# Simulation of self-dual U(1) lattice gauge theory with electric and magnetic matter

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M.A., Gattringer, C., Iqbal, N., Sulejmanpasic, T.
*Phase structure of self-dual lattice gauge theories in 4d.*
J. High Energ. Phys. 2022, 149 (2022) [arXiv:2203.14774]

M.A., Gattringer, C., Sulejmanpasic, T.
*Self-dual U(1) lattice field theory with a θ-term.*
J. High Energ. Phys. 2022, 120 (2022) [arXiv:2201.09468]

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# Motivation

- Duality is a powerful tool relating weak and strong coupling regimes, allowing for non-perturbative insights.
- We study a fully self-dual lattice model with U(1) gauge fields and electric matter as well as magnetic matter (coupled in a local way).
- Spontaneous breaking of self-duality as a function of the matter coupling parameter has been conjectured.
- We find a first order transition line with two end points.
- In this project we numerically explore possible self-duality breaking, using Monte Carlo simulations based on a worldline formulation.

# U(1) pure lattice gauge theory: *Villain formulation*

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu} F_{x,\mu\nu}}$$

**Villain variables** $n_{x,\mu\nu} \in \mathbb{Z}$
(assigned to the plaquettes)

$F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$

$(dA)_{x,\mu\nu} = A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu}$

$\sum_{\{n\}} = \prod_{x\in\Lambda} \prod_{\mu<\nu} \sum_{n_{x,\mu\nu}\in\mathbb{Z}}$

Link variables $U_{x,\mu} \;=\; e^{iA_{x,\mu}}$ are invariant under the shift $A_{x,\mu} \to A_{x,\mu} \,+\, 2\pi\, k_{x,\mu},\; k_{x,\mu} \in \mathbb{Z}$

Exterior derivatives transform as $(dA)_{x,\mu\nu} \;\to\; (dA)_{x,\mu\nu} \,+\, 2\pi\, (dk)_{x,\mu\nu}$

Summation over the Villain variables $n_{x,\mu\nu}$ cancels the shifts $(dk)_{x,\mu\nu}$

# U(1) pure lattice gauge theory: *Villain formulation*

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu}F_{x,\mu\nu}}$$

- strength field tensor
  $F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$

- shift symmetry
  $(dA)_{x,\mu\nu} \;\to\; (dA)_{x,\mu\nu} \;+\; 2\pi\;(dk)_{x,\mu\nu}$

One can gauge the center symmetry imposing the ‘closedness’ constraint to *eliminate monopoles* :

$$\boxed{d^2 = 0 \to \;(dn)_{x,\mu\nu\rho} \;=\; 0 \quad \forall\,(x,\mu<\nu<\rho).}$$

** C.Gattringer, T.Sulejmanpasic*
*Nucl. Phys. B943 (2019)*
*arXiv:1901.02637*

The constraints are:

- implemented on the cubes
- necessary for **self-duality**
- correspond to **absence of monopoles** in the U(1) gauge theory with Villain action

# U(1) pure lattice gauge theory: *Villain formulation*

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu}F_{x,\mu\nu}} \prod_{x} \prod_{\mu<\nu<\rho} \delta\Big((dn)_{x,\mu\nu\rho}\Big)$$

Constraints on Villain variables in the partition sum are introduced with the Kronecker deltas.

**In integral representation:**

$$Z(\beta) \;=\; \int D[A^e] \int D[A^m] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_x \sum_{\mu<\nu} (F^e_{x,\mu\nu})^2} e^{i\sum_x \sum_{\mu<\nu<\rho} A^m_{x,\mu\nu\rho}(dn)_{x,\mu\nu\rho}}$$

*Introducing a magnetic gauge field that naturally lives on links $\tilde{x},\mu$ of the dual lattice*

- Electric gauge field $A^e_{x,\mu}$ describes the photon dynamics.
- Magnetic gauge field $\widetilde{A}^m_{\tilde{x},\mu} \in R$ lives on the dual lattice and removes monopoles.

# U(1) pure lattice gauge theory: *Villain formulation*

Switching to the dual lattice and using the Poisson resummation one finds

$$Z(\beta) \;=\; c\, Z(\widetilde{\beta}), \qquad \widetilde{\beta} \;=\; \frac{1}{4\pi^2 \beta}$$

Theory is self-dual!

The self-duality relation obviously ***maps the weak and strong coupling*** regions of $Z(\beta)$ onto each other.

# Generalization to coupling magnetic & electric matter

$$Z(\beta, J_e, J_m) \;\equiv\; \int D[A^e] \int D[A^m] \; B_\beta[A^e, A^m] \; Z[A^e, J_e] \; \widetilde{Z}\big[\widetilde{A}^m, J_m\big]$$

We have coupled electric and magnetic matter using U(1)-valued matter fields (also possible to couple complex-valued bosonic fields or fermions)

$\phi^e_x = e^{\, i\varphi^e_x}$ with $\varphi^e_x \in [-\pi, \pi)$

$$Z[A^e, J_e] \equiv \int D[\phi^e] \, e^{\, J_e S_e[\phi^e, \, A^e]}$$

$$\mathrm{S}_e[\phi^e, A^e] \equiv \frac{1}{2} \sum_{x,\mu} \left[ \phi^{e\,*}_x \, e^{iA^e_{x,\mu}} \, \phi^e_{x+\hat{\mu}} + c.c. \right] = \sum_{x,\mu} \cos\left( \varphi^e_{x+\hat{\mu}} - \varphi^e_x + A^e_{x,\mu} \right).$$

# Generalized theory remains self dual

$$Z(\beta, J_e, J_m) \;\equiv\; \int D[A^e] \int D[A^m] \; B_\beta[A^e, A^m] \; Z[A^e, J_e] \; \widetilde{Z}\big[\widetilde{A}^m, J_m\big]$$

As before ***self-duality*** with the corresponding relations :

$$Z(\beta, J_e, J_m) \;=\; c\, Z\big(\widetilde{\beta}, \widetilde{J}_e, \widetilde{J}_m\big)$$

$$\text{with} \quad \widetilde{\beta} \;=\; \tfrac{1}{4\pi^2\beta} \;, \widetilde{J}_e \;=\; J_m \;, \widetilde{J}_m \;=\; J_e$$

Derivatives of relate observables in $\ln Z$ *strong and weak coupling region*

$$\langle F^2\rangle_\beta \equiv -\frac{1}{3V}\frac{\partial}{\partial\beta}\ln Z(\beta) \qquad \boxed{\beta\,\langle F^2\rangle_{\beta,J_e,J_m} \;+\; \widetilde{\beta}\,\langle F^2\rangle_{\widetilde{\beta},\widetilde{J}_e,\widetilde{J}_m} \;=\; 1}$$

$$\langle s_{e,m}\rangle_J \equiv -\frac{1}{4V}\frac{\partial}{\partial J_{e,m}}\ln Z(\beta) \qquad \boxed{\langle s_e\rangle_{\beta,J_e,J_m} \;=\; \langle \widetilde{s}_m\rangle_{\widetilde{\beta},\widetilde{J}_e,\widetilde{J}_m}}$$

# Simulate theory in the self-dual point

$$\beta = \widetilde{\beta} = \beta^* = \frac{1}{2\pi}$$

$J_e \; = \; J_m \; = \; \widetilde{J}_e \; = \; \widetilde{J}_m \; = \; J$ $\quad\Longrightarrow\quad$ The only remaining parameter is coupling J

- $\beta^* \, \langle F^2 \rangle_{\beta^*,\, J} \; + \; \beta^* \, \langle F^2 \rangle_{\beta^*,\, J} \; = \; 1 \quad\Longrightarrow\quad \langle F^2 \rangle_{\beta^*,\, J} \; = \; \pi \quad \forall \, J$

- $\langle s_e \rangle_{\beta^*,\, J} \; = \; \langle \widetilde{s}_m \rangle_{\beta^*,\, J} \quad \forall \, J$

***Can self-duality be broken spontaneously as a function of J ?***

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- $\langle s_e \rangle_{\beta^*,\,J} \; = \; \langle \widetilde{s}_m \rangle_{\beta^*,\,J} \quad \forall\, J$

***Can self-duality be broken spontaneously as a function of J ?***

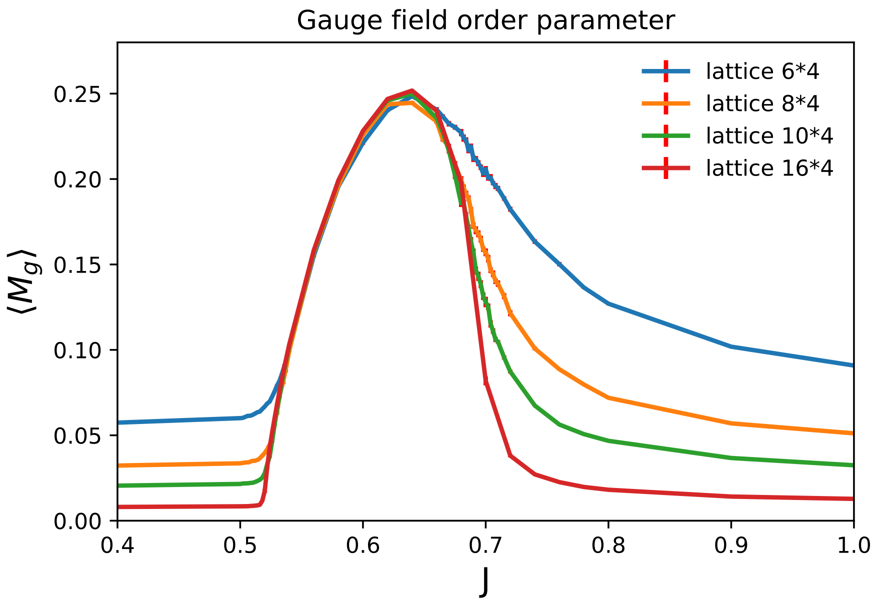
*Study behaviour of the observables*

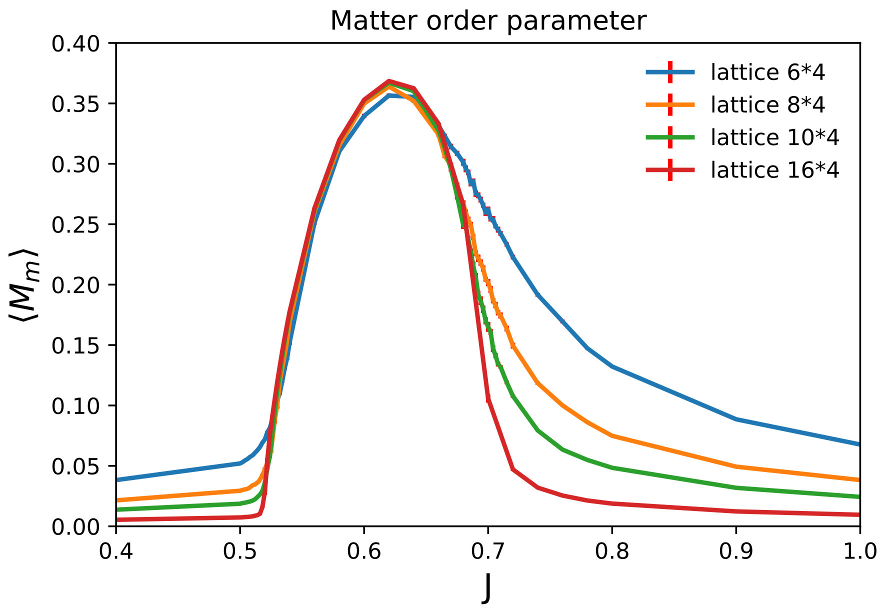
Self-duality breaking parameter, susceptibility, Binder cumulant

# Defining the **order parameters** of self-duality breaking

$$M_g \equiv |F^2 - \pi|$$

$$M_m \equiv |s_e - s_g|$$



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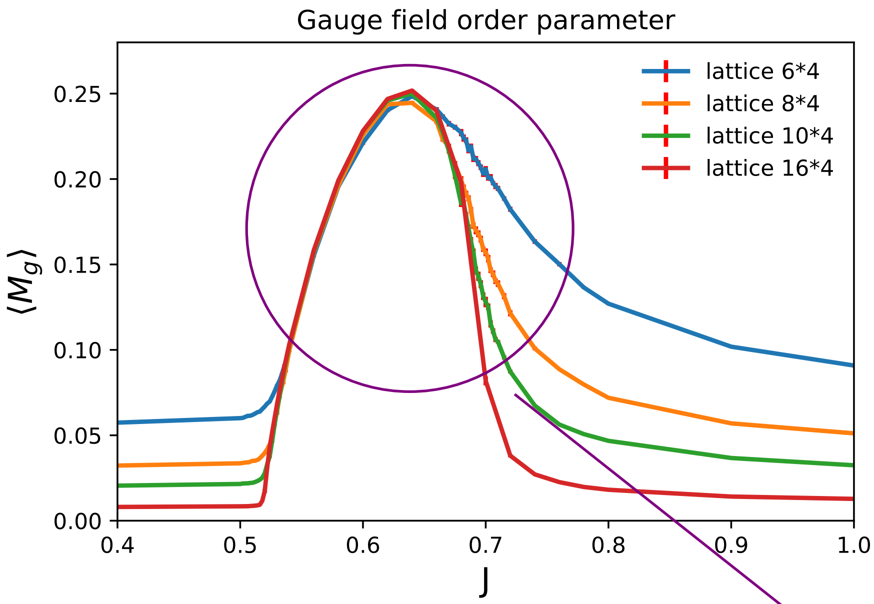
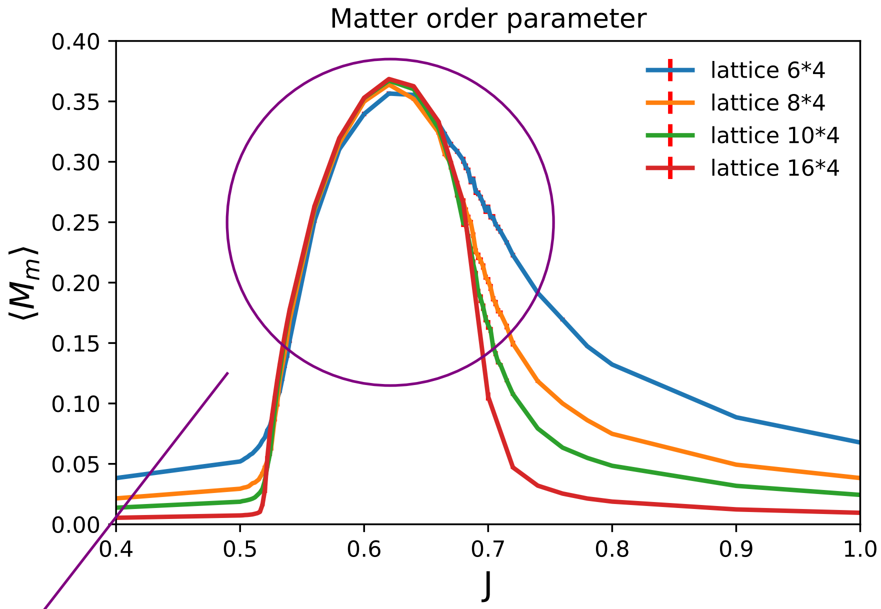


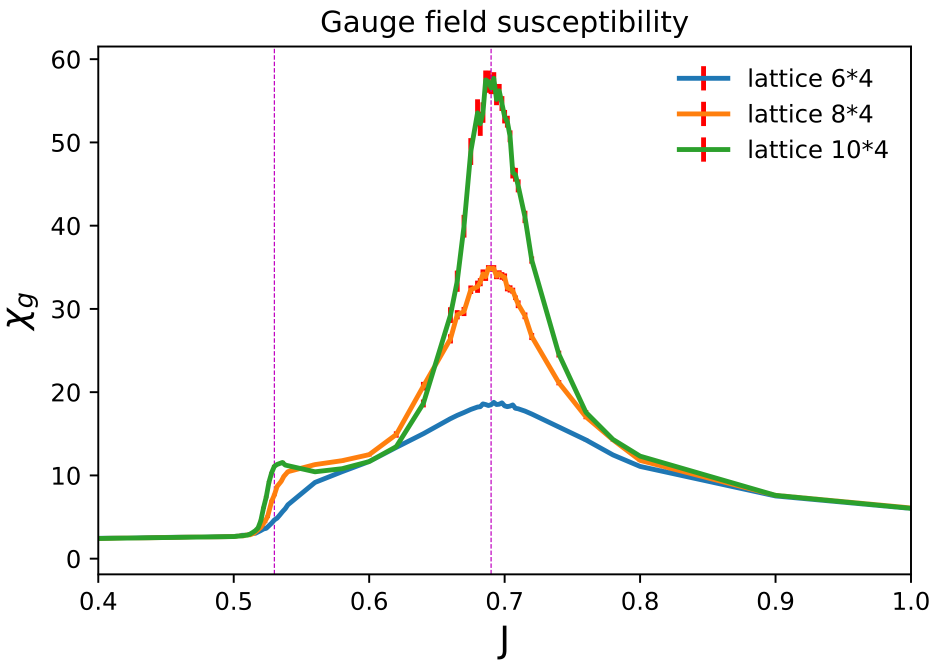



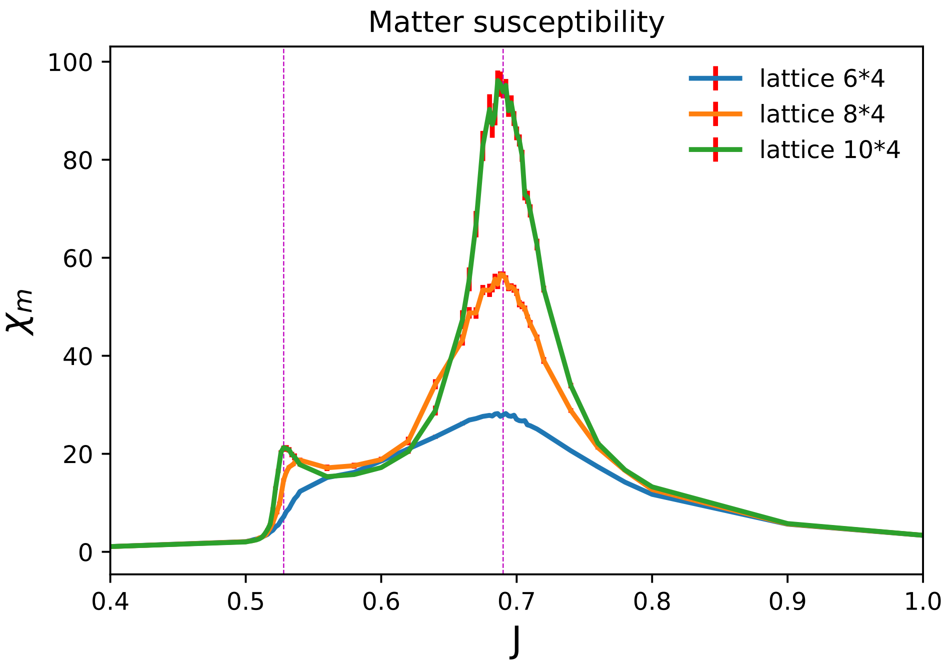

$$\langle M_g \rangle_{\beta^*,\,J} \neq 0 \quad \langle M_m \rangle_{\beta^*,\,J} \neq 0$$

***Spontaneous breaking of self-duality***

# Defining **susceptibilities**

$$\chi_g \equiv V \left\langle (M_g - \langle M_g \rangle_{\beta^*,\,J})^2 \right\rangle_{\beta^*,\,J} \qquad \chi_m \equiv V \left\langle (M_m - \langle M_m \rangle_{\beta^*,\,J})^2 \right\rangle_{\beta^*,\,J}$$



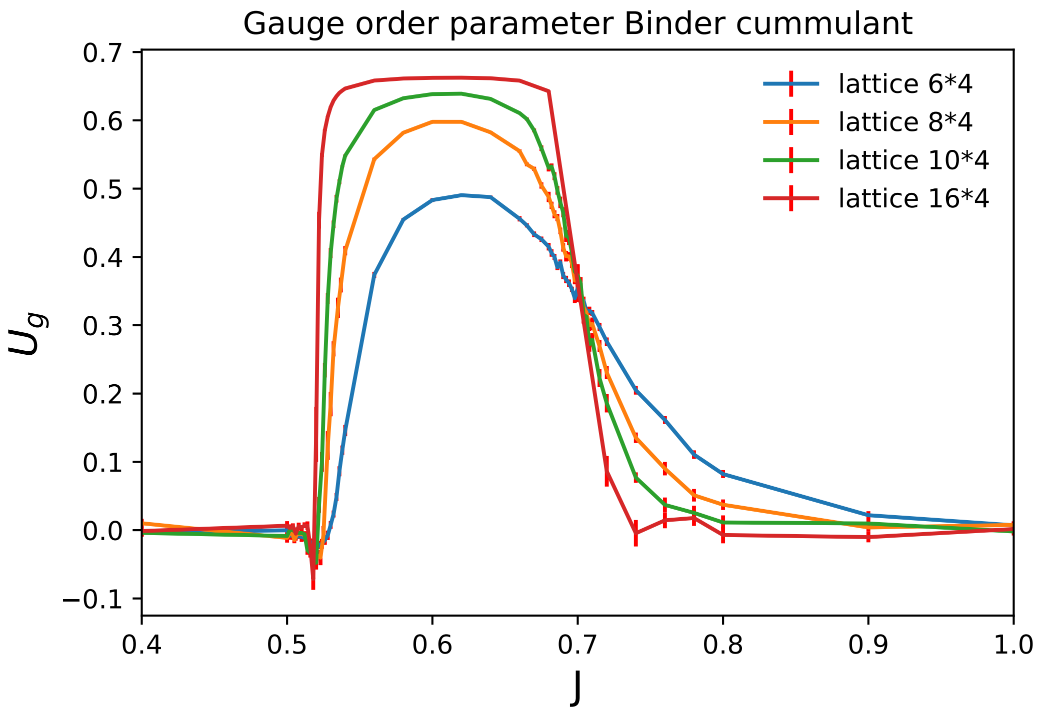



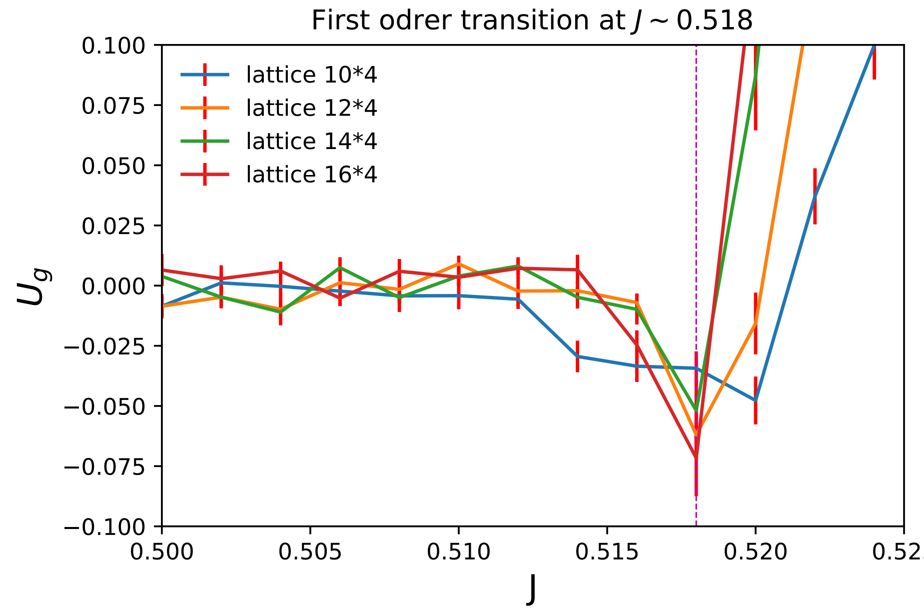

***First order*** phase transition at J ~ 0.52

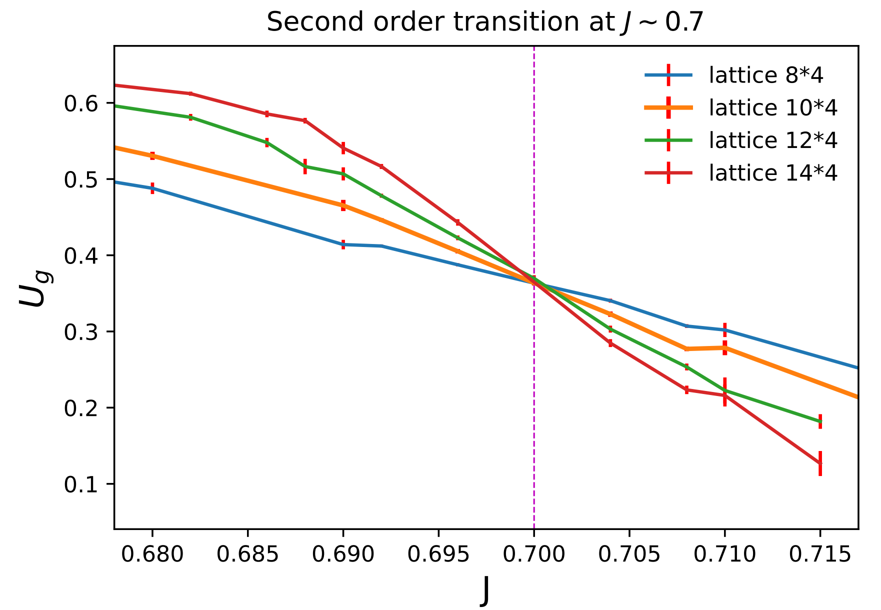
***Second order*** phase transition at J ~ 0.69

# **Binder cumulant** of the gauge order parameter

$$U_g \equiv 1 - \frac{\left\langle (M_g)^4 \right\rangle_{\beta^*,\, J}}{3 \left\langle (M_g)^2 \right\rangle_{\beta^*,\, J}^2}$$

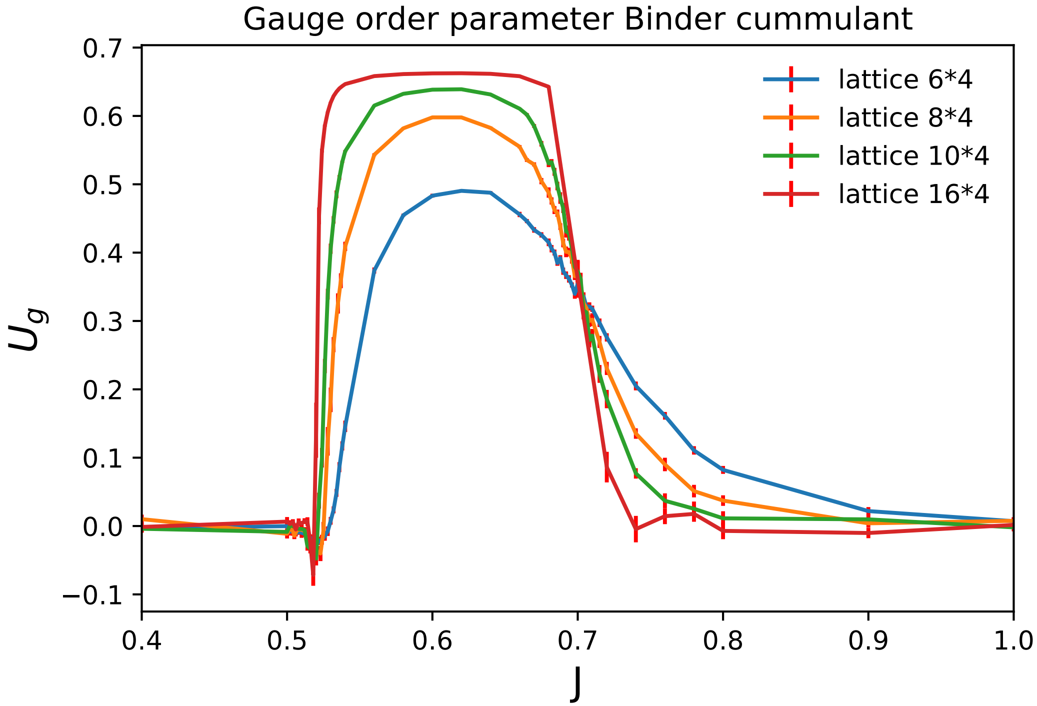


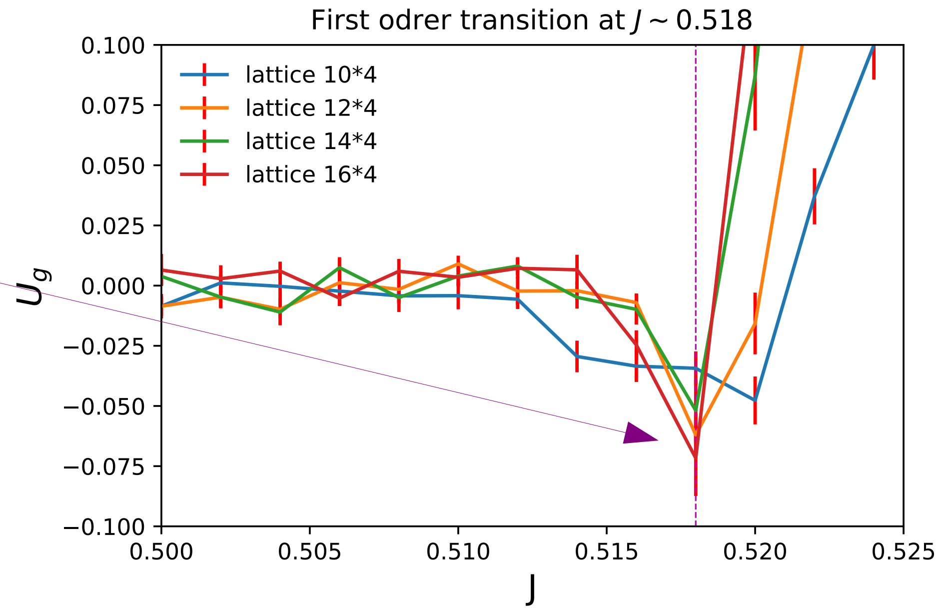



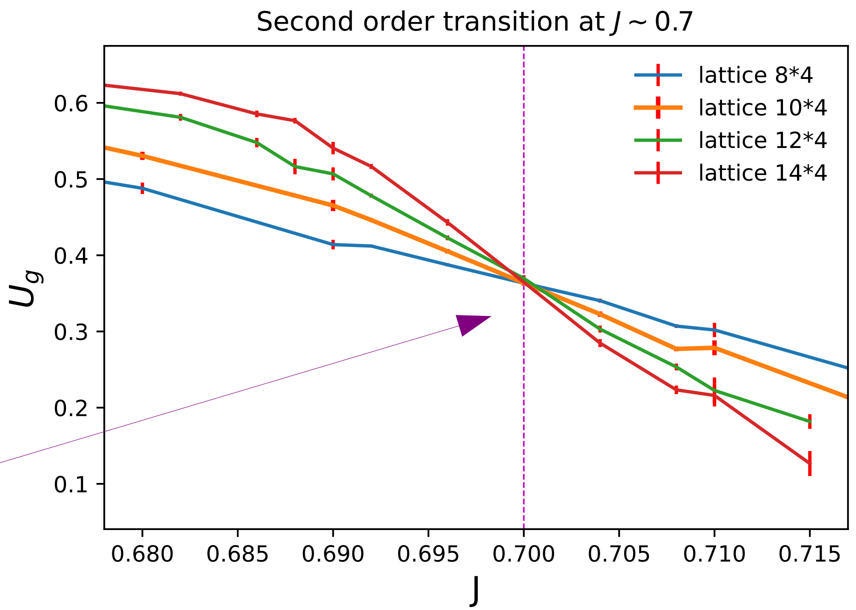



# **Binder cumulant** of the gauge order parameter

Develops a minimum at a first order transition

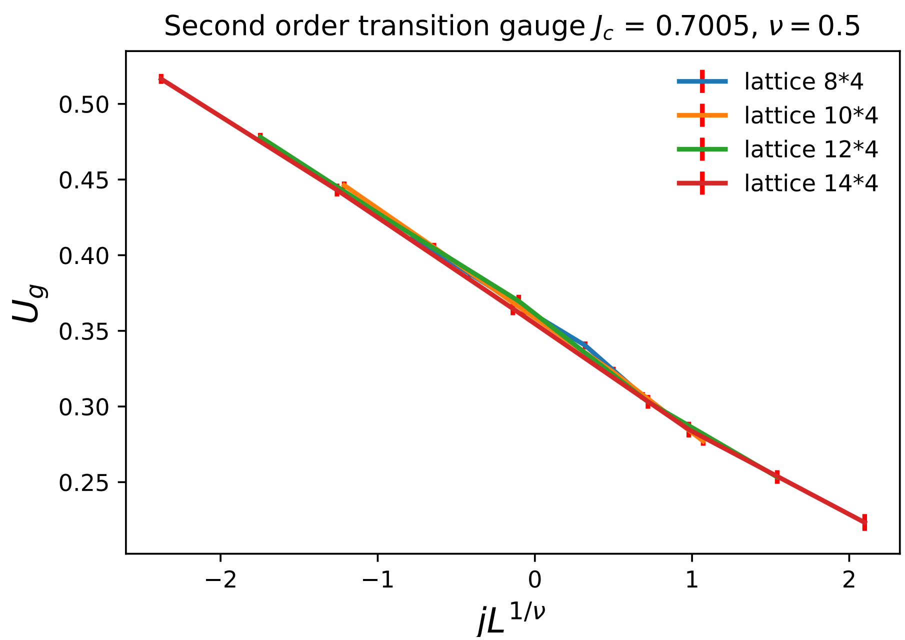


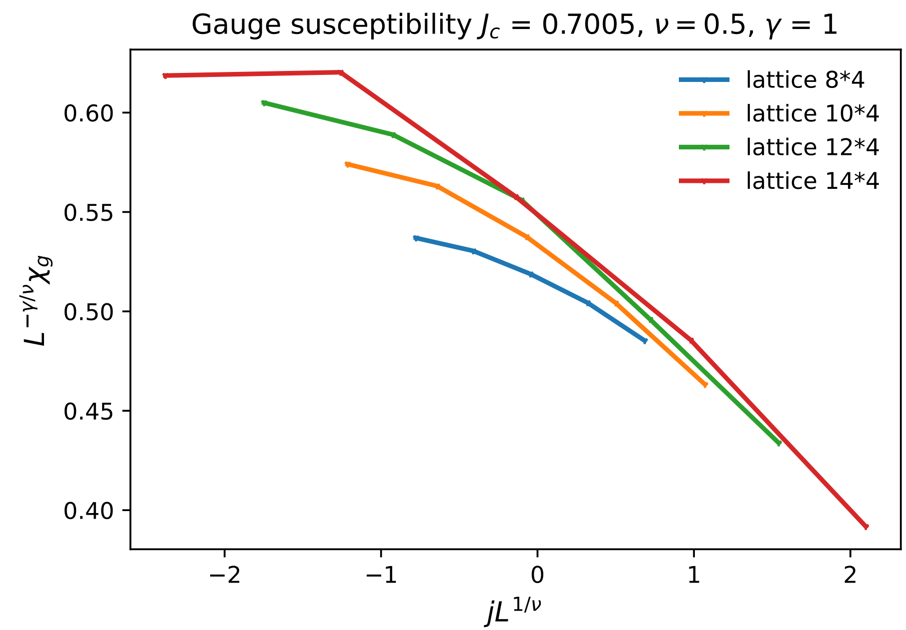



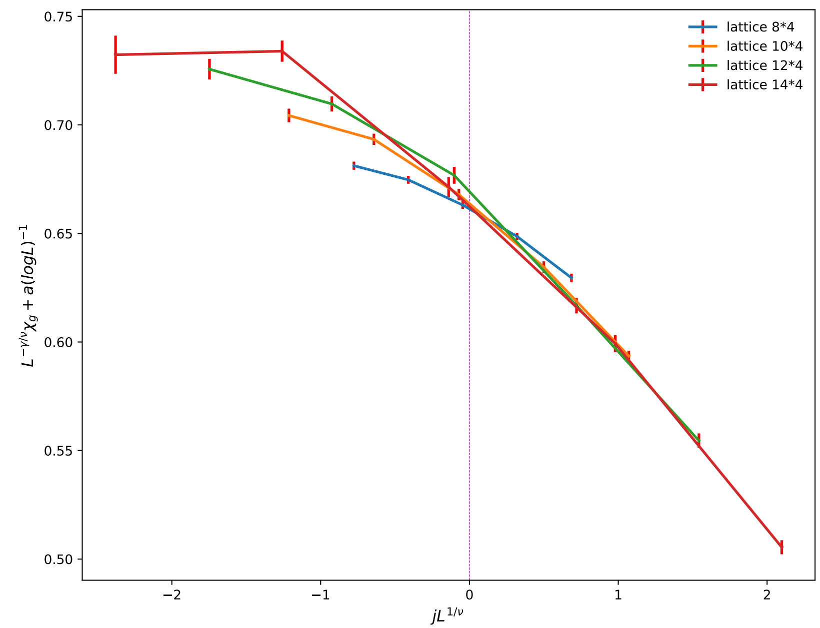




The intersection point converges to a critical point in the infinite lattice

# **Critical exponents:** mean field values



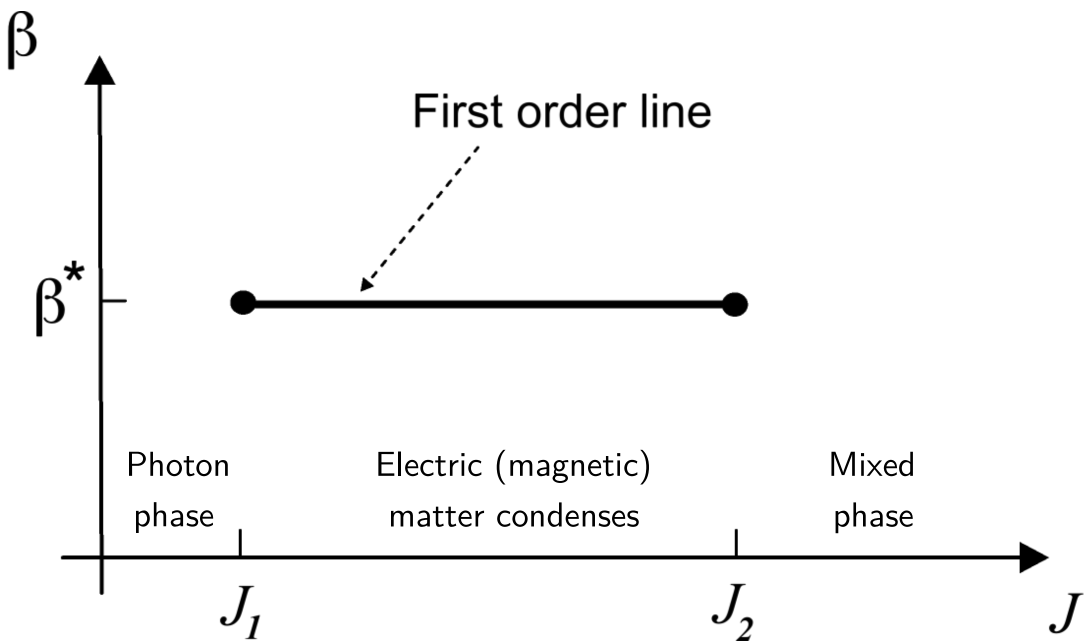




$\nu = 0.5 \pm 0.01$

$\gamma = 1$ (including subleading logarithmic corrections)

# Phase diagram

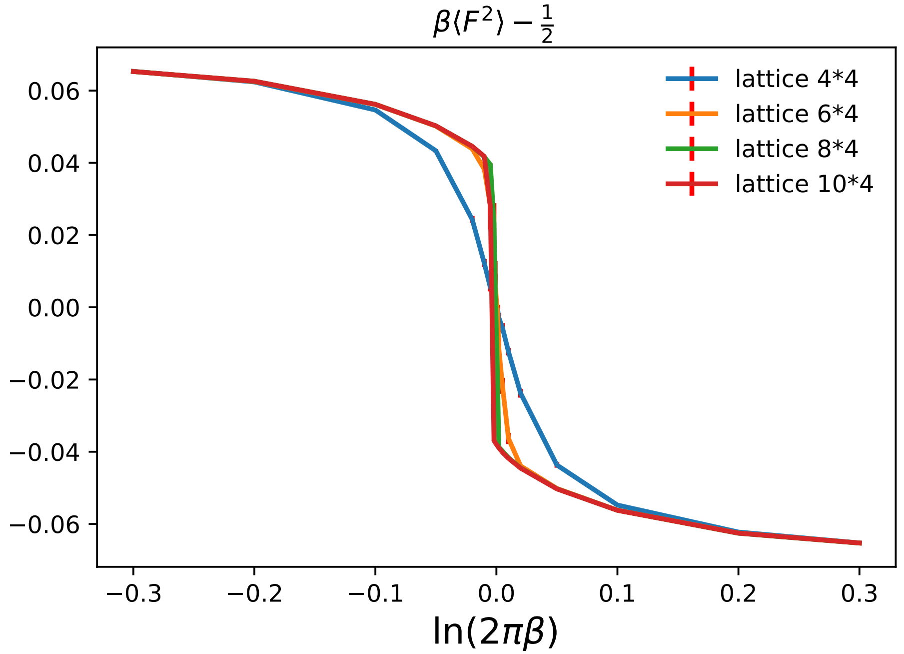
The self-dual coupling
$\beta^* = \frac{1}{2\pi}$

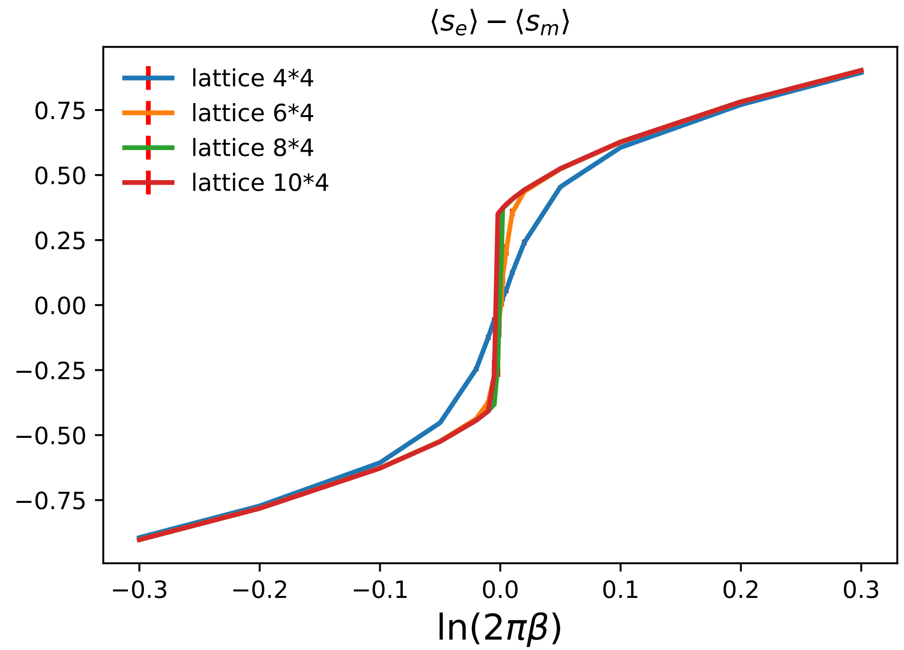



Two end points $J_1 \sim 0.518$ and $J_2 \sim 0.7$

# Cross-check of the transition at J=0.61

Crossing the 1st order line in vertical direction.



# Conclusions

- We study self-dual U(1) gauge theory with electric and magnetic matter.
- Use worldline representation for numerical simulation of self-duality relations.
- We observe spontaneous breaking of self-duality at $\beta^* = \frac{1}{2\pi}$ as a function of $J$ in the region with two endpoints.
- Binder cummulants allow an assessment of the nature of the endpoints:

  the first order at $J_1 \sim 0.518$ and the second order at $J_2 \sim 0.7$.
- Second order transition established with *mean field critical exponents*.

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Thank you for your attention!