Lattice Non-Linear Sigma Model on the Supersphere

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Rethinking
Quantum Field Theory

The Non-Linear Sigma Model

String theory and supersymmetry are crucial in finding a description of quantum gravity (e.g. Superstrings).

String theories defined in a curved space-time take the form of a field theory, more precisely a **Non-Linear Sigma Model (NLSM)**.

The dynamical fields Φ are coordinates in a 2-dimensional worldsheet with their values in a Riemannian (super)manifold.

$$\Phi : \Sigma \to \mathcal{M}.$$

Check their behavior at arbitrary coupling numerically using **lattice QFT**, comparing with non-perturbative results from different methods (i.e. integrability).

The Non-Linear Sigma Model

String sigma models are very hard to study on the lattice (Diffeos invariance is broken by the lattice and power divergences arise) [Bianchi, Forini, Leder et al 19], [Bliard, IC, Forini, Patella 22] .

Look at simpler **toy models** in the realm of supersymmetric NLSM that share key features with string sigma models.

Under certain conditions, NLSM are renormalizable and even completely solvable.

In lattice QFT, NLSM are a widely used tool applied to a big number of physical problems (statistical mechanics, QCD toy models. . .).

Starting point: work with a simpler **supersymmetric NLSM** and check its properties and behavior on the lattice.

Non-Linear Sigma Model on the Supersphere

A possible toy model is the **Supersphere/$\boldsymbol{OSP(N+2m|2m)}$-invariant NLSM** in two dimensions, supersymmetric extension of the $O(N)$ NLSM model.

$$\mathcal{S} = \frac{1}{g}\int d^2x\,\partial_\mu \Phi \cdot \partial^\mu \Phi,$$

Φ is a superfield, with bosonic and fermionic degrees of freedom

- $\Phi = (\phi_1, \ldots, \phi_{N+2m}, \psi_1, \ldots, \psi_{2m})$;
- $\Phi \in \frac{OSP(N+2m|2m)}{OSP(N+2m-1|2m)} \equiv S_{N+2m-1|2m}$,

  $$\Phi \cdot \Phi = \sum \phi_n^2 + \sum J_{\alpha\beta}\psi_\alpha\psi_\beta = 1, \quad J_{\alpha\beta} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix};$$
- $Z = \int \mathcal{D}_S\Phi \exp\left[-\mathcal{S}\right]$;
- $\mathcal{D}_S\Phi = \mathcal{D}\phi\mathcal{D}\psi\,\delta(\phi^2 + J_{\alpha\beta}\psi_\alpha\psi_\beta)$.

Some properties of the Supersphere NLSM

Conjecture: the S-matrix of these theories is determined by continuation of the $O(N)$ counterpart for $N + 2m > 1$ and $N < 2$ [Saleur, Kaufmann 01].

For $N > 1$:

- Renormalization properties of the $OSP(N + 2m|2m)$ model on the lattice are similar to the $O(N)$ NLSM.
  To calculate physical quantities, we only need to renormalize g;
- At all orders the beta functions $\beta_{OSP} = \beta_{O(N)}$.

The $OSP(3|2)$-invariant model

Particular case is $N = m = 1$. This is the $\boldsymbol{OSP(3|2)}$-invariant model.

We have proven that the 2-point correlation functions are all identical and together with the $\langle\varphi_n^4\rangle$, equivalent to the **Ising Model**.

However, there are some observables that are unique of the supersphere model.

Examples are

$$\langle\phi_n\phi_n\bar{\psi}\psi\rangle \qquad \langle\phi_n\phi_n\phi_m\phi_m\rangle \qquad \langle\bar{\psi}\psi\bar{\psi}\psi\rangle$$

We want to check these properties and behavior of the theory at different values of the coupling with lattice simulations.

The theory on the lattice

Discretized action:

$$S = \frac{1}{g}\sum_{x} \partial_{\mu}^{f}\Phi \cdot \partial^{f\ \mu}\Phi = \frac{1}{g}\sum_{x,\mu,n}\left[2 - \left(\bar{\psi}_{x+\mu}\psi_{x} + \bar{\psi}_{x+}\psi_{x+\mu} + 2\phi^{n}_{x+\mu}\phi^{n}_{x}\right)\right].$$

Using:

- $\phi_n = \rho\varphi_n \qquad$ with $\varphi^2 = 1$.
- $\mathcal{Z} = \int \mathcal{D}\varphi\mathcal{D}\psi\,\delta(\varphi^2 - 1)e^{-S_{\text{eff}}}$.

$$S_{\text{eff}} = \sum_{x}\bar{\psi}\psi + \frac{2}{g}\sum_{x}\left[1 - \varphi^{n}_{x+\mu}\varphi^{n}_{x} + \bar{\psi}_{x}\psi_{x}\varphi^{n}_{x}\left(\varphi^{n}_{x+\mu} + \varphi^{n}_{x-\mu}\right)\right.$$
$$\left. - \bar{\psi}_{x+\mu}\psi_{x+\mu}\bar{\psi}_{x}\psi_{x}\varphi^{n}_{x+\mu}\varphi^{n}_{x} - \bar{\psi}_{x}\left(\psi_{x+\mu} + \psi_{x-\mu}\right)\right].$$

The theory on the lattice

Apply Hubbard-Stratonovich transformation:

$$e^{\alpha\xi^2} = \sqrt{\frac{\pi}{\alpha}} \int dA\, e^{-\alpha(A^2 + 2A\xi)},\ \xi = \bar{\psi}\psi\varphi \quad \text{for every multi-index } (x, \mu, n),$$

and integrate out the fermions to get

$$S_{\text{eff}} = \sum_x \frac{2}{g}\left(1 - \varphi^n_{x+\mu}\varphi^n_x - \frac{1}{2}A^{(\mu,n)\,2}_x\right) + \sum_{x,y} \chi^T_x (\mathcal{K}\mathcal{K}^T)^{-1}(x,y)\,\chi_y.$$

$$\mathcal{K}(x,y) = \delta_{xy} + \frac{2}{g}\left[\varphi^n_x\left(\varphi^n_{x+\mu} + \varphi^n_{x-\mu}\right)\delta_{xy} + (A^{\mu,n}_x + A^{\mu,n}_{x-\mu})\varphi^n_{n,x}\delta_{xy} - \frac{1}{2}(\delta_{x-\mu,y} + \delta_{x+\mu,y})\right].$$

Sign problem

$\det\mathcal{K} \in \mathbb{R}$ but is not positive definite.

The theory on the lattice

Pseudo-fermion action is non-local. We have worked with a **Hybrid Monte-Carlo** algorithm.

Bosonic fields need to satisfy the constraint $\varphi^2 = 1$.

Molecular dynamics: use the rigid rotor hamiltonian to build a symplectic integrator that takes into account the constraints of the system.

$$\begin{cases} \pi_{1/2} = \pi_0 + \frac{\tau}{2} Q_0 \, F(q_0) \\ q_1 = \cos(\tau|\pi_{1/2}|)\, q_0 + \sin(\tau|\pi_{1/2}|)\, \frac{\pi_{1/2}}{|\pi_{1/2}|} \\ \pi_1 = \cos(\tau|\pi_{1/2}|)\, \pi_{1/2} - \sin(\tau|\pi_{1/2}|)\, |\pi_{1/2}| q_0 + \frac{\tau}{2}\, Q_1 \, F(q_1) \end{cases}$$

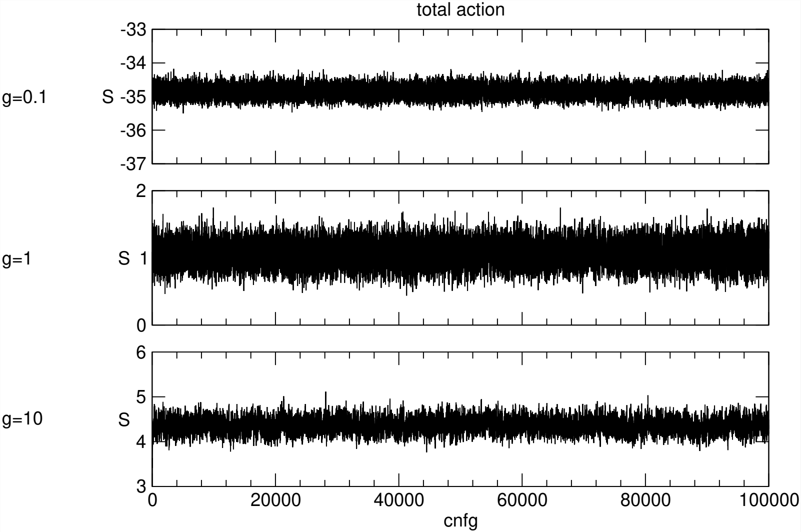
$\boldsymbol{q}$ identifies a field (bosons φ or the auxiliary field A);

$\boldsymbol{Q}$: projectors on the plane orthogonal to q;

$\boldsymbol{F} = -\frac{\partial S}{\partial \boldsymbol{q}}$.

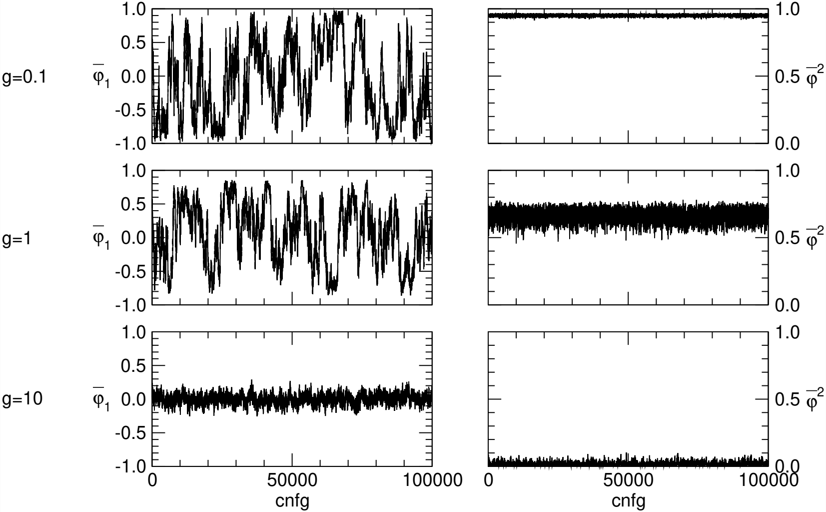
Some numerical results

Results are for a 16×16 lattice at different values of the coupling ($g = 0.1, 1.0, 10$).

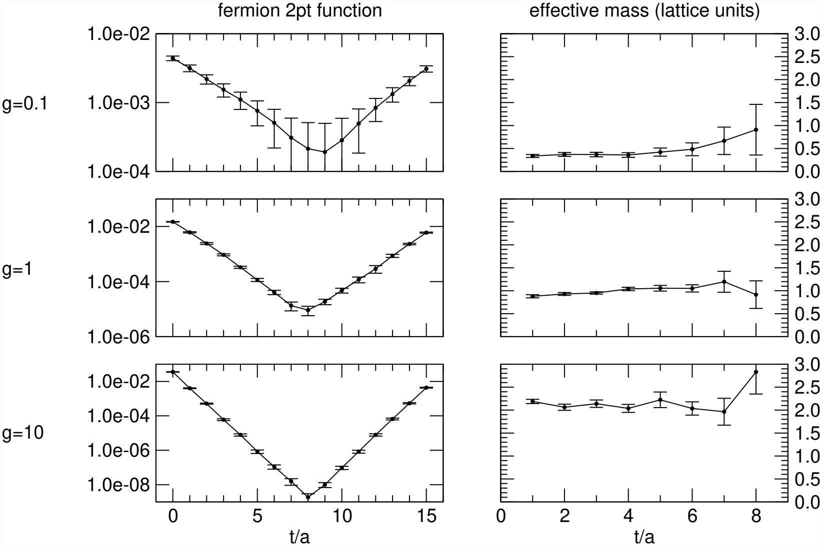


Some numerical results

$$\bar{\varphi}_1 = \frac{1}{V}\sum_x \varphi_{1\,x} \qquad \bar{\varphi}^2 = \sum_n (\frac{1}{V}\sum_x \varphi_{n\,x})^2$$



Some numerical results



Outlook

- Compute two-point functions of the bosons and of the conserved currents;
- Find the effective masses;
- Analyze the sign problem in discretized path integral;
- Take the limit $V \to \infty$;
- Confront with results in the $O(N)$ case known from the literature and with theoretical predictions;
- In the future, we also want to look at OSP-invariant models with a higher number of degrees of freedom (i.e. $OSP(4|2), OSP(5|2)\ldots$);
- $OSP(4|2)$ is scale invariant and would move the analysis towards string theory.

Thank you!