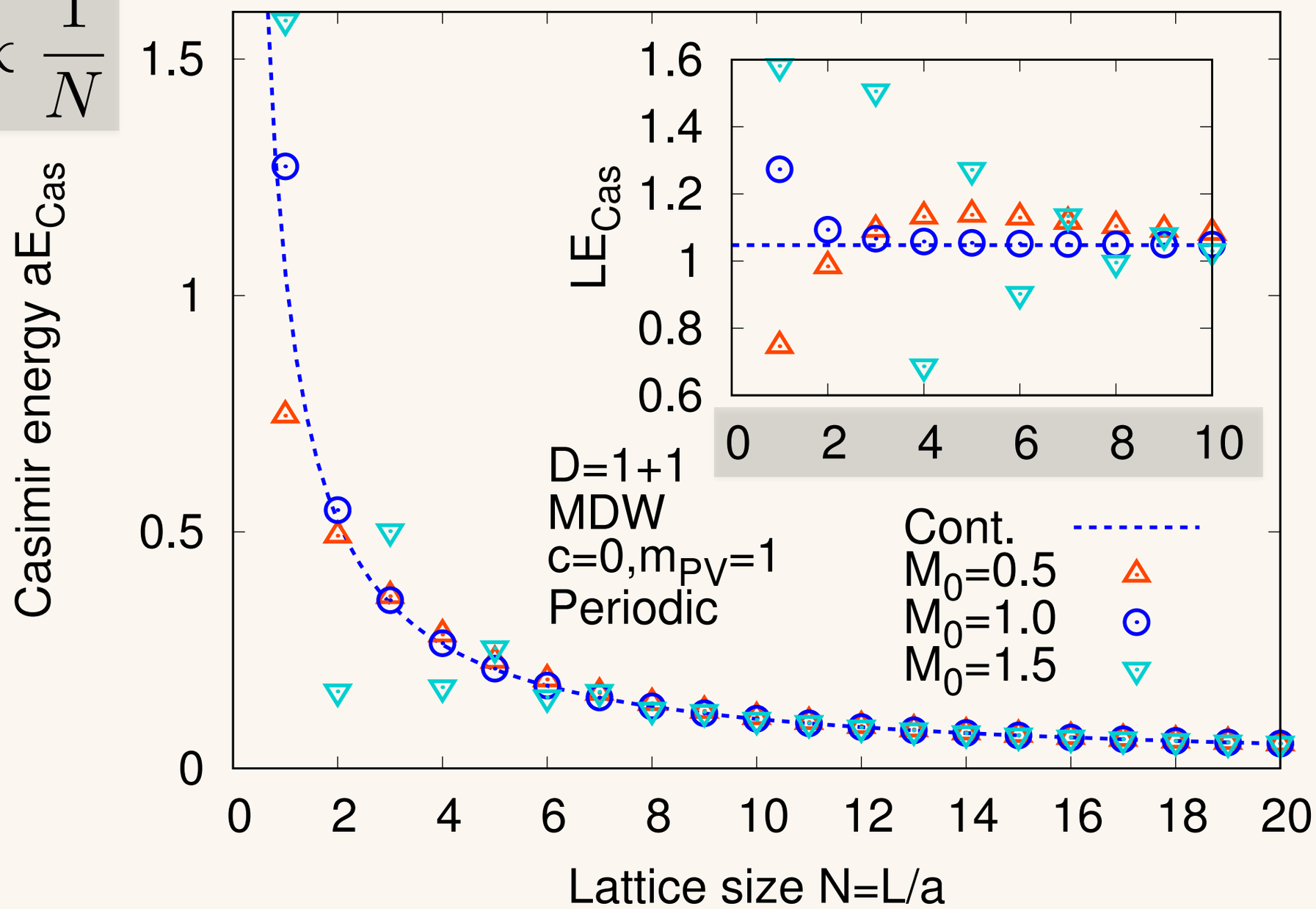
$$\rightarrow \quad E_{\rm Cas} = \frac{\pi}{3L} + O(a^2) \qquad (N = L/a)$$

→ Lattice reg. reproduces Casimir energy in the continuum.

◉ Numerical calc. of Domain-wall fermion

◇ Domain-wall fermion (Numerical)

$$aE_{\mathrm{Cas}} \propto \frac{1}{N}$$



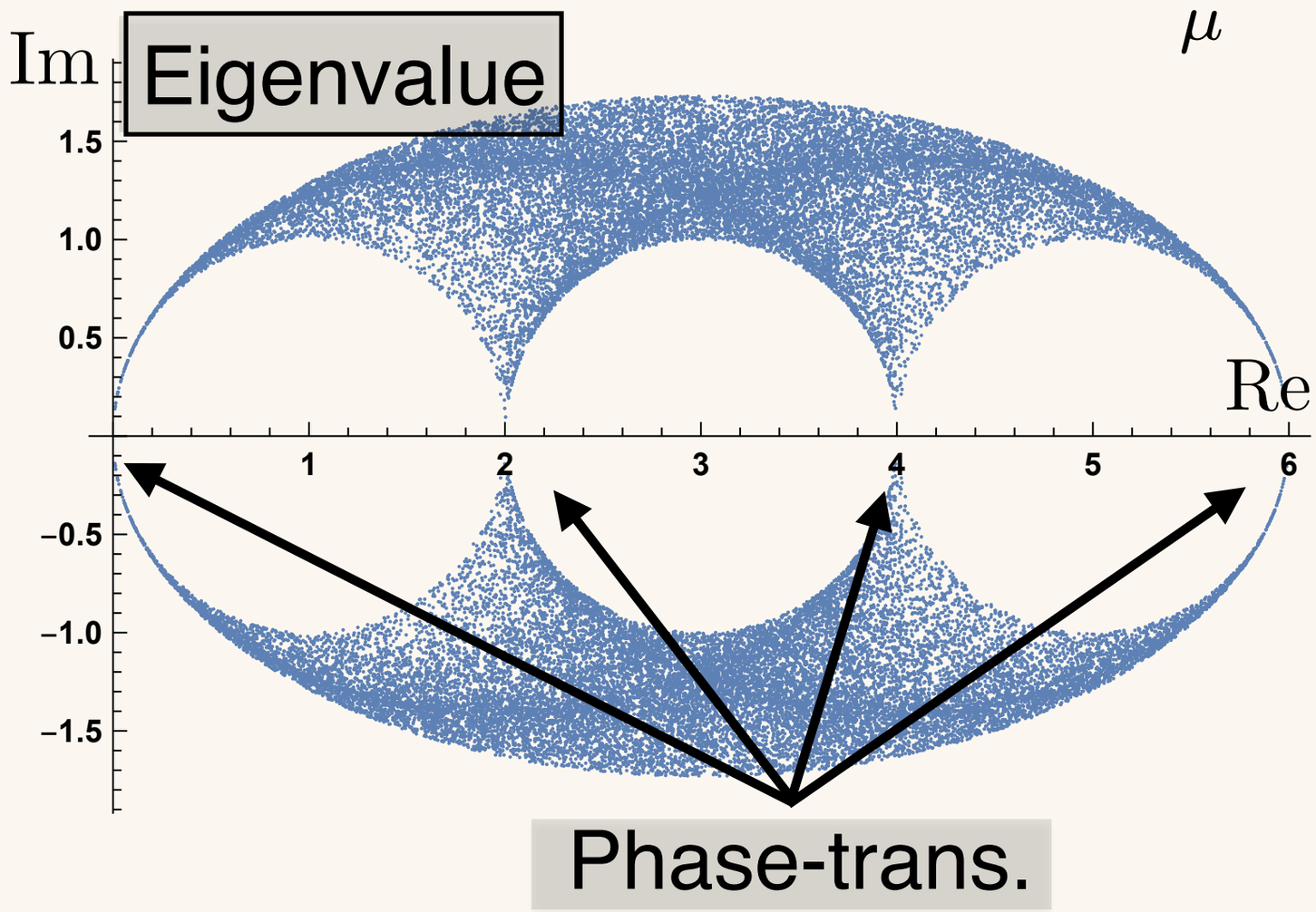
→ Lattice "artifact" is small. $(10 \lesssim N)$

◉ Topological phase-transition

◇ For cond-mat., we consider more general parameters.

→ Domain-wall hight M_0 in domain-wall fermion.

$$aD_\mu^{\mathrm{DW}} = i\gamma_\mu \sin ap_\mu + \sum_\mu (1 - \cos ap_\mu) - M_0$$



◇ For practical calc.,

$$0 < M_0 < 2$$

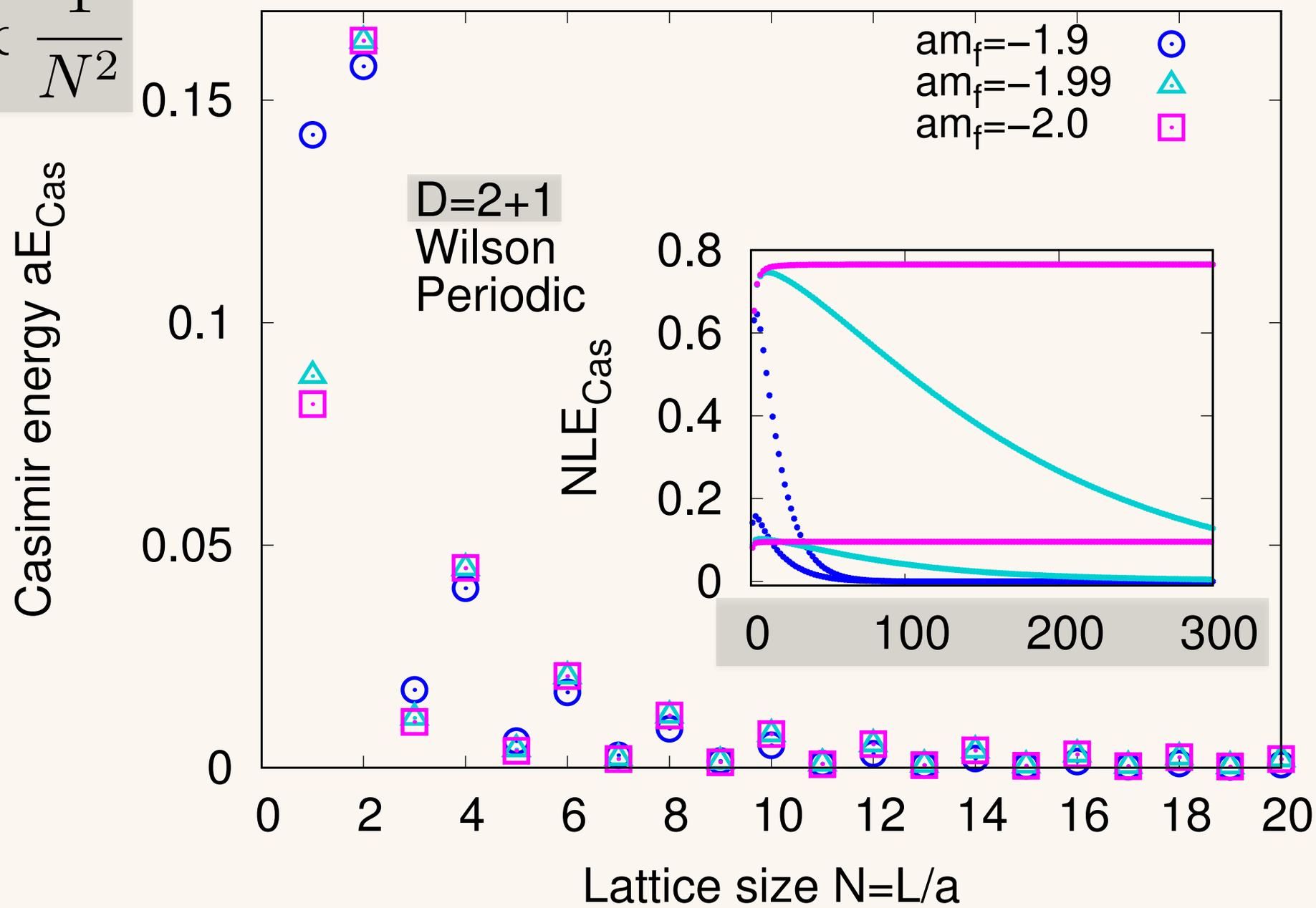
◇ Phase transitions occur at

$$M_0 = 0, 2, 4, 6$$

(Note:this is for 2+1 dim.)

◉ Negative mass Wilson fermion

$$aE_{\mathrm{Cas}} \propto \frac{1}{N^2}$$



→ Lattice "effect" is enhanced around phase trans. point

$(100 \lesssim N)$

◉ Summary

- We define and calculate Casimir energy with lattice regularization.

- Our calculation can reproduce the continuum result with other regularization scheme.

- For lattice simulations, lattice “artifact” is sufficiently small for free-fermions with the reasonable number of lattices.

- For cond-mat, lattice “effect” can be enhanced and detected around topological phase-transition point of domain-wall fermion.

◉ Abel-Plana formula

$f(z), g(z)$: meromorphic function for $a \leq x \leq b$ in $z = x + iy$

$z_{f,k}$ is the pole of $f(z)$ in $a < x < b$

$$\int_a^b f(x)dx = R[f(z), g(z)] - \frac{1}{2}\int_{-i\infty}^{+i\infty} [g(u) + \sigma(z)f(u)]_{u=a+z}^{u=b+z} dz, \ \sigma(z) \equiv \mathrm{sgn}(\mathrm{Im}\, z),$$

$$R[f(z), g(z)] = \pi i \left[\sum_k \mathrm{Res}_{z_g,k}\, g(z) + \sum_k \sigma\left(z_{f,k}\right) \mathrm{Res}_{z=z_{f,k}}\, f(z) \right]$$

[A.A. Saharian, arXiv:hep-th/0002239 (2000)]
[A.A. Saharian, arXiv:0708.1187 (2007)]

↓ (For $a, b \in \mathbb{Z}$)

$$\sum_{n=\lceil a \rceil}^{\lfloor b \rfloor} f(n) - \int_a^b dx f(x) - \left(\frac{1}{2} f(a) + \frac{1}{2} f(b) \text{ if } a, b \in \mathbb{Z} \right)$$

$$= i\int_0^\infty dy \frac{f(a+iy)}{e^{2\pi(y-ia)} - 1} - i\int_0^\infty dy \frac{f(a-iy)}{e^{2\pi(y+ia)} - 1} - i\int_0^\infty dy \frac{f(b+iy)}{e^{2\pi(y-ib)} - 1} + i\int_0^\infty dy \frac{f(b-iy)}{e^{2\pi(y+ib)} - 1}.$$

◉ Analytical calculation

$$\sum_{n=\lceil a\rceil}^{\lfloor b\rfloor} f(n) - \int_a^b dx f(x) - \left(\frac{1}{2}f(a) + \frac{1}{2}f(b) \text{ if } a,b\in\mathbb{Z}\right)$$

$$= i\int_0^\infty dy \frac{f(a+iy)}{e^{2\pi(y-ia)}-1} - i\int_0^\infty dy \frac{f(a-iy)}{e^{2\pi(y+ia)}-1} - i\int_0^\infty dy \frac{f(b+iy)}{e^{2\pi(y-ib)}-1} + i\int_0^\infty dy \frac{f(b-iy)}{e^{2\pi(y+ib)}-1}.$$

◇ Wilson fermion $\sqrt{D_{\rm W}^\dagger D_{\rm W}} = 2\sqrt{\sin^2\left(\frac{\pi n}{N}\right)}.$

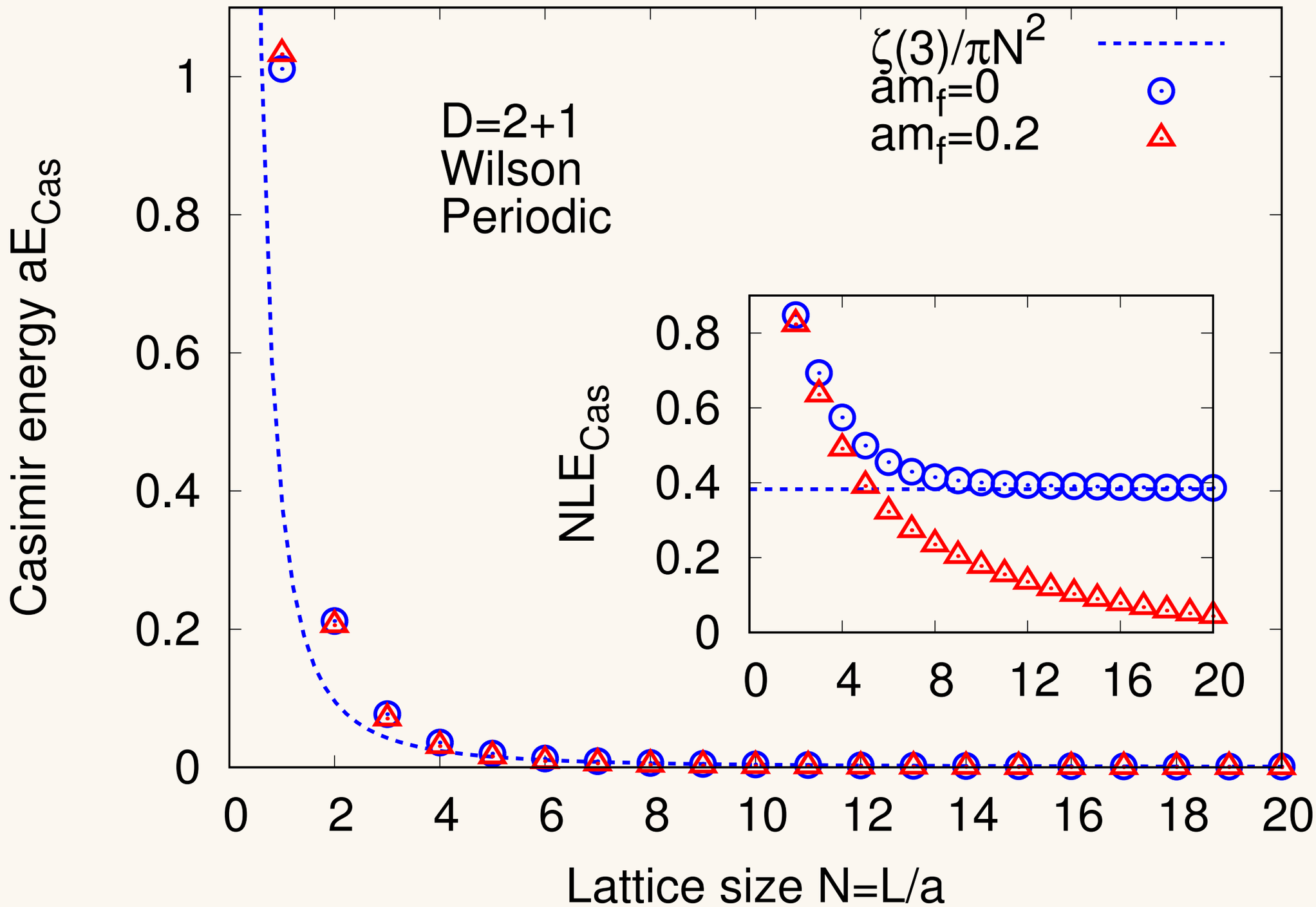
$$\sqrt{\sin^2\left(\frac{\pi}{N}(0+\epsilon\pm iy)\right)} = \pm i\sinh\left(\frac{\pi y}{N}\right) \qquad \sqrt{\sin^2\left(\frac{\pi}{N}(N-\epsilon\pm iy)\right)} = \mp i\sinh\left(\frac{\pi y}{N}\right)$$

$$\Downarrow$$

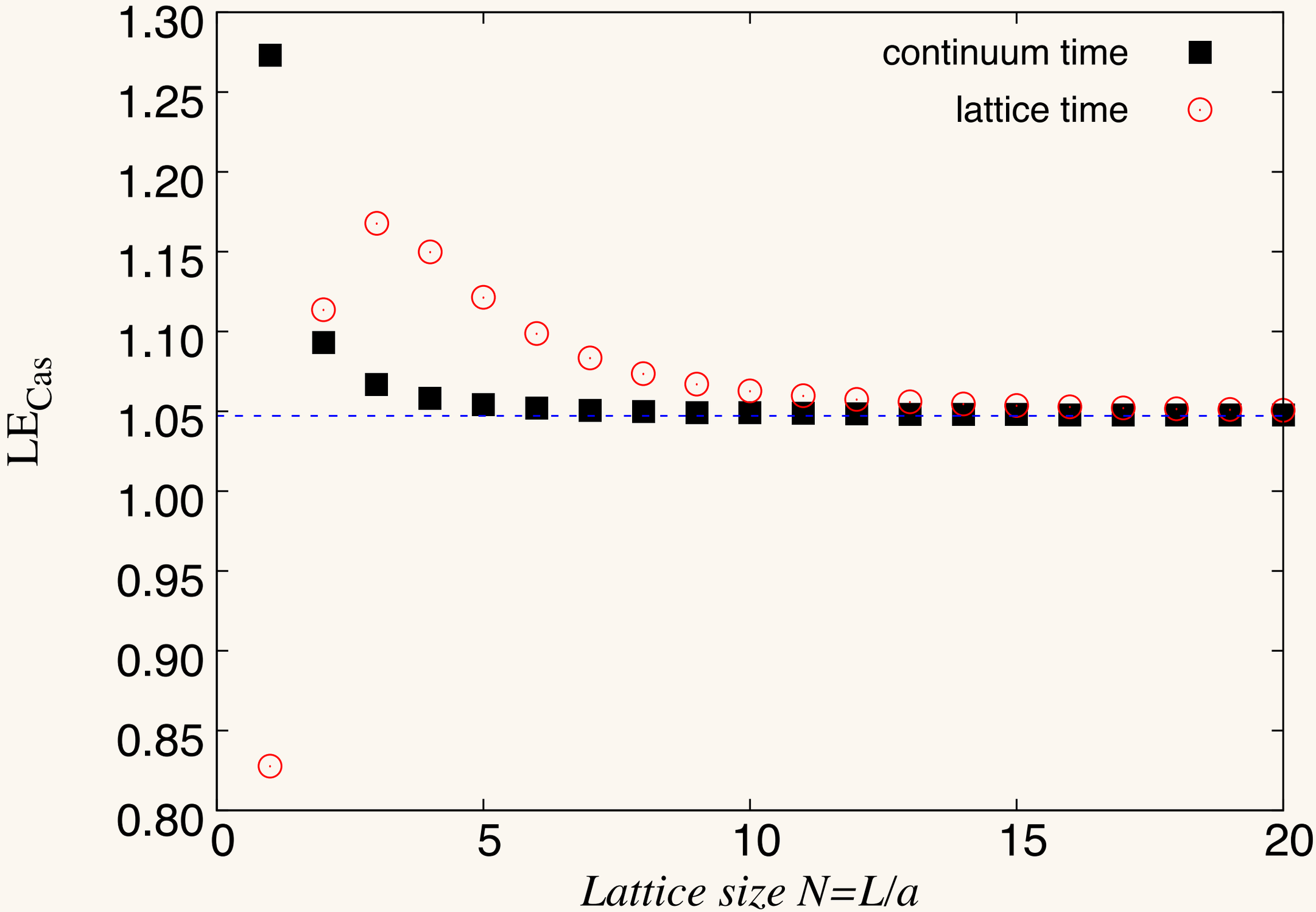
$$i\int_0^\infty \frac{dy}{e^{2\pi y}-1}\left(f(z)\big|_{z=\epsilon-iy}^{z=\epsilon+iy} - f(z)\big|_{z=N-\epsilon-iy}^{z=N-\epsilon+iy}\right)$$

$$= -8\int_0^\infty \frac{dy\sinh(\pi y/L)}{e^{2\pi y}-1} = -\frac{4N}{\pi} + 2\cot\left(\frac{\pi}{2N}\right),$$

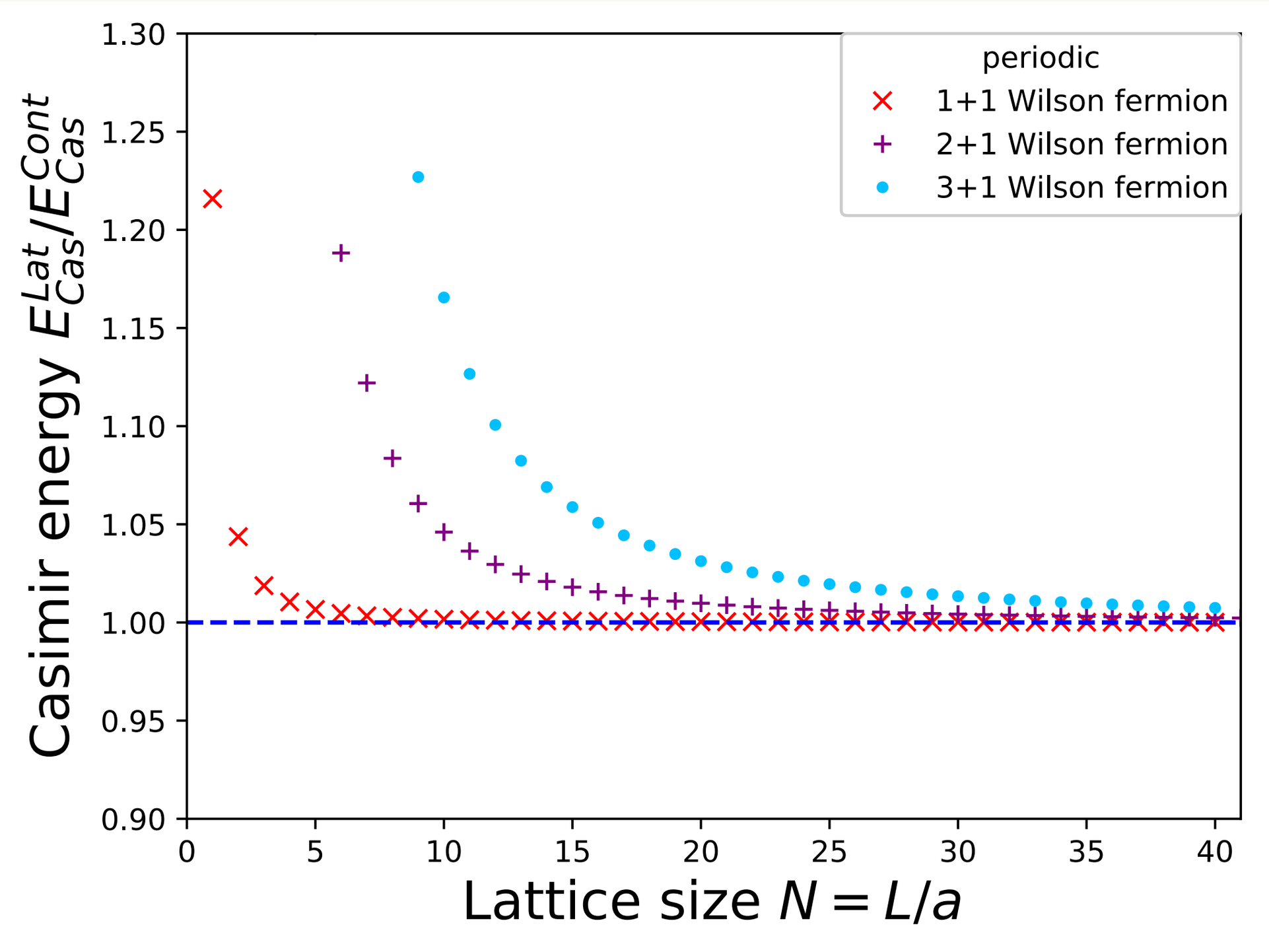
◉ Wilson fermion (2+1 dim)



◉ Wilson fermion (1+1 dim with lattice time)



◉ Wilson fermion (dependence of dimension)



◉ Domain-wall fermion near the phase trans.