Reformulation of anomaly inflow on the lattice and construction of lattice chiral gauge theories

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Outline

Goal **Construction of lattice chiral gauge theories (LCGTs)**

- Analyze the integrability condition of the chiral determinant of overlap fermions
- Derive necessary and sufficient conditions to construct LCGTs (without gauge anomalies)

Discussion **Reformulate the modern theory of anomalies on the lattice**

- Dai-Freed theorem $\longleftrightarrow$ 5-dim lattice DW fermions
- APS index theorem $\longleftrightarrow$ 6-dim lattice DW fermions

  $\Longrightarrow$ Bordism invariance of the lattice η invariant
- Triviality of the lattice η invariant $\Longrightarrow$ Integrability conditions

Introduction

Fermions on the lattice

- (d-dim) Wilson fermions

$$D_{\mathrm{w}} = -\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger$$

- (d-dim) Domain-wall (DW) fermions

$$\begin{aligned} X_{\mathrm{w}} &= D_{\mathrm{w}} - \frac{m_0}{a}, \quad m_0 \in (0,2) \\ H_{\mathrm{w}} &= \overline{\gamma}\left(D_{\mathrm{w}} - \frac{m_0}{a}\right), \quad d = \mathrm{even} \end{aligned}$$

- (d-dim) Overlap fermions

$$D_{\mathrm{ov}} = \frac{1}{2a}\left(1 + X_{\mathrm{w}} \frac{1}{\sqrt{X_{\mathrm{w}}^\dagger X_{\mathrm{w}}}}\right)$$

- Admissibility condition $\|1 - P_{\mu\nu}(x))\| < \frac{2}{5d(d-1)}$

Chiral symmetry on the lattice and gauge anomaly

- Ginsparg-Wilson relation $\{\gamma_5, D\} = 2D\gamma_5 D$

$$\Rightarrow \hat{\gamma}_5 = \gamma_5(1-2D)\ ,\ \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

- Lattice Weyl fermion

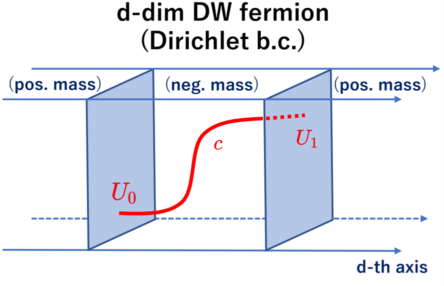
$$\psi_-(x) = \hat{P}_-\psi_-(x)\ ,\ \psi_-(x) = \sum v_i(x)c_i\ ,\ \hat{P}_- v_i(x) = v_i(x)$$

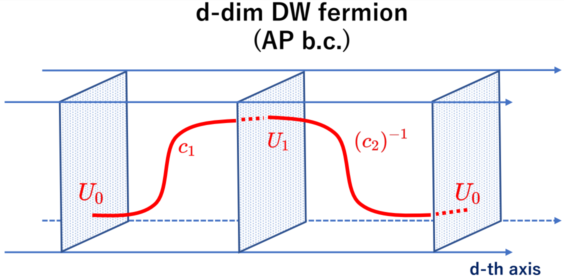
$$\overline{\psi}_-(x) = \overline{\psi}_-(x)P_+\ ,\ \overline{\psi}_-(x) = \sum \overline{c}_k \overline{v}_k(x)\ ,\ \overline{v}_k(x)P_+ = \overline{v}_k(x)$$

- Partition function $\mathrm{e}^{\Gamma_W[U]} = \int \mathcal{D}[\psi_-]\,\mathcal{D}[\bar{\psi}_-]\,\mathrm{e}^{-S_W} = \det(\bar{v}D_{\mathrm{ov}}v)$

$\Longrightarrow$ Phase ambiguity of the partition function ...lattice gauge anomaly (due to Ginsparg-Wilson relation)

- Reverse the sign of the mass along the d-th axes and take $+m \to \infty$
- Define different (d-1)-dim gauge fields(U_0, U_1) on each boundaries so that we get chiral fermions ($U_0 \sim$ reference gauge field)
- Interpolate them with a path c

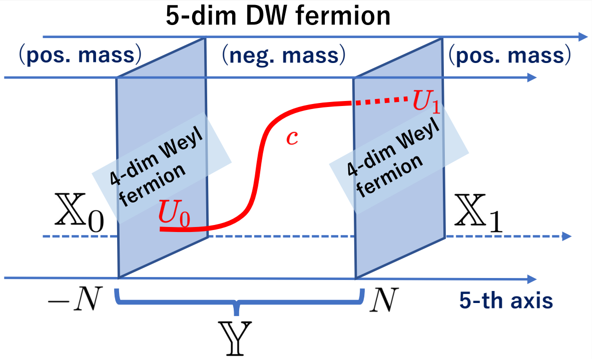




Construction of LCGTs

5-dim DW fermions and Dai-Freed theorem

Partition function of 5-dim DW fermions with Dirichlet b.c.
$+$ Chiral determinant of 4-dim Weyl fermions on the boundaries



5-dim DW fermions and Dai-Freed theorem

5-dim DW fermions $+$ 4-dim Weyl fermions

$$\lim_{N\to\infty} \frac{\det X_{\rm w}^{(5)}\Big|^{c}_{\rm Dir}}{\left|\det X_{\rm w}^{(5)}\Big|^{c\cdot c^{-1}}_{\rm AP}\right|^{1/2}} = \det\left(\overline{v}D_{\rm ov}v^1\right)\det\left(\overline{v}D_{\rm ov}v^0\right)^* \frac{\det\left(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0\right)}{\left|\det\left(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0\right)\right|}$$

$$=: \exp\left(\Gamma\left(\mathbb{X}_1\cup\overline{\mathbb{X}}_0\right)\right)\exp\left(i2\pi\eta_{\rm DF}\left(\mathbb{Y}|_{\rm Dir}\right)\right)$$

Boundary part $\exp\left(\Gamma\left(\mathbb{X}_1\cup\overline{\mathbb{X}}_0\right)\right)$ and the bulk part $\exp\left(i2\pi\eta_{\rm DF}\left(\mathbb{Y}|^{c}_{\rm Dir}\right)\right)$ cancel out the dependence of $\{v_i\}$

$\Longrightarrow$ Anomaly inflow based on Dai-Freed theorem

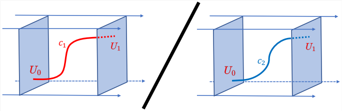
Anomaly $\leftrightarrow$ bulk dependency $\exp\left(i2\pi\eta_{\rm DF}\left(\mathbb{Y}|_{\rm Dir}\right)\right)$

The lattice η invariant and integrability conditions

- Define the lattice η invariant with the phase of 5-dim overlap fermions [T.Aoyama and YK, 1999]:

$$e^{i2\pi\eta\left(\mathbb{Y}|_{\mathrm{Dir/AP}}\right)} := \lim_{N\to\infty}\left[\frac{\det D_{\mathrm{ov}}^{(5)}\Big|_{\mathrm{Dir/AP}}}{\left|\det D_{\mathrm{ov}}^{(5)}\Big|_{\mathrm{Dir/AP}}\right|}\right]^2 = \lim_{N\to\infty}\frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir/AP}}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir/AP}}\right|}$$

- Bulk dependence



$$= \frac{e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}^{c_1}\right)}}{e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}^{c_2}\right)}} = e^{i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}^{c_1 c_2^{-1}}\right)}$$

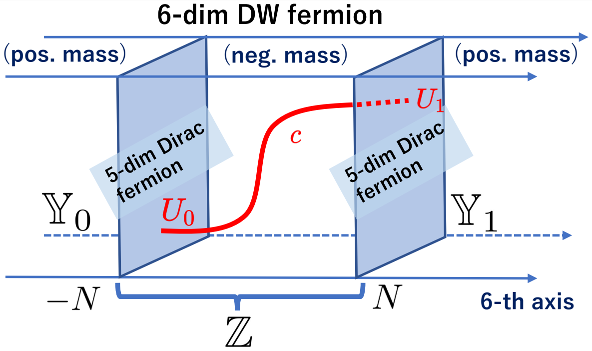
$\Longrightarrow$ Bulk independency $\quad e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}^{c_1}|_{\mathrm{Dir}}\right)} = e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}^{c_2}|_{\mathrm{Dir}}\right)}$

"$e^{i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right)} = 1$ for arbitrary gauge configurations "

(Integrability condition)

6-dim DW fermions and APS index theorem

Partition function of 6-dim DW fermions with Dir. b.c.
$+$ Determinant of 5-dim overlap fermions on the boundaries



6-dim DW fermions and APS index theorem

6-dim DW fermions + 5-dim overlap fermions

$$\lim_{N\to\infty} \frac{\det X_{\rm w}^{(6)}\Big|^{c}_{\rm Dir}}{\left|\det X_{\rm w}^{(6)}\Big|^{cc^{-1}}_{\rm AP}\right|^{1/2}} = \det D_{\rm ov}^{(5)}\Big|_{\rm Y_1} \left\{\det D_{\rm ov}^{(5)}\Big|_{\rm Y_0}\right\}^{*} \times \frac{\det \mathcal{T}|^{c}_{\rm APS}}{|\det \mathcal{T}|^{c}_{\rm APS}|}$$

(w/ $\mathcal{T}|^{c}_{\rm APS} \equiv \frac{1}{2}\Phi_1 \cdot \prod_{t\in\tilde{c}} T_t^{(5)} \cdot \Phi_0$, Φ_i: basis in $\mathbb{Y}_i$)

$$\text{L.H.S.} \quad = \quad (-1)^{I\left(\mathbb{Z}|_{\rm Dir}\right)} \quad , \quad I\left(\mathbb{Z}|_{\rm Dir}\right) = -\frac{1}{2}\,{\rm Tr}\left\{ H_{\rm w}^{(6)}/\sqrt{H_{\rm w}^{(6)2}}\Big|_{\rm Dir}\right\}$$

$$\text{Boundary} \quad = \quad \exp\left(i\pi\eta\left(\mathbb{Y}|_{\rm AP}\right)\right)\exp\left(-i\pi\eta\left(\mathbb{Y}|_{\rm AP}\right)\right)$$

[H.Fukaya et. al.,2020]

$$\text{Bulk} \quad =: \quad \exp\left(i\pi P\left(\mathbb{Z}|^{c}_{\rm APS}\right)\right)$$

$\Longrightarrow$ APS index theorem

$$I\left(\mathbb{Z}|_{\rm Dir}\right) = P\left(\mathbb{Z}|^{c}_{\rm APS}\right) + \eta\left(\mathbb{Y}_1|_{\rm AP}\right) - \eta\left(\mathbb{Y}_0|_{\rm AP}\right)$$

The cohomological problem

- $P\left(\mathbb{Z}|^{c}_{\mathrm{APS}}\right)$ can be expressed with the lattice topological field $q^{(6)}(z) := -\frac{1}{2}\,\mathrm{tr}\left\{ \left.\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\right|_{\mathrm{AP}} \right\}(z,z)$:

$$P\left(\mathbb{Z}|^{c}_{\mathrm{APS}}\right) = \lim_{N\to\infty} \sum_{y,s\in c} q^{(6)}(z)$$

- **(The cohomological problem)** Assumption: $q^{(6)}$ can be expressed with the lattice Chern character and gauge invariant currents:

$$q^{(6)}(z) = \hat{c}_3(z) + \partial^*_\mu k_\mu(z)$$

- Under the perturbative anomaly cancellation condition $\hat{c}_3(z) = 0$ (i.e. $\Sigma_R \,\mathrm{Tr}_R \left[T^a \left\{T^b, T^c\right\}\right] = 0$):

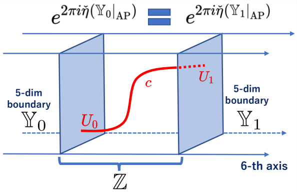
$$P\left(\mathbb{Z}|^{c}_{\mathrm{APS}}\right) = \sum_{y} k_s(z)|_{\mathbb{Y}_1} - \sum_{y} k_s(z)|_{\mathbb{Y}_0}$$

Bordism invariance of the lattice η invariant

- Redefine the lattice η invariant:

$$\check{\eta}\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right) = \eta\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right) + \sum_{y} k_s(y,s)|_{\mathbb{Y}_{1,0}}$$

$$\Longrightarrow \quad e^{2\pi i\check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right)} e^{-2\pi i\check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right)} = (-1)^{2I\left(\mathbb{Z}|^{c}_{\mathrm{Dir}}\right)} = 1 \text{ (from APS index theorem)}$$



Bordism Invariance

- d-dim manifolds Y and Y' are **bordant** if there exist a (d+1)-dim manifold Z such that $\partial Z = Y_1 \sqcup Y_2$
- An amount $\alpha(Y)$ is **bordism invariant** if $\alpha(Y_1) = \alpha(Y_2)$ for bordant Y_1 and Y_2

$\Longrightarrow$ $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)}$ is "bordism" invariant

Able to evaluate $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)}$ in arbitrary gauge configurations

Looking back on our discussion

1. **Problem** Gauge anomaly of 4-dim lattice chiral fermion
   Solution Anomaly inflow with 5-dim object $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$

2. **Problem** 5-dim dependency
   Solution $\exp(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_1}_{\mathrm{Dir}}\right))/\exp(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_2}_{\mathrm{Dir}}\right)) = \exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right))$
   $\Longrightarrow$ If $\exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right)) = 1$ for any gauge configurations, it doesn't depend on the bulk! (Integrability condition)

3. **Problem** Calculating $\exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right))$ for any gauge configurations
   Solution Redefining $\check{\eta}$ $\cdots$ "bordism" invariant ($\leftarrow$Cohomological problem)
   $\Longrightarrow$ Only need to calculate it on representatives of "bordism" equivalent class

Summary: Construction of LCGTs

- We derived two conditions to construct LCGTs:

Condition-1 Triviality of $q^{(6)}(z)$ under the perturbative anomaly cancellation condition

Condition-2 $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)} = 1$ for representative gauge configurations of "bordism" equivalent class following the admissibility condition on the 5-dim lattice space

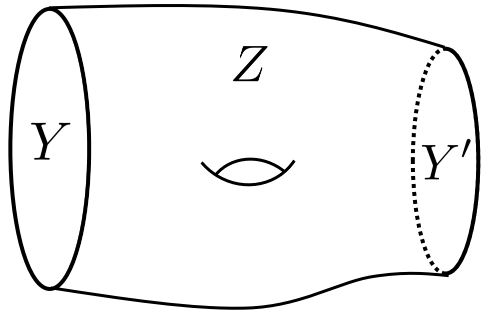
- It is known that Condition-1 holds in $U(1)$ and $SU(2) \times U(1)$
[M.Luscher, 1999] [YK and Y.Nakayama,2001]
$\longrightarrow$We further confirmed that Condition-2 also holds in those cases

- We gave a proof of Condition-1 in generic non-Abelian gauge theories
[JWP and YK, in preparation]

Thank you!

Bordism

- d-dim manifolds Y and Y' are **bordant** if there exist a (d+1)-dim manifold Z such that $\partial Z = Y_1 \sqcup Y_2$
- An amount $\alpha(Y)$ is **bordism invariant** if $\alpha(Y_1) = \alpha(Y_2)$ for bordant Y_1 and Y_2



$SU(2) \times U(1)$ chiral gauge theory

Consider a 5-dim lattice space $\mathbb{L}^5(\mathbb{L}_{\mathrm{P}}^4 \times \mathbb{L}_{\mathrm{AP}})$

Gauge DOF[ref?]

1. U(1) magnetic flux $m_{\mu\nu}$ ... Defined on $\mathbb{L}_{\mathrm{P}}^2$ among $\mathbb{L}_{\mathrm{P}}^4$
2. U(1) Wilson line W_μ ... Defined on $\mathbb{L}_{\mathrm{P}}^4$ and wraps along $\mathbb{L}_{\mathrm{AP}}$ direction
3. SU(2) Instanton ϕ ... Defined on either $\mathbb{L}_{\mathrm{P}}^4$ or $\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}$

Result: $SU(2) \times U(1)$ chiral gauge theory

U(1) magnetic flux	U(1) winding	SU(2) instanton	determinant phase
0	0	on $\mathbb{L}_{\mathrm{P}}^4$	$1 = 1^4$
0	0	on $\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	$1 = (-1)^4$
1	0	0	1
1	1	0	1
1	0	on $\mathbb{L}_{\mathrm{P}}^4$	1
1	0	on $\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	1
1	1	on $\mathbb{L}_{\mathrm{P}}^4$	1
1	1	on $\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	1

$\Longrightarrow$ Condition-2 confirmed!

Proof of Condition-1

Setup: 6-dim, $\hat{c}_3 = 0$, $\underline{q(x) = \partial^*_\mu k_\mu(x)}$

- Taking complete axial gauge, we define link variables with a product of some independent plaquette variables (from a reference point x_0)

$$\hat{P}\left(x, \mu, \nu; x^{(0)}\right) \sim \prod_{\tau} \prod_{y=x_0}^{x} \hat{U}(y, \tau)$$

- Introduce a parameter s:

$$\hat{P}_s\left(x, \mu, \nu; x^{(0)}\right) \sim \prod_{\tau} \prod_{y=x_0}^{x} \hat{U}^s(y, \tau) \quad , \quad q_s(x) = q(x)|_{\hat{U} \to \hat{U}_s}$$

Proof of Condition-1

- Taylor expansion of $q_s(x)$:

$$q_s(x) = \sum_{n=5}^{\infty} \frac{s^n}{n!} q_0^{(n)}(x)$$

- We can rewrite $q_0^{(j)}(x)$ with gauge-invariant currents form lower degree

$\Longrightarrow$ We can show the triviality of the original field $q(x)$ with those currents

Details will be discussed in our paper in prep.