Loop-string-hadron formulation of an SU(3) gauge theory with dynamical quarks

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with

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Outline

- Motivation
- Loop-string-hadron (LSH) formulation in SU(2) : Features and advantages
- LSH Formulation for SU(3) lattice gauge theory in 1+1 D
- Conclusions and Outlook

Non-Abelian gauge theories

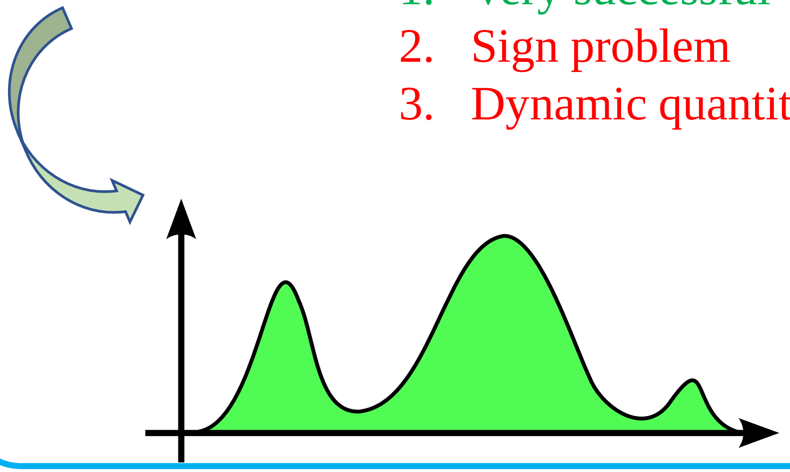
- Strong force → Quantum chromodynamics (QCD) $SU_C(3)$
- Strongly coupled at low energies → Requires nonperturbative calculations

Path Integral Formulation

- Lagrangian in a discretized space-time
- Wick rotate to Euclidean spacetime
- Monte-Carlo sampling of the functional integral

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{Z_0} \int \mathcal{D}[\bar{\psi}, \psi]\mathcal{D}[A]\ e^{-\int d^4x_E \mathcal{L}_E} \mathcal{O}$$

1. Very successful
2. Sign problem
3. Dynamic quantities



Hamiltonian Formulation

- Hamiltonian in a discretized space and continuous time

$$\langle \hat{\mathcal{O}}(t) \rangle = \langle 0 | e^{iHt} \hat{\mathcal{O}}(0) e^{-iHt} | 0 \rangle$$

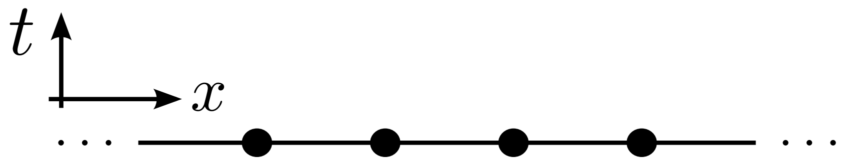
1. No sign problem
2. Both static and dynamic quantities
3. Hilbert space scales exponentially with the system size

Quantum simulation and
Quantum computation

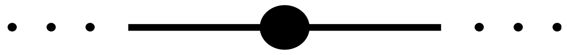
Hamiltonian formulation of non-Abelian lattice gauge theories

Example: SU(2) gauge theory in 1+1 D with dynamical matter field

➢ Kogut-Susskind (KS) Hamiltonian

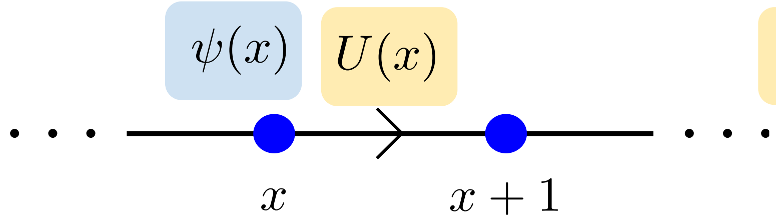


Temporal Gauge



Staggered Fermion

Matter → Even sites
Anti-matter → Odd sites



2×2 Unitary matrix

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$$

$$H_M = m_0 \sum_x (-1)^x \psi^\dagger \psi$$

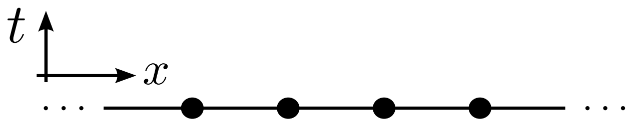
From staggering

$$H_I = \frac{1}{2a} \sum_x [\psi^\dagger(x) U(x) \psi(x+1) + \text{h.c.}]$$

Hamiltonian formulation of non-Abelian lattice gauge theories

Example: SU(2) gauge theory in 1+1 D with dynamical matter field

- Kogut-Susskind (KS) Hamiltonian



Staggered Fermion

Matter → Even sites
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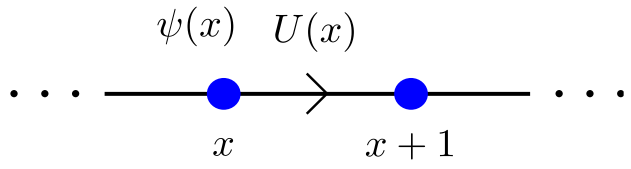


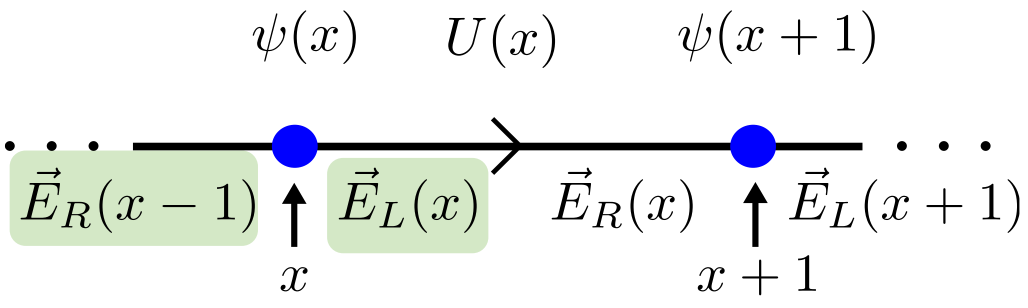
$$H_M = m_0 \sum_x (-1)^x \psi^\dagger \psi$$

From staggering

$$H_I = \frac{1}{2a} \sum_x [\psi^\dagger(x) U(x) \psi(x+1) + \text{h.c.}]$$

- Hilbert space of link operator ↔ Quantum spherical rigid rotor in body and space frame

Equivalent of generators of rotations in body and space frame



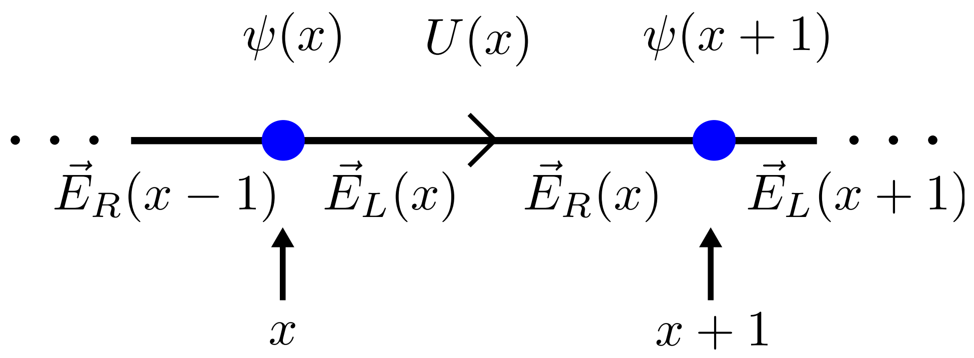
with $\quad E_L(x)^2 = E_R(x)^2$

$$H_E = \frac{g^2 a}{2} \sum_x E_L(x)^2$$

$$H_{(KS)} = H_E + H_I + H_M$$

Hamiltonian formulation of non-Abelian lattice gauge theories

Example: SU(2) gauge theory in 1+1 D with dynamical matter field



- Gauss's law operator: Full generator of local gauge transformations

$$G^a(x) = -E_L^a(x) + E_R^a(x-1) + \tfrac{1}{2}\psi^\dagger(x)\sigma^a\psi(x)$$

Pauli matrices $a = 1, 2, 3$

- Physical Hilbert space: Gauge invariant states

$$G^a(x)|\mathrm{Phys}\rangle = 0$$

Non-commutative Gauss' law constraints

- Construct the Hilbert space of physical states for simulation:
  - ❖ Different formulations for mapping the vacuum and excited field states to a state basis
  - ❖ Truncation of the Hilbert space depending on its formulation
  - ❖ Different (dis)advantages of different formulations

Comparison with other formulations

Davoudi, Raychowdhury, and Shaw
Phys. Rev. D 104, 074505

Angular momentum formulation

Purely fermionic formulation

Purely bosonic formulation

KS Hamiltonian of SU(2) LGT in 1+1 D

Many more formulations, see
Bauer, Davoudi, et Al.
arXiv:2204.03381
For other approaches

Loop-string-hadron formulation*

* Raychowdhury and Stryker
Phys. Rev. D 101, 114502

- DOF: Local gauge singlets namely loops, strings, and hadrons
- Generalizable for higher dimensions*
- Hamiltonian is made from operators that act directly on local basis states
- Abelian Gauss' law

Digital Quantum Simulation:
Jesse Stryker
Thursday 11:50 AM in Algorithms

Analog Quantum Simulation:
Dasgupta and Raychowdhury
Phys. Rev. A 105, 023322

LSH Formulation for SU(3) lattice gauge theory in 1+1 D

A step towards quantum simulation of QCD

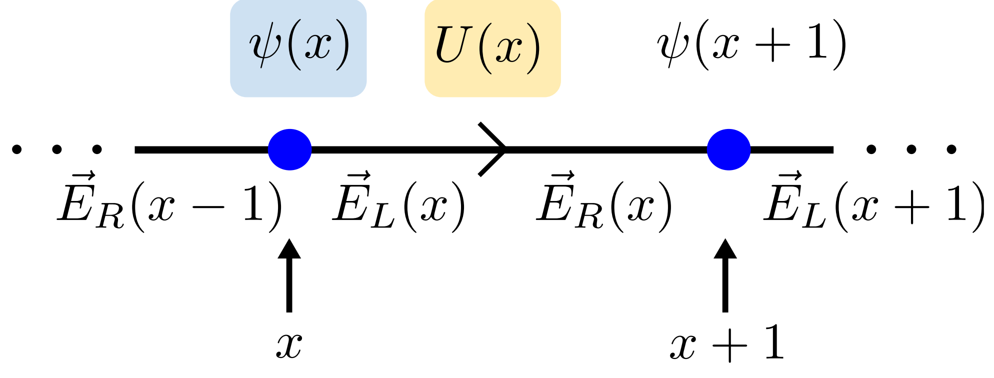
Kogut-Susskind Hamiltonian for SU(3)

Kogut-Susskind Hamiltonian for SU(3)

3×3 Unitary matrix

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \end{pmatrix}$$

3 colors



with $E_L(x)^2 = E_R(x)^2$

- Gauss' law operators

$$G^a(x) = -E^a_L(x) + E^a_R(x-1) + \tfrac{1}{2}\psi^\dagger(x)\lambda^a\psi(x)$$

Gell-Mann matrices

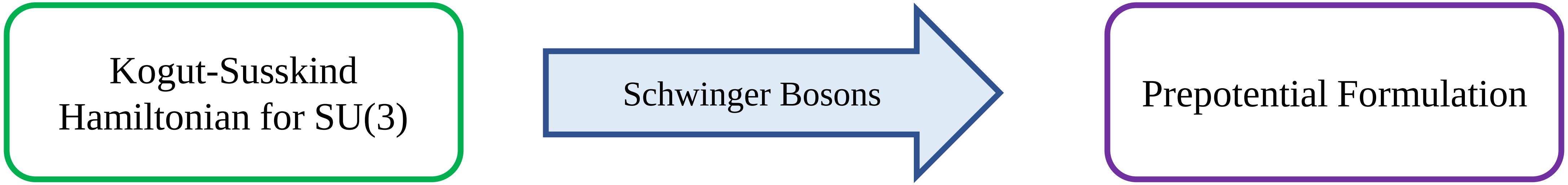
$$a = 1, 2, \cdots, 8$$

- Physical Hilbert space is given by states that satisfy: $G^a(x)|\text{Phys}\rangle = 0, \ \ \forall\, a$

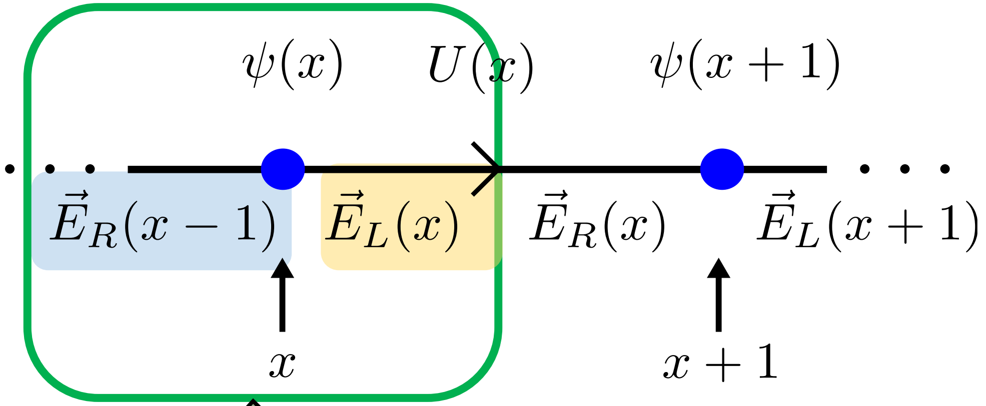
Non-commutative Gauss' law constraints

LSH Formulation for SU(3) lattice gauge theory in 1+1 D

A step towards quantum simulation of QCD



Prepotential Formulation



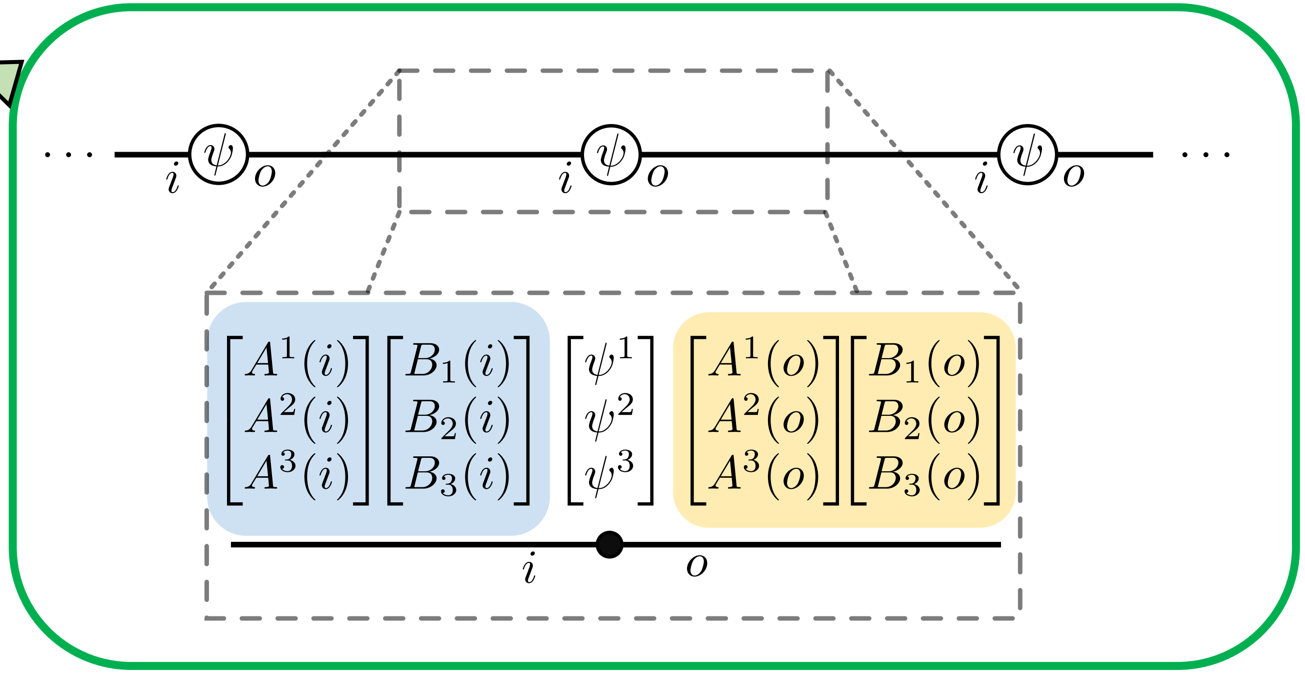
with $E_L(x)^2 = E_R(x)^2$

➢ Prepotential formulation of KS Hamiltonian:

❖ Electric field and link operator → Schwinger bosons

❖ Commutation relations remain the same

❖ $SU(3)$ is a rank 2 group $\Longrightarrow$ Requires two triplets of Schwinger bosons



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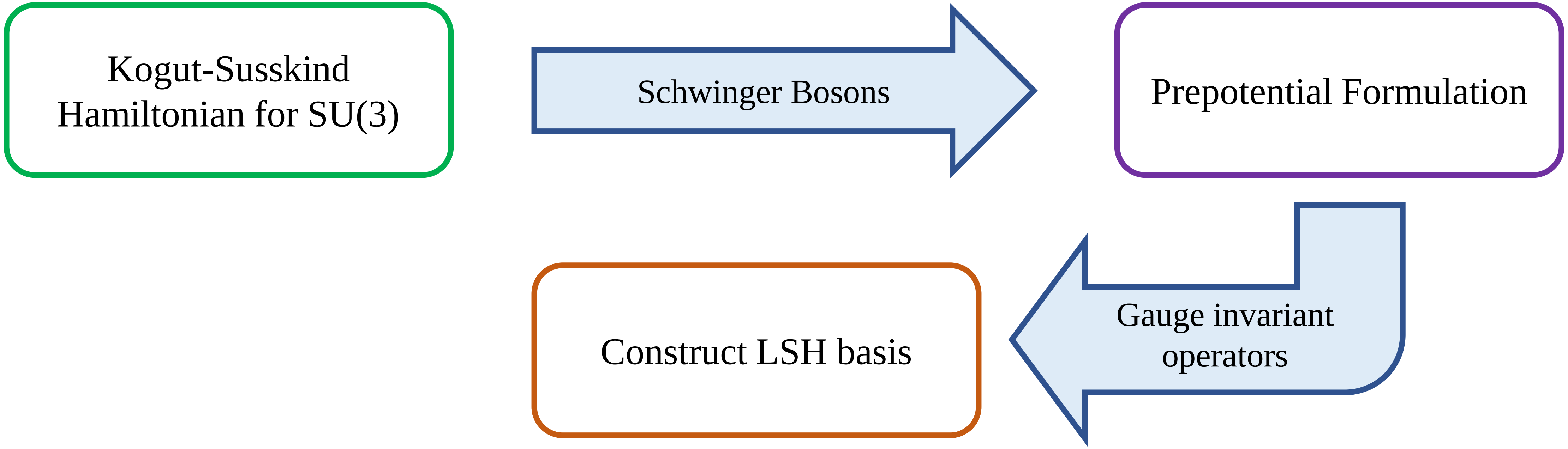
Anishetty, Mathur, and Raychowdhury J.Math.Phys. 50 (2009) 053503

Anishetty, Mathur, and Raychowdhury J.Phys.A 43 (2010) 035403

Anishetty, Mathur, and Raychowdhury J.Math.Phys. 51 (2010) 093504

LSH Formulation for SU(3) lattice gauge theory in 1+1 D

A step towards quantum simulation of QCD



Construct LSH basis

Local degrees of freedom

$\underline{3}$	$\underline{3^*}$
$A^\dagger_\alpha(i)$	$A^\alpha(i)$
$A^\dagger_\alpha(o)$	$A^\alpha(o)$
$B_\alpha(i)$	$B^{\dagger\alpha}(i)$
$B_\alpha(o)$	$B^{\dagger\alpha}(o)$
$\psi^\dagger_\alpha$	ψ^α

- Local singlets using

$\delta^\alpha_\beta \equiv \cdot$ or $\epsilon^{\alpha\beta\gamma} \equiv \wedge$

- Only a subset of them is required to reconstruct the Hamiltonian
- Basis is constructed from all possible gauge invariant combinations of creation operators

Bosonic: $A^\dagger(i) \cdot B^\dagger(o)$

$B^\dagger(i) \cdot A^\dagger(o)$

Fermionic + Bosonic : $\psi^\dagger \cdot B^\dagger(i)$

$\psi^\dagger \cdot B^\dagger(o)$

$\psi^\dagger \cdot (A^\dagger(i) \wedge A^\dagger(o))$

$\psi^\dagger \cdot (\psi^\dagger \wedge A^\dagger(i))$

$\psi^\dagger \cdot (\psi^\dagger \wedge B^\dagger(o))$

Fermionic: $\psi^\dagger \cdot (\psi^\dagger \wedge \psi^\dagger)$

LSH Formulation for SU(3) lattice gauge theory in 1+1 D

Quantum numbers and the state basis

$$|n_P, n_Q, \nu_i, \nu_m, \nu_o\rangle \propto \left[A^\dagger(i)\cdot B^\dagger(o)\right]^{n_P}\left[B^\dagger(i)\cdot A^\dagger(o)\right]^{n_Q}|0,0,\nu_i,\nu_m,\nu_o\rangle$$

Local degrees of freedom

$\underline{3}$	$\underline{3^*}$
$A^\dagger_\alpha(i)$	$A^\alpha(i)$
$A^\dagger_\alpha(o)$	$A^\alpha(o)$
$B_\alpha(i)$	$B^{\dagger\alpha}(i)$
$B_\alpha(o)$	$B^{\dagger\alpha}(o)$
$\psi^\dagger_\alpha$	ψ^α

- Local singlets using

  $\delta^\alpha_\beta \equiv \cdot$ or $\epsilon^{\alpha\beta\gamma} \equiv \wedge$

- Only a subset of them is required to reconstruct the Hamiltonian

- Basis is constructed from all possible gauge invariant combinations of creation operators

Loops: Bosonic

$\left[A^\dagger(i)\cdot B^\dagger(o)\right]^{n_P} \rightarrow$ $\}n_P$

$\left[A^\dagger(o)\cdot B^\dagger(i)\right]^{n_Q} \rightarrow$ $\}n_Q$

	ν_i	ν_m	ν_o
$\lvert\Omega\rangle \rightarrow$	0	0	0

Strings: Fermionic + Bosonic

	ν_i	ν_m	ν_o
$\psi^\dagger\cdot B^\dagger(o)\lvert\Omega\rangle \rightarrow$	0	0	1
$\psi^\dagger\cdot A^\dagger(i)\wedge A^\dagger(o)\lvert\Omega\rangle \rightarrow$	0	1	0
$\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(o)\lvert\Omega\rangle \rightarrow$	0	1	1
$\psi^\dagger\cdot B^\dagger(i)\lvert\Omega\rangle \rightarrow$	1	0	0
$\psi^\dagger\cdot B^\dagger(i)\,\psi^\dagger\cdot B^\dagger(o)\lvert\Omega\rangle \rightarrow$	1	0	1
$\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(i)\lvert\Omega\rangle \rightarrow$	1	1	0

Hadron: Fermionic

	ν_i	ν_m	ν_o
$\psi^\dagger\cdot\psi^\dagger\wedge\psi^\dagger\lvert\Omega\rangle \rightarrow$	1	1	1

LSH Formulation for SU(3) lattice gauge theory in 1+1 D

Abelian Gauss' laws and non-locality

$$|n_P, n_Q, \nu_i, \nu_m, \nu_o\rangle \propto \left[A^\dagger(i)\cdot B^\dagger(o)\right]^{n_P}\left[B^\dagger(i)\cdot A^\dagger(o)\right]^{n_Q}|0,0,\nu_i,\nu_m,\nu_o\rangle$$

Number operators

$$\hat{P}_i(x) = \hat{n}_P(x) + \hat{\nu}_m(x)\left[1-\hat{\nu}_o(x)\right]$$

$$\hat{P}_o(x) = \hat{n}_P(x) + \hat{\nu}_o(x)\left[1-\hat{\nu}_m(x)\right]$$

$$\hat{Q}_i(x) = \hat{n}_Q(x) + \hat{\nu}_i(x)\left[1-\hat{\nu}_m(x)\right]$$

$$\hat{Q}_o(x) = \hat{n}_Q(x) + \hat{\nu}_m(x)\left[1-\hat{\nu}_i(x)\right]$$

Abelian Gauss' laws

$$\hat{P}_o(x) = \hat{Q}_i(x+1) = \hat{P}(x)$$

$$\hat{Q}_o(x) = \hat{P}_i(x+1) = \hat{Q}(x)$$

Non-locality is only through these Abelian Gauss' laws

$$\left[A^\dagger(i)\cdot B^\dagger(o)\right]^{n_P} \rightarrow$$

$$\left[A^\dagger(o)\cdot B^\dagger(i)\right]^{n_Q} \rightarrow$$

$$|\Omega\rangle \rightarrow$$

$$\psi^\dagger\cdot B^\dagger(o)|\Omega\rangle \rightarrow$$

$$\psi^\dagger\cdot A^\dagger(i)\wedge A^\dagger(o)|\Omega\rangle \rightarrow$$

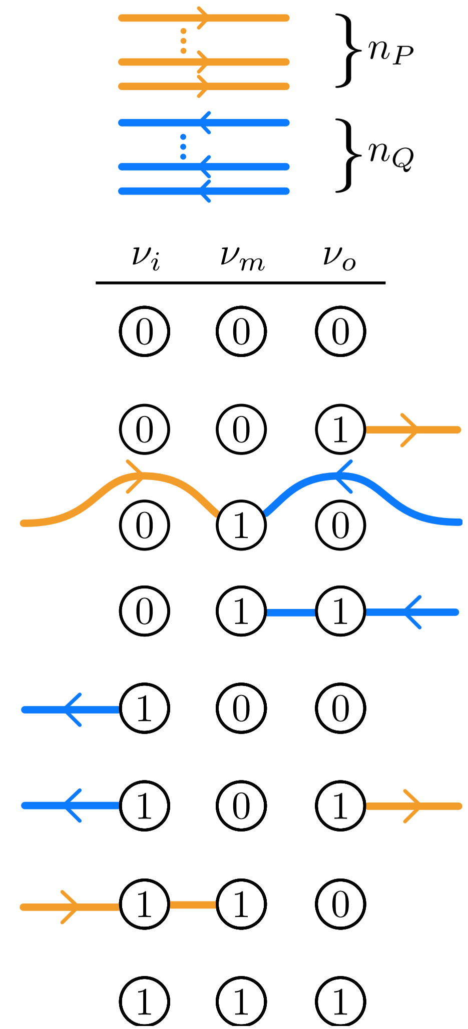
$$\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(o)|\Omega\rangle \rightarrow$$

$$\psi^\dagger\cdot B^\dagger(i)|\Omega\rangle \rightarrow$$

$$\psi^\dagger\cdot B^\dagger(i)\,\psi^\dagger\cdot B^\dagger(o)|\Omega\rangle \rightarrow$$

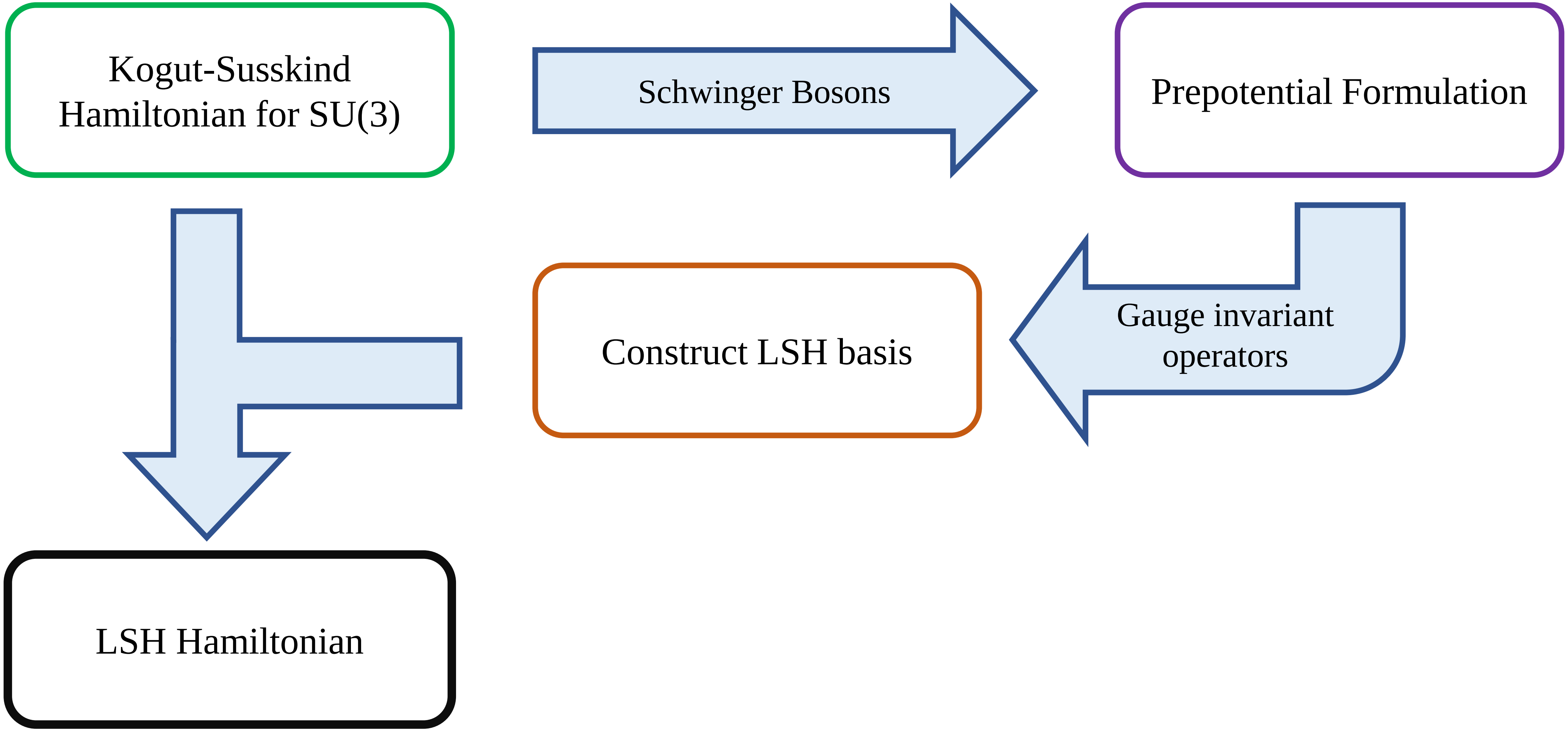
$$\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(i)|\Omega\rangle \rightarrow$$

$$\psi^\dagger\cdot\psi^\dagger\wedge\psi^\dagger|\Omega\rangle \rightarrow$$



LSH Formulation for SU(3) lattice gauge theory in 1+1 D

A step towards quantum simulation of QCD



LSH Formulation for SU(3) lattice gauge theory in 1+1 D

LSH Hamiltonian

Numerical benchmark: Purely fermionic formulation

Fermion-like number operators

$$\hat{\nu}_i(x),\ \hat{\nu}_m(x),\ \hat{\nu}_o(x)$$

$$H_M = m_0 \sum_x (-1)^x \left[\hat{\nu}_i(x) + \hat{\nu}_m(x) + \hat{\nu}_o(x)\right]$$

Fermion-like and loop number operators

$$\hat{P}_o(x) = \hat{n}_P(x) + \hat{\nu}_o(x)\left[1 - \hat{\nu}_m(x)\right]$$

$$\hat{Q}_o(x) = \hat{n}_Q(x) + \hat{\nu}_m(x)\left[1 - \hat{\nu}_i(x)\right]$$

$$\hat{P}_o(x) = \hat{Q}_i(x+1) = \hat{P}(x)$$

$$\hat{Q}_o(x) = \hat{P}_i(x+1) = \hat{Q}(x)$$

$$H_E = \frac{g^2 a}{2} \sum_x \frac{1}{3}\left[\hat{P}(x)^2 + \hat{Q}(x)^2 + \hat{P}(x)\hat{Q}(x)\right] + \hat{P}(x) + \hat{Q}(x)$$

Ladder operators

$\hat{\chi}_o,\ \hat{\chi}_o^\dagger \rightarrow\ \hat{\nu}_o$ quantum number

$\hat{\Lambda}_P^-,\ \hat{\Lambda}_P^+ \rightarrow\ P$ Total gauge flux quantum number

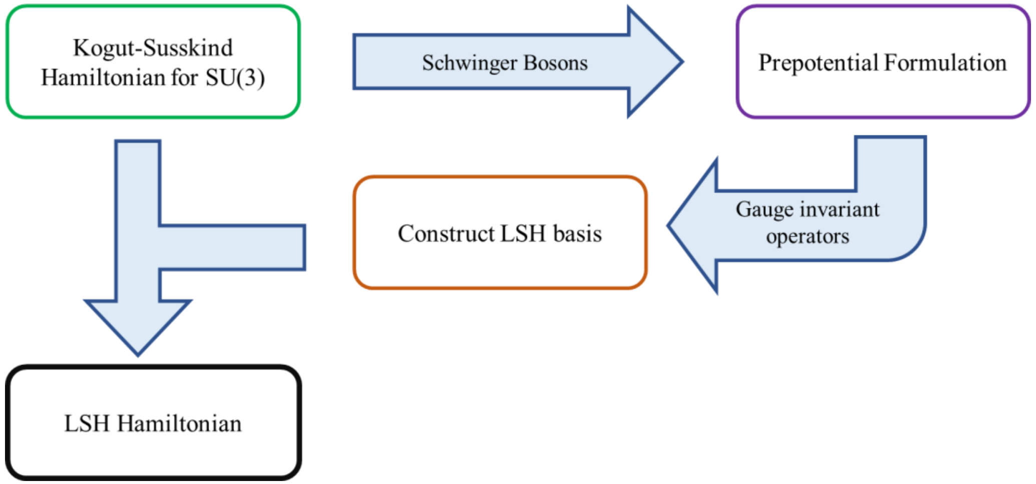
$$H_I = \frac{1}{2a} \sum_{x=0} \left\{ \hat{\chi}_o^\dagger (\hat{\Lambda}_P^+)^{\hat{\nu}_m} \sqrt{1 - \frac{\hat{\nu}_m}{\hat{n}_P + 2}} \sqrt{1 - \frac{\hat{\nu}_i}{\hat{n}_P + \hat{n}_Q + 3}} \right\}_x$$

$$\times \left\{ \sqrt{1 + \frac{\hat{\nu}_m}{\hat{n}_P + 1}} \sqrt{1 + \frac{\hat{\nu}_i}{\hat{n}_P + \hat{n}_Q + 2}}\ \hat{\chi}_o (\hat{\Lambda}_P^+)^{1-\hat{\nu}_m} \right\}_{x+1} + \text{h.c.} + \cdots$$

Conclusions

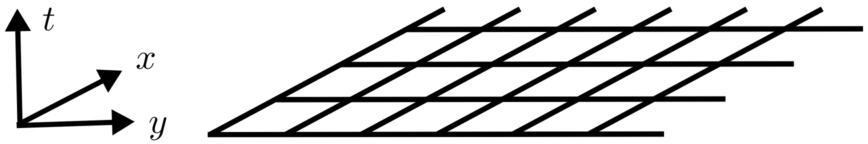
**Preliminary Results
-Kadam, Raychowdhury,, and Stryker

- ❑ Desirable features of SU(2) LSH are seen in SU(3) LSH Hamiltonian
  - ✓ Local gauge singlet basis
  - ✓ Non-locality through Abelian Gauss' laws
  - ✓ Hamiltonian as building blocks that act on local basis states



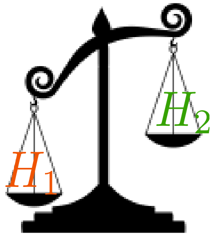
Outlook

- ❑ Generalization to higher dimensions → QCD



- ❑ Digital quantum circuit

- ❑ Comparison with other formulations



Thank You !

Appendix Slides

Numerical Benchmark

Comparing with purely fermionic formulation

- In 1+1 D with open boundary condition: No dynamical gauge dof
- Gauge field can be removed by fixing the gauge as:

$$\psi(x) \to \psi'(x) = \left[\prod_{r<x} U(r)\right] \psi(x) \quad \text{implies} \quad U(x) \to U'(x) = \mathbf{1}$$

- Electric field can be rewritten using Gauss' law:

$$G^a(x)|\text{Phys}\rangle = 0 \quad \text{and} \quad G^a(x) = -E_L^a(x) + E_R^a(x-1) + \tfrac{1}{2}\psi^\dagger(x)\lambda^a\psi(x)$$

implies $$E_L^a(x) = \epsilon^a + \frac{1}{2}\sum_{r=0}^{x-1}\psi^\dagger(r)\lambda^a\psi(r)$$

- Hamiltonian is reformulated in only fermionic dofs
- Hilbert space is spanned by fermionic occupation number basis

Q	(P_{out}, Q_{out})	$d(P_{out}, Q_{out})$	Eigenvalue
0	(0, 0)	1	0.000
1	(1, 0)	3	-0.387 1.721
2	(0, 1)	3	-1.535 0.868 3.333
2	(2, 0)	6	1.333
3	(0, 0)	1	-3.858 -0.497 2.137 4.884
3	(1, 1)	8	-0.081 2.747
4	(1, 0)	3	-2.562 1.089 4.140
4	(0, 2)	6	1.333
5	(0, 1)	3	-1.277 2.610
6	(0, 0)	1	0.000

TABLE I. Eigenvalues for $N = 2$ with $\mu a = 1$ are shown in this table. The eigenvalues are categorized according to according to P_{out}, Q_{out}, and Q quantum numbers of their corresponding eigenstates. $d_{(P_{out},Q_{out})}$ is the dimension of the (P_{out}, Q_{out}) irreducible representation of the outgoing chromo-electric flux given by Eq. 82 which is also the multiplicity of degenerate eigenvalues in the purely fermionic formulation.

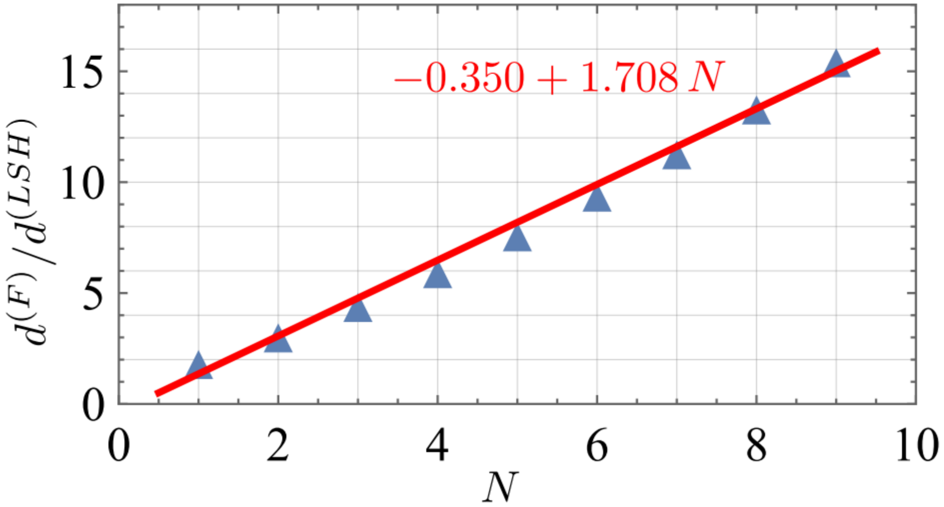


FIG. 6. The ratio of the dimension of the Hilbert space in the fully fermionic formulation with OBC, $d^{(F)}$, to the dimension of the physical Hilbert space withing the LSH formulation, $d^{(LSH)}$ is plotted for a range of lattice size values, N. Blue triangles denote the numerical values, and the red line indicates the emperical fit.

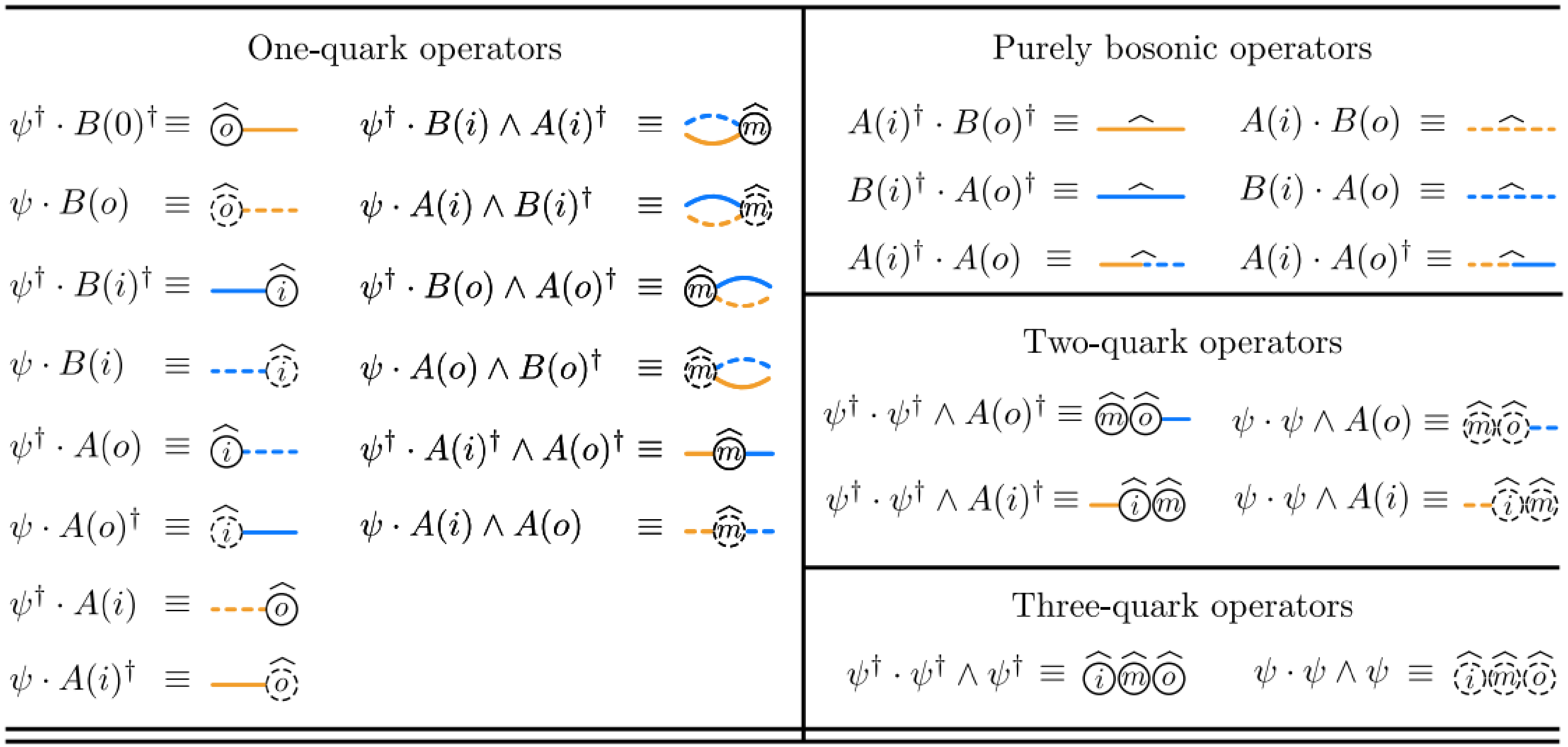

One-quark operators
$\psi^\dagger \cdot B(0)^\dagger \equiv$
$\psi \cdot B(o) \equiv$
$\psi^\dagger \cdot B(i)^\dagger \equiv$
$\psi \cdot B(i) \equiv$
$\psi^\dagger \cdot A(o) \equiv$
$\psi \cdot A(o)^\dagger \equiv$
$\psi^\dagger \cdot A(i) \equiv$
$\psi \cdot A(i)^\dagger \equiv$
$\psi^\dagger \cdot B(i) \wedge A(i)^\dagger \equiv$
$\psi \cdot A(i) \wedge B(i)^\dagger \equiv$
$\psi^\dagger \cdot B(o) \wedge A(o)^\dagger \equiv$
$\psi \cdot A(o) \wedge B(o)^\dagger \equiv$
$\psi^\dagger \cdot A(i)^\dagger \wedge A(o)^\dagger \equiv$
$\psi \cdot A(i) \wedge A(o) \equiv$
Purely bosonic operators
$A(i)^\dagger \cdot B(o)^\dagger \equiv$
$A(i) \cdot B(o) \equiv$
$B(i)^\dagger \cdot A(o)^\dagger \equiv$
$B(i) \cdot A(o) \equiv$
$A(i)^\dagger \cdot A(o) \equiv$
$A(i) \cdot A(o)^\dagger \equiv$
Two-quark operators
$\psi^\dagger \cdot \psi^\dagger \wedge A(o)^\dagger \equiv$
$\psi \cdot \psi \wedge A(o) \equiv$
$\psi^\dagger \cdot \psi^\dagger \wedge A(i)^\dagger \equiv$
$\psi \cdot \psi \wedge A(i) \equiv$
Three-quark operators
$\psi^\dagger \cdot \psi^\dagger \wedge \psi^\dagger \equiv$
$\psi \cdot \psi \wedge \psi \equiv$

FIG. 3.