Asymptotic lattice spacing dependence of spectral quantities in lattice QCD with Wilson or Ginsparg-Wilson quarks

Based on Eur.Phys.J.C 80 (2020) 3, 200; Phys.Lett.B 829 (2022) 137069; arXiv:2206.03536 and ongoing work.

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LATTICE 2022, Bonn, 13 August 2022

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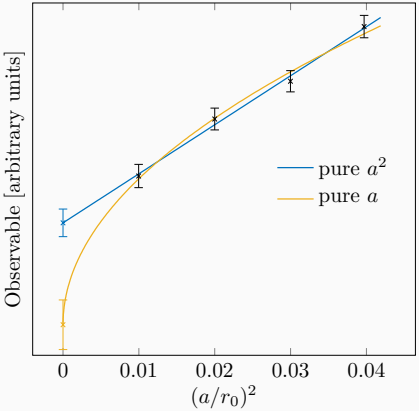
Motivation: Continuum extrapolation

Renormalisation Group Invariant (RGI) quantity $\mathcal{P}$



Commonly used ansatz

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 + a^{n_{\min}}\text{const.} + \mathrm{O}(a^{n_{\min}+1}).$$

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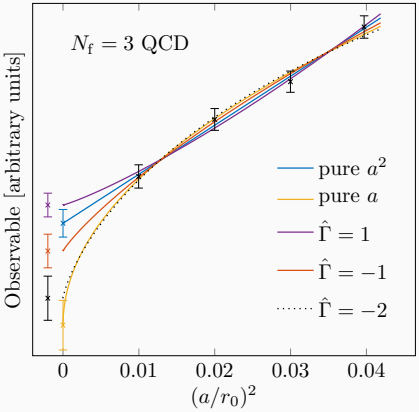

In an asymptotically free theory, like QCD, leading lattice artifacts are of the form (up to factors of $\log \bar{g}(1/a)$)

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 + a^{n_{\min}} \sum_i [\bar{g}^2(1/a)]^{\hat{\Gamma}_i} c_i + \mathrm{O}(a^{n_{\min}+1}, a^{n_{\min}} \bar{g}^{2\hat{\Gamma}_i+2}(1/a), \ldots)$$

$\hat{\Gamma}_i$ can be negative and distinctly nonzero

$\Rightarrow$ impact on convergence.

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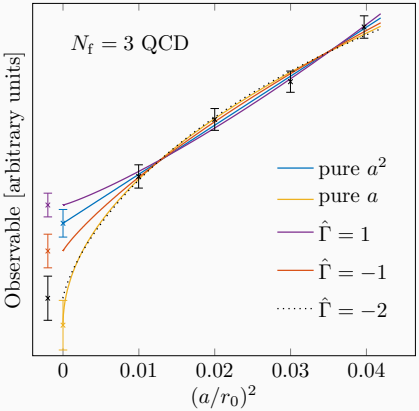

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$\hat{\Gamma}_i$ can be negative and distinctly nonzero

$\Rightarrow$ impact on convergence.

Warning example: 2d O(3) non-linear sigma model $\min \hat{\Gamma}_i = -3$ [Balog et al., 2009, 2010]

$\Rightarrow$ **Compute $\hat{\Gamma}_i$ in QCD to gain better control over continuum extrapolation.**

Symanzik Effective Theory (SymEFT)

Describe lattice spacing dependence in terms of a **continuum** Effective Field Theory

[Symanzik, 1980, 1981, 1983a,b]

$$S_{\mathrm{Sym}} = S_{\mathrm{QCD}} + a^{n_{\min}} \int \mathrm{d}^4 x \sum_j \bar{\omega}_j(g_0) \mathcal{O}_j(x) + \dots$$

with **on-shell** operator basis $\mathcal{O}_j$ compatible with symmetries of lattice formulation and matching coefficients $\bar{\omega}_j$.

The leading asymptotic lattice spacing dependence can then be written as

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 - a^{n_{\min}} \sum_j [2 b_0 \bar{g}^2(1/a)]^{\hat{\Gamma}_j} \hat{c}_j \delta \mathcal{P}_{j;\mathrm{RGI}} + \dots$$

$\hat{\Gamma}_j$ are related to 1-loop anomalous dimensions of irrelevant operators with mass-dimension $[\mathcal{O}] = 4 + n_{\min}$.

Basis sufficient for spectral quantities.

For non-spectral quantities also contributions from discretised local fields must be included.

Minimal operator basis at $\mathrm{O}(a)$

Relevant basis [Sheikholeslami, Wohlert, 1985] for unimproved Wilson quarks.

$$\frac{i}{4}\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$$

$$\frac{\mathrm{tr}(m)}{g_0^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \quad \bar{\Psi}m^2\Psi \quad \mathrm{tr}(m)\bar{\Psi}\Psi \quad \mathrm{tr}(m)^2\bar{\Psi}\Psi \quad \mathrm{tr}(m^2)\bar{\Psi}\Psi$$

Basis commonly used to perform non-perturbative $\mathrm{O}(a)$ improvement of Wilson quarks [Lüscher et al., 1997].

Minimal operator basis at $\mathrm{O}(a^2)$

pure gauge [Lüscher, Weisz, 1985a] $\mathrm{O}(a)$ improved [Sheikholeslami, Wohlert, 1985]

Wilson-like [Sheikholeslami, Wohlert, 1985]

$$\frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}) \qquad \sum_\mu \bar{\Psi}\gamma_\mu D_\mu^3 \Psi \qquad g_0^2(\bar{\Psi}\Gamma\Psi)^2 \qquad g_0^2(\bar{\Psi}\Gamma T^a\Psi)^2$$

$$\frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu})$$

$$\Gamma \in \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, i\sigma_{\mu\nu}\}$$

$$\frac{i}{4}\bar{\Psi} m \sigma_{\mu\nu} F_{\mu\nu}\Psi \qquad \frac{i\mathrm{tr}(m)}{4}\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi \qquad \frac{\mathrm{tr}(m^2)}{g_0^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \qquad \frac{\mathrm{tr}(m)^2}{g_0^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$$

$$\bar{\Psi} m^3 \Psi \qquad \mathrm{tr}(m)\bar{\Psi} m^2 \Psi \qquad \mathrm{tr}(m^2)\bar{\Psi} m \Psi \qquad \mathrm{tr}(m)^2\bar{\Psi} m \Psi$$

$$\mathrm{tr}(m^3)\bar{\Psi}\Psi \qquad \mathrm{tr}(m^2)\mathrm{tr}(m)\bar{\Psi}\Psi \qquad \mathrm{tr}(m)^3\bar{\Psi}\Psi \qquad \Psi = (\mathrm{u}, \mathrm{d}, \mathrm{s}, \ldots)$$

Non-perturbative improvement impractical due to $(7, 13)$ **massless + 11 massive operators (GW,Wilson)!**

Renormalisation Group

Leading lattice artifacts are parametrised as

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 - a^{n_{\min}} \sum_j c_j^{\mathcal{O}}(\bar{g}^2(1/a))\delta\mathcal{P}_j^{\mathcal{O}}(1/a) + \mathrm{O}(a^{n_{\min}+1}).$$

The remaining scale dependence of $\delta\mathcal{P}_j^{\mathcal{O}}(1/a)$ is governed by RGE

$$\mu\frac{\mathrm{d}\delta\mathcal{P}_i^{\mathcal{O}}(\mu)}{\mathrm{d}\mu} = -\left[\gamma_0^{\mathcal{O}}\bar{g}^2(\mu) + \mathrm{O}(\bar{g}^4)\right]_{ij}\delta\mathcal{P}_j^{\mathcal{O}}(\mu).$$

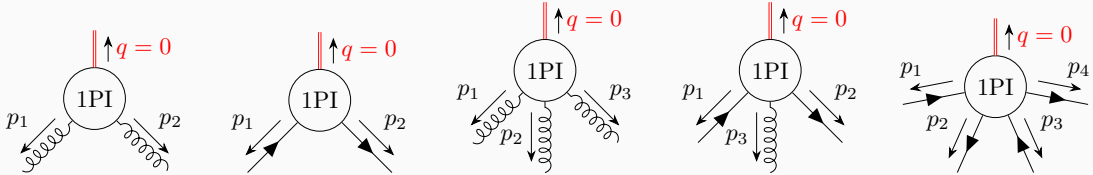
Making a change of basis $\mathcal{O} \to \mathcal{B}$ such that $\gamma_0^{\mathcal{B}} = \mathrm{diag}\left((\gamma_0)_1, \ldots, (\gamma_0)_n\right)$ allows to rewrite

$$\delta\mathcal{P}_j^{\mathcal{B}}(1/a) = \left[2b_0\bar{g}^2(1/a)\right]^{\hat{\gamma}_j}\delta\mathcal{P}_{j;\mathrm{RGI}}^{\mathcal{B}} \times \left[1 + \mathrm{O}(\bar{g}^2(1/a))\right], \quad \hat{\gamma}_j = \frac{(\gamma_0)_j}{2b_0}.$$

If $\gamma_0^{\mathcal{O}}$ is non-diagonalisable we can bring it into Jordan normal form. This will give rise to terms with factors of $\log\bar{g}(1/a)$. (relevant for quenched and mixed actions)

Computational strategy

Collect 1-loop UV poles of 1PI graphs with operator insertion $\tilde{\mathcal{O}}(q=0)$ in $D=4-2\varepsilon$ dimensions and background field gauge ['t Hooft, 1975; Abbott, 1981, 1982; Lüscher, Weisz, 1995].



Yields relevant part of 1-loop mixing matrix via

needed additionally

$$\begin{pmatrix}\mathcal{O}\\ \mathcal{E}\end{pmatrix}_{\overline{\text{MS}}} = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}\mathcal{E}} \\ 0 & Z_{\mathcal{E}\mathcal{E}}\end{pmatrix}\begin{pmatrix}\mathcal{O}\\ \mathcal{E}\end{pmatrix} \Rightarrow \mu\frac{\mathrm{d}Z_{\mathcal{O}\mathcal{O}}}{\mathrm{d}\mu}Z^{-1}_{\mathcal{O}\mathcal{O}} = -\gamma_0^{\mathcal{O}}\bar{g}^2 + \mathrm{O}(\bar{g}^4)$$

with class of EOM-vanishing operators $\mathcal{E}$.

Tools: QGRAF [Nogueira, 1993, 2006], FORM [Vermaseren, 2000]

`https://github.com/nikolai-husung/Symanzik-QCD-workflow`

Matching

Taking also (TL) matching into account yields $c_j^{\mathcal{B}}(\bar{g}^2) = [2b_0\bar{g}^2]^{n_j}\hat{c}_j \times \{1 + \mathrm{O}(\bar{g}^2)\}$

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 - a^{n_{\min}} \sum_j \hat{c}_j [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_j} \delta\mathcal{P}^{\mathcal{B}}_{j;\mathrm{RGI}} \times \{1 + \mathrm{O}(\bar{g}^2)\} + \mathrm{O}(a^{n_{\min}+1}),$$

$$\hat{\Gamma}_j = \hat{\gamma}_j + n_j, \quad n_j \in \mathbb{N} \cup \{0\}$$

as final form of the asymptotic lattice spacing dependence. $\Rightarrow$ Collection of $(\hat{c}_j, \hat{\Gamma}_j)$.

The leading order coefficients $\hat{c}_j$ depend on the precise formulation of the lattice action

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L}_{\mathrm{QCD}} + a^{n_{\min}} \sum_j \hat{c}_j [2b_0\bar{g}^2(1/a)]^{n_j} \mathcal{B}_{j;\mathrm{R}} + \mathrm{O}(a^{n_{\min}+1}, a^{n_{\min}}\bar{g}^{2n_j+2}, \ldots)$$

and we assume that the 1-loop coefficients do not vanish, i.e. $n_j \in \{0, 1\}$.

Results found and how to interpret them

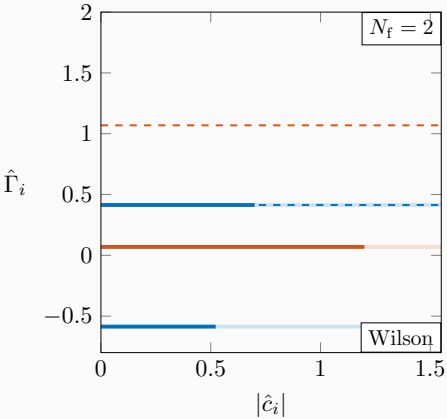
Conventions:

1. Normalise vector forming diagonal basis by dominant entry.
   Caveat: massless case might shift $\hat{c}_i$ due to dominant massive mixing.
2. Combine non-vanishing basis elements with degenerate eigenvalues $\hat{\Gamma}_{\rm deg}$ but no logs at LO and normalise by dominant matching coefficient $\hat{c}_{\rm max}$, i.e.

$$\mathcal{B}_{\rm deg} = \frac{1}{\hat{c}_{\rm max}} \sum_{i:\,\hat{\Gamma}_i = \hat{\Gamma}_{\rm deg}} \hat{c}_i \mathcal{B}_i$$

3. Assume $|\delta\mathcal{P}^{\mathcal{B}}_{i;\rm RGI}| \sim |\delta\mathcal{P}^{\mathcal{B}}_{j;\rm RGI}|\ \forall i, j$.

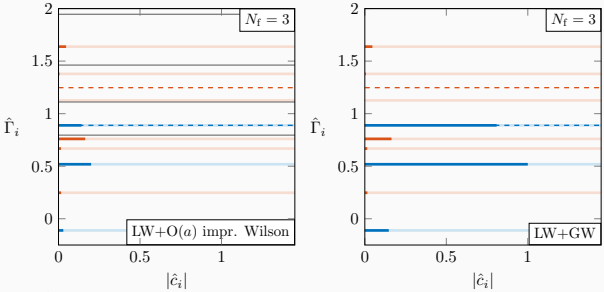
Results found and how to interpret them (Example: unimproved Wilson at $\mathrm{O}(a)$)



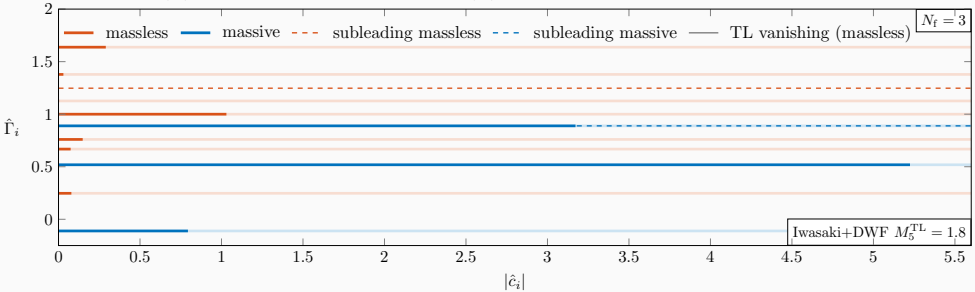
- Dashed lines indicate subleading corrections of leading massive/massless contribution $\hat{\Gamma}_\mathrm{min}$.
- Potential 1-loop massless contributions having vanishing TL coefficients are indicated by a gray line.
- Faded lines introduced to make severely suppressed $\hat{c}_i$ visible.
- Massive $\hat{\Gamma}_\mathrm{min} \approx -0.59 \gg -3$.
- Axes flipped compared to [NH, P. Marquard, R. Sommer, 2022].

massless — massive - - - subleading massless - - - subleading massive —— TL vanishing (massless)

$N_\mathrm{f} = 3$ results at $\mathrm{O}(a^2)$



- Same leading $\hat{\Gamma}_i$ for Wilson and GW, $\hat{\Gamma}_{\mathrm{min}} \gg -3$.
- TL matching coefficients can have vastly different orders of magnitude!



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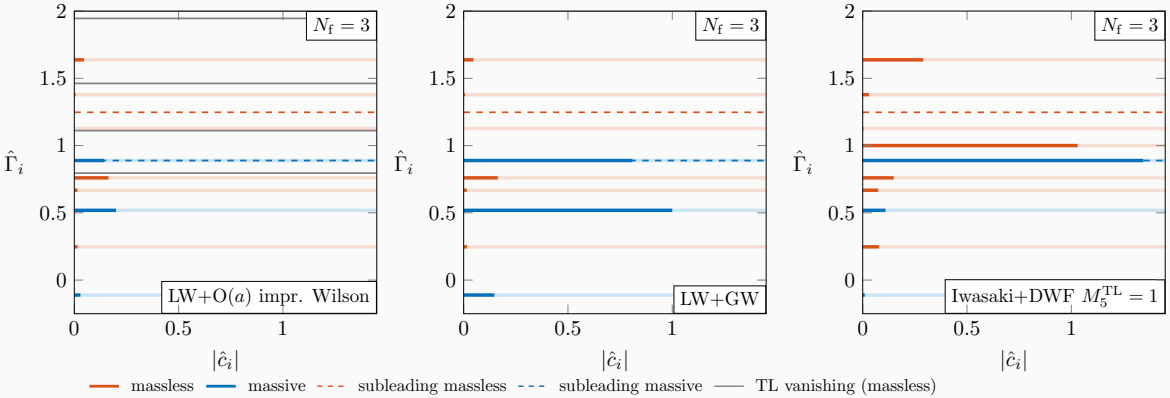

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  $\Rightarrow$ tune DWF s.t. $M_5(g_0) = 1 + \mathrm{O}(g_0^2)$.

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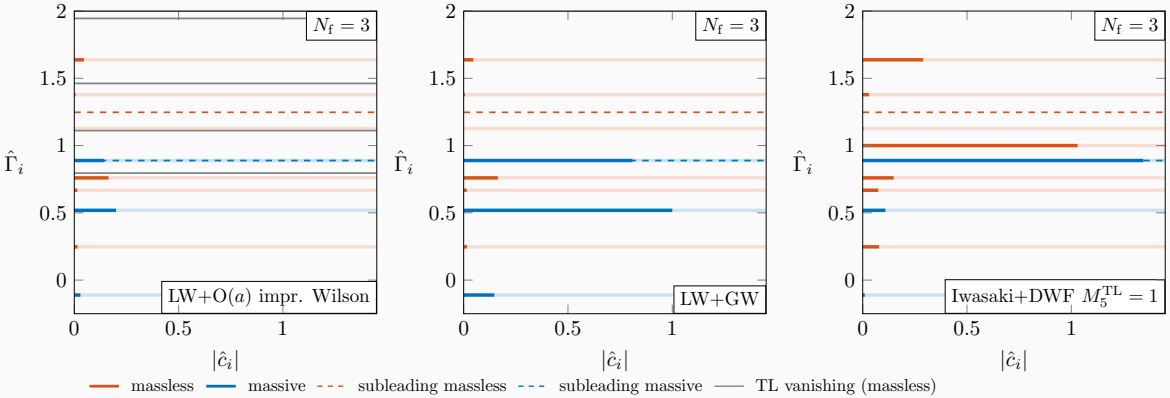
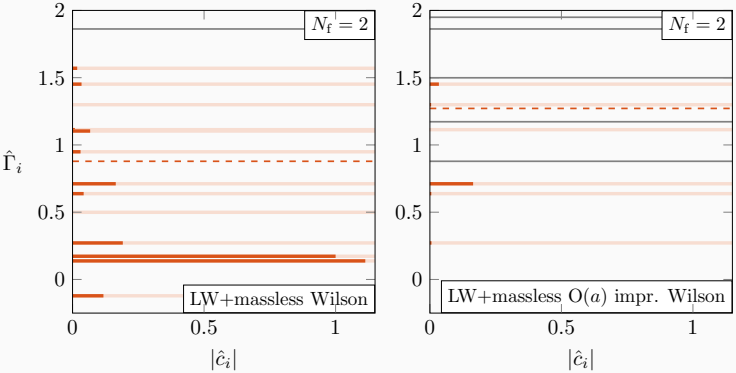


TL Symanzik improvement suggests (using LW gauge action [Lüscher, Weisz, 1985b])

$$\hat{D}_\mathrm{Wilson} \to \hat{D}_\mathrm{Wilson} - \frac{a^2}{12} \sum_\mu \gamma_\mu \{\nabla_\mu + \nabla_\mu^*\} \nabla_\mu^* \nabla_\mu$$

which suppresses all massless $\mathrm{O}(a^2)$ contributions at tree-level [DeGrand et al., 1995].

Effect of explicit $\mathrm{O}(a)$ improvement: $N_\mathrm{f} = 2$ massless Wilson quarks at $\mathrm{O}(a^2)$



Symmetry of QCD in FV

$\bar{\Psi} \to i\bar{\Psi}\gamma_5\tau^j, \quad \Psi \to i\gamma_5\tau^j\Psi$

with Pauli matrix τ^j.

$S_1 = a\int d^4x\, i[\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi](x)$

$\langle \ldots S_1\rangle \to -\langle \ldots S_1\rangle = 0$

$\langle \ldots S_1^2\rangle \to +\langle \ldots S_1^2\rangle \neq 0$

$\Rightarrow$ Double insertion of S_1 impacts $\mathrm{O}(a^2)$ lattice artifacts.

This applies also to maximally twisted tmQCD relying on automatic $\mathrm{O}(a)$ improvement.

Summary $\mathrm{O}(a^2)$ lattice artifacts

Indicators for asymptotic lattice spacing dependence (assume $|\delta\mathcal{P}^{\mathcal{B}}_{i;\mathrm{RGI}}| \sim |\delta\mathcal{P}^{\mathcal{B}}_{j;\mathrm{RGI}}|\ \forall i,j$):

1. Value of $\min_i \hat{\Gamma}_i$ (here $N_\mathrm{f} \leq 4$).
   $\Rightarrow$ Distinctly negative value worsens convergence compared to classical a^n power law.
   - $\hat{\Gamma}_{\mathrm{min}} \gg -3$ in contrast to O(3) model.
   - $\hat{\Gamma}_{\mathrm{min}} \gtrsim 0.2$ for massless quarks.
   - Slightly negative $\hat{\Gamma}_{\mathrm{min}} \gtrsim -0.2$ for massive quarks.

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2. Density of spectrum for the powers $\Delta\hat{\Gamma}_{ij} = \hat{\Gamma}_i - \hat{\Gamma}_j$.
   $\Rightarrow$ Determines overall suppression of higher power corrections in $\bar{g}^2(1/a)$.
   - Dense spectrum due to presence of 4-fermion operators. May expect complicated lattice artifacts with cancellations and pile ups. **Even denser spectrum for mixed actions.**

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3. Hierarchy of matching coefficients $\hat{c}_i$.
   $\Rightarrow$ If $|\hat{c}_i| \gg |\hat{c}_j|$ for $\hat{\Gamma}_i > \hat{\Gamma}_j$ suppression by power $\bar{g}^{2\Delta\hat{\Gamma}_{ij}}(1/a)$ may be undone in range of lattice spacings available.

   TL matching coefficients can have vastly different orders of magnitude!

Not covered but available

- $\hat{\gamma}_i$ for any N_c and N_f including the (partially) quenched case. arXiv:2206.03536
  Pure gauge $\mathrm{O}(a^2)$ [NH, P. Marquard, R. Sommer, 2020], $N_\mathrm{f} = 3, 4$ [NH, P. Marquard, R. Sommer, 2022].
- Extension to differing discretisations of dynamical flavours and mixed actions.
  arXiv:2206.03536
- Gradient flow in pure gauge $\mathrm{O}(a^2)$: 3rd operator with $\hat{\gamma}_\mathrm{min} = 0$, see PhD thesis [NH, 2021].

Outlook

- Leading asymptotic behaviour is now known and should be incorporated into continuum extrapolations (of spectral quantities) e.g.:
  - through use of dominant $\hat{\Gamma}$ in extrapolations,
  - or vary $\hat{\Gamma}$ in the range of 1-loop anomalous dimensions,
  - ...

  Best practice must still be worked out. **Be careful when doing extrapolations!**
- **Stay tuned for:** Enlarged spectra for some local fermion bilinears.

  $\Rightarrow$ Additional set of powers $\hat{\Gamma}_i$ for each local field involved in non-spectral quantity.
- **Possible directions for future research:**
  - Gradient flow for full QCD (unflowed quarks) requires inclusion of two additional operators,
  - staggered quarks require **additional** operators in the minimal basis compared to GW quarks due to flavour changing interactions,
  - ...

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