’t Hooft anomalies for staggered fermions

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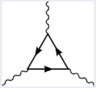
Anomalies - continuous symmetries

Perturbative gauge anomalies



$\rightarrow$ inconsistent unless anomaly cancels.

Perturbative 't Hooft anomalies

Imagine gauging **global symmetry**

Non-zero anomaly $\overset{\text{RG invariant}}{\rightarrow}$ **massless** states in I.R

- Massless (composite) fermions
- SSB $\rightarrow$ Goldstones $\leftarrow$ chiral symmetry in QCD

Constrains IR phases/dynamics...

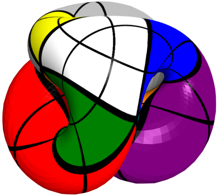
Non-perturbative anomalies:

eg. $SU(2)$ with 1 Weyl fermion (Witten)

Recent developments

Non-perturbative 't Hooft anomalies for **discrete** symmetries

- eg. reflections (parity) R.
- R flips orientation globally. Gauge: flip locally.
- Expose anomaly: consider non-trivial background gauge field. Use **non-orientable** manifold.
- Anomaly: partition function Z acquires a phase.
- $\rightarrow$ **new** constraints on IR phases.



RP^2

In this talk ...

Anomalies of this type exist for staggered fermions !

Features:

- Anomalies are **exact** on lattice.
- Gravitational in nature $\leftarrow$ curved space.
- Involve $\gamma_5 \times \gamma_5 = \epsilon(x)$ **twisted** (flavored) chiral op.
- Constrain IR phases of staggered fermions in flat space

To see them:

Generalize staggered fermions to random triangulations of curved spaces.
$\rightarrow$ (discrete) **Kähler–Dirac** fermions in **non-trivial background gravitational field**

Kähler–Dirac fermions generalize staggered fermions

What are Kähler–Dirac fermions ?

- Fermions $\Psi = (\Psi, \Psi_i, \Psi_{ij}, \ldots)$ live on sites, links, triangles, p-simplices of triangulation.
- Kähler–Dirac equation: $(K + m)\Psi = 0$

  $$K^2 = \Box \textbf{ alternative to Dirac}$$

  $K = \sum_p I_p - I_p^T$ where I_p is $N_{p+1} \times N_p$ incidence matrix of **oriented** triangulation.
  eg K returns (signed) list of link fields that lie on boundary of given triangle.
- **On hypercubic lattice Kähler–Dirac $\equiv$ staggered fermions.**
  Describe 4 Dirac fermions in continuum

Discrete Kähler–Dirac fermions allow us to couple staggered fermions to gravity $\rightarrow$ anomaly

Symmetries of Kähler–Dirac fermions

Operator Γ

$$(\mathrm{p-simplex\ field}) \quad \stackrel{\Gamma}{\rightarrow} (-1)^p \quad (\mathrm{p-simplex\ field})$$

$[\Gamma, K]_+ = 0$. Generates exact $U(1)$ symmetry of $m = 0$ theory

$$\Psi \rightarrow e^{i\alpha\Gamma}\Psi$$

$$\overline{\Psi} \rightarrow \overline{\Psi} e^{i\alpha\Gamma}$$

Can be used to project out dof:

$$\Psi_\pm = \frac{1}{2}(1 \pm \Gamma)\Psi$$

reduced Kähler–Dirac field

(cf **reduced** staggered fermion)

$$\overline{\Psi} K \Psi \rightarrow \overline{\Psi}_+ K \Psi_- + \overline{\Psi}_- K \Psi_+ \leftarrow \mathrm{drop}$$

Using $\Phi = \begin{pmatrix} \overline{\Psi}_+ \\ \Psi_- \end{pmatrix} \stackrel{RKD}{\rightarrow} \Phi^T K \Phi$ (Note: $\Gamma \rightarrow \epsilon(x)$ for staggered)

Perturbative gravitational anomaly for $U_\Gamma(1)$

Under Γ

$$\delta S_{\rm KD} = 0 \text{ when } m = 0$$

But measure not invariant

$$D\Psi D\overline{\Psi} = \prod_p d\psi_p d\overline{\psi}_p \to e^{2iN_0\alpha} e^{-2iN_1\alpha} .. e^{2i(-1)^d N_d\alpha} \prod_p d\psi_p d\overline{\psi}_p$$
$$= \; e^{2i\chi\alpha} D\overline{\Psi} D\Psi \quad \chi = \text{ Euler characteristic}$$

Anomaly in even dimensions

Compactify $R^{2n} \to S^{2n}$ with $\chi = 2$. Breaks $U(1) \to Z_4$
Index theorem: $\chi = n_+ - n_-$
$n_\pm$ is number of zero modes with $\Gamma = \pm 1$

Note

Lattice calc. agrees with continuum
Example of QM anomaly for finite number dof ...

Immediate consequence

If $U(1)$ local ...

Anomaly implies theory no longer gauge invariant !

Gauging $U(1)_\Gamma \equiv$ coupling **reduced** Kähler–Dirac fermions to gravity.
Inconsistent !

analogy

ABJ anomaly for Dirac implies cannot gauge axial symmetry
$\rightarrow$ cannot couple Weyl fermions to U(1)

But this is not the end of the anomaly story for Kähler–Dirac fermions ..

Reflection symmetry

A 't Hooft anomaly ...

Staggered theory invariant under a reflection $R : x_1 \to -x_1$.
Gauge this - use **non-orientable triangulation**
Natural choice - real projective plane
$RP^n \sim S^n/Z_2$ with $\chi = 1$

Reveals **non-perturbative** mixed 't Hooft anomaly between Z_4 and R

To see this ..

Decompose Ψ into 2 reduced fields:
Couple to scalar field σ to generate eg four fermion terms
Under $Z_4 : \Psi \to i\Gamma\Psi \text{ and } \sigma \to -\sigma$

$$\text{Fermion op } M_F = K\delta^{ab} + \sigma\epsilon^{ab} \leftarrow \textit{real}, \textit{antisymmetric}.$$

Non-perturbative anomaly

Formally $Z_F = \mathrm{Pf}(M_F(\sigma))$

Easy to see that $\det M_F(\sigma) = \det M_F(-\sigma)$

BUT $\mathrm{Pf}(M_F) = \pm\sqrt{\det(M_F)}$

Define $\mathrm{Pf}(M_F) = \prod_{\mathrm{sgn}(-i\lambda_i)>0} \lambda_i(\sigma_0)$ some ref σ_0

Require Pfaffian be smooth function of $\lambda_i(\sigma)$

Consider $\sigma = (1-s)\sigma_0 - s\sigma_0$

Focus on the single zero mode $K\Phi = 0$ on RP^n near $s = \frac{1}{2} + \delta s$

$\lambda = \pm i\delta s\sigma_0$. Spectral flow as $0^- \leq \delta s < 0^+$

Partition function Z_F changes sign. (Non-perturbative) anomaly !

like 1 Weyl in $SU(2)$ - Witten ...

Cancelling the anomaly

Anomaly avoided if eigenvalues cross in pairs

i.e for multiples of 2 Kähler–Dirac (staggered) fields.
In 4d flat space and in continuum limit:
2 Kähler–Dirac $=$ 8 (massless) Dirac fermions $\equiv$ 16 Majorana

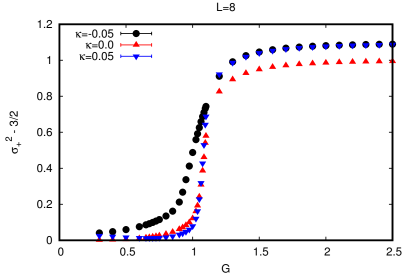
Key observation:

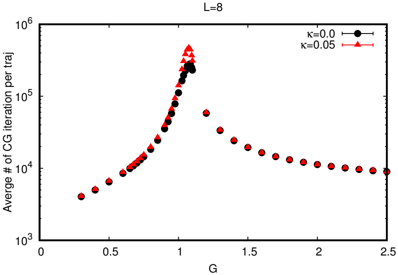
Cancellation of all ’t Hooft anomalies **necessary** condition to give mass to fermions without breaking symmetries.

Agrees with results from topological superconductors and Dai-Freed theorem for Weyl fermions

Massive symmetric phase for 2 staggered fermions

Higgs-Yukawa model: $S = \sum \chi(\eta.\Delta)\chi + \frac{1}{2}\sigma^2 - \kappa\sigma\Box\sigma + G\sigma\chi\chi$





Evidence for direct, continuous phase transition between massless and massive phases with no symmetry breaking (S.C et al. PRD98 (2018) 114514)

Summary

- Kähler–Dirac fermions admit (mixed) gravitational anomalies which **survive discretization.**
- Viewed as 't Hooft anomalies yield constraints on possible IR phases of staggered fermions - in even dimension **non-perturbative** anomaly cancels for multiples of 2 staggered fields (4 reduced staggered) $\rightarrow$ 8 continuum Dirac
- Cancellation of all 't Hooft anomalies **necessary condition for symmetric mass generation (SMG).**
- Explains recent results on phase diagram of certain staggered fermion models.

What's is SMG good for ?

Use SMG to gap mirrors in lattice models targeting chiral gauge theories ..?

Thanks!