Absence of inhomogeneous phases in $2+1$-dimensional Four-fermion models

Marc Winstel, Laurin Pannullo, Marc Wagner

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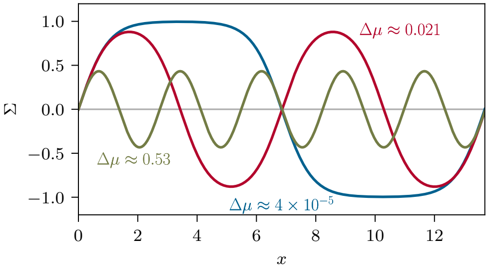
- Simple strong-interaction models feature so-called inhomogeneous, chiral phase
- Chiral condensate breaks translational invariance spontaneously $\langle\bar{\psi}\psi\rangle = f(\mathbf{x})$
- Indications for such phases and related phenomena found in QCD[1]



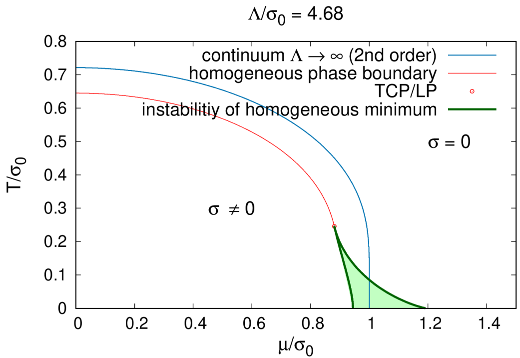
$1+1$-dimensional Gross-Neveu model in the mean-field approximation[2,3]

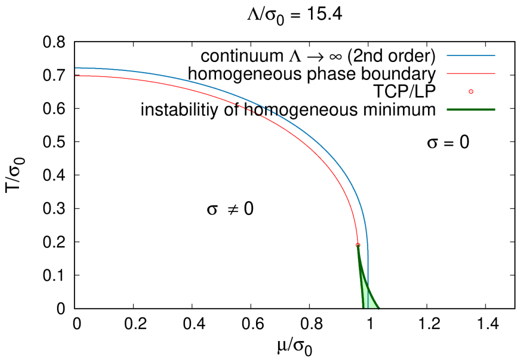
[1] W.-j. Fu, J. M. Pawlowski, F. Rennecke, *Phys. Rev. D* **2020**, *101*, 054032.
[2] M. Thies, K. Urlichs, *Phys. Rev. D* **2003**, *67*, 125015.
[3] A. Koenigstein, L. Pannullo, S. Rechenberger, M. Winstel, M. J. Steil, **2021**.

- In mean-field models inhomogeneous phases are:
  - Established in $1+1$ dimensions
  - Also found in $3+1$ dimensions[4], but the results are questionable ...
    - In the renormalizable Quark-Meson model the action gets unbounded when renormalizing[5]
    - In non-renormalizable NJL model the results depend on the regularizations scheme $\Rightarrow$ see Laurin Pannullos talk yesterday
- In $2+1$ dimensions we refer to Refs.[6][7][8]
- In short: Inhomogeneous phases are found at finite regulator values depending on regularization scheme, but vanish when $\Lambda \to \infty$

[4] M. Buballa, S. Carignano, *Prog. Part. Nucl. Phys.* **2015**, *81*, 39–96.
[5] S. Carignano, M. Buballa, B.-J. Schaefer, *Phys. Rev. D* **2014**, *90*, 014033.
[6] M. Buballa, L. Kurth, M. Wagner, M. Winstel, *Phys. Rev. D* **2021**, *103*, 034503.
[7] R. Narayanan, *Phys. Rev. D* **2020**, *101*, 096001.
[8] L. Pannullo, M. Wagner, M. Winstel, *Symmetry* **2022**, *14*, 265.

Motivation: 2+1-dimensional GN model





- Studied $2+1$-dim. Gross-Neveu model, which is a Four-fermion model with scalar $(\bar{\psi}\psi)^2$ channel

$\Rightarrow$ Can this result be transferred to more involved models in 2+1 dimensions?

A more general Four-fermion model?

- In principle, stability analysis applies to every kind of interaction-channel
- 4×4 Dirac Algebra allows for 16 possible bilinears, i.e.

$$\{\gamma_A\}_{A=1,\ldots,16} = \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45} \equiv \mathrm{i}\gamma_4\gamma_5, \gamma_\mu, \tfrac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu], \mathrm{i}\gamma_\mu\gamma_4, \mathrm{i}\gamma_\mu\gamma_5, \}$$

- Vector interactions do not exhibit chiral condensation in the mean-field approximation[9]
  $\Rightarrow$ Inhomogeneous chiral phases are not expected

[9] G. Parisi, *Nucl. Phys. B* **1975**, *100*, 368–388.

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$$\{\gamma_B\}_{B=1,\ldots,16} = \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45}\}$$

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CRC-TR 211

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- All kind of chemical potentials can be included, i.e. $\mu_B, \mu_{45}, \mu_5, \mu_4$ with corresponding structures in Dirac space, but also isospin potential μ_I

[9] G. Parisi, *Nucl. Phys. B* **1975**, *100*, 368–388.

A more general Four-Fermion model

- Bosonization of Four-Fermion models leads to the action

$$S_{\mathrm{FF}}[\bar{\psi}, \psi, \vec{\phi}] = \int \mathrm{d}^3x \, \left(N_c \, \frac{\vec{\phi}(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x})}{2\lambda} \; + \; \bar{\psi}(x) \, Q \, \psi(x) \right)$$

$$Q = \not{\partial} + \gamma_0 \mu + M + \sum_j c_j \phi_j(\mathbf{x})$$

- Equivalent action after integrating fermions out

$$S_{\mathrm{eff}}[\vec{\phi}]/N_c = \int \mathrm{d}^3x \; \frac{\vec{\phi}(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x})}{2\lambda} \; - \; \mathrm{Tr} \ln Q$$

- M contains all type of allowed mass terms, set $M = 0$ for chiral limit

$$\langle \phi_j \rangle \sim \langle \, \bar{\psi} c_j \psi \, \rangle$$

A more general Four-Fermion model

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- Bosonization of Four-Fermion models leads to the action $c_j \in \{\gamma_B\}_{B=1,\dots,16}$

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The stability analysis

- Analyze the stability of the homogeneous ground state $\vec{\phi} = \vec{\bar{\phi}}$

$$\phi_j = \bar{\phi}_j + \delta\phi_j(\mathbf{x})$$

- Mean-field approximation $\Rightarrow$ Homogeneous ground state via optimization of $S_{\text{eff}}[\bar{\phi}_j]$
- Compute corrections to the action due to perturbation $\delta\phi_j(\mathbf{q})$
- Second order corrections determine whether action is lowered by the perturbation
- Supplementary: Minimization with respect to $\phi_j\,(\mathbf{x})$ via LFT

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$$S_{\text{eff}}[\sigma]/N_c = \frac{1}{2\lambda}\int \mathrm{d}^3x\,(\bar\sigma + \delta\sigma(\mathbf{x}))^2 - \mathrm{Tr}(\ln(\underbrace{\not\partial + \gamma_0\mu + \bar\sigma}_{\equiv \bar Q} + \delta\sigma(\mathbf{x})))$$

- Expand $S_{\text{eff}}[\bar\sigma + \delta\sigma]$ in powers of $\delta\sigma$ yields

$$S^{(2)}_{\text{eff}}/N_c = \frac{\beta}{2\lambda}\int d^2x\,\left(\delta\sigma(\mathbf{x})\right)^2 + \frac{1}{2}\mathrm{Tr}\left(\bar{\mathrm{Q}}^{-1}\delta\sigma\bar{\mathrm{Q}}^{-1}\delta\sigma\right)$$

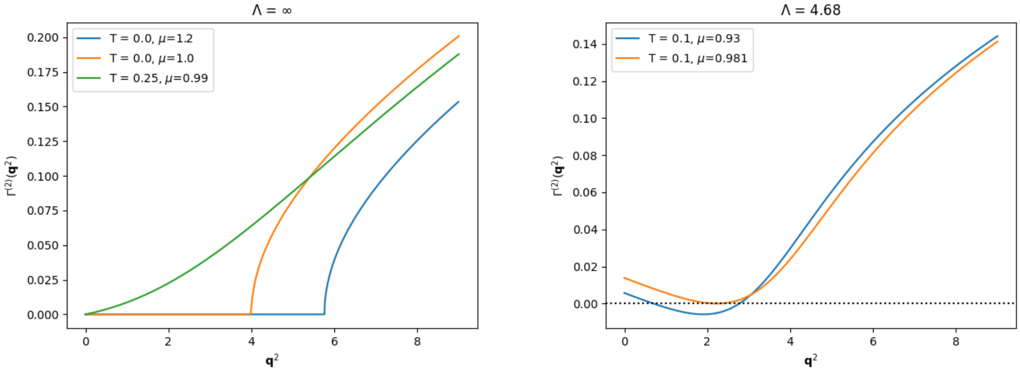
- Evaluating traces and fourier transform gives

$$S^{(2)}_{\text{eff}}/N_c = \frac{1}{2}\beta\int \frac{d^2q}{(2\pi)^2}|\delta\tilde\sigma(\mathbf{q})|^2\Gamma^{(2)}(\mathbf{q}^2)$$

$$\Gamma^{(2)}(\mathbf{q}^2) = \frac{1}{\lambda} - \ell_1 \underbrace{-\frac{1}{2}(\mathbf{q}^2 + 4\bar\sigma^2)\ell_2(\mathbf{q}^2,\bar\sigma,\mu^2)}_{L_2(\mathbf{q}^2,\bar\sigma,\mu)}$$

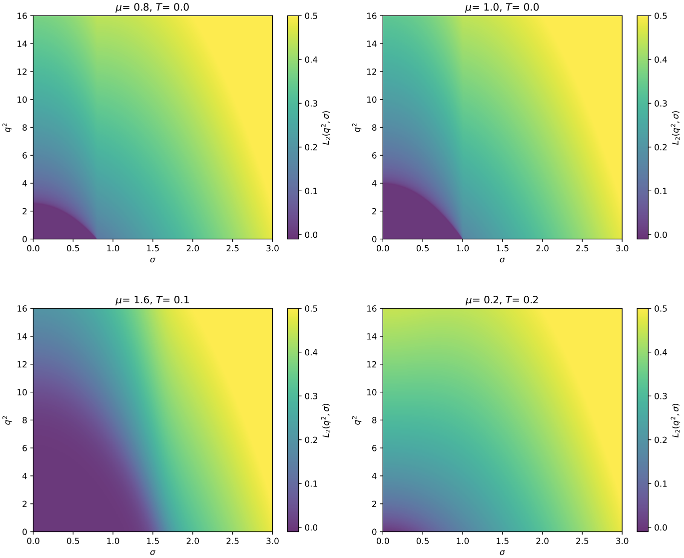
- $L_2(\mathbf{q}^2)$ is monotonically increasing $\forall \mu, T, \bar\sigma$

The two-point function in the GN model



- $T = 0, \bar{\sigma} = 0 : \; L_2 \sim \Theta(\frac{q^2}{4} - \mu^2) \, q \arctan\left(\sqrt{\frac{q^2}{4} - \mu^2}/\mu\right)$

Momentum dependence of $\Gamma^{(2)}$ - Explore $\mu, T, \bar{\sigma}$

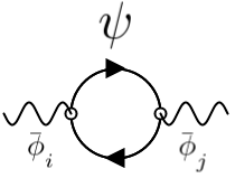


Analyzing the two-point function for other models

- The stability analysis can be applied to all Four-fermion channels

$$S_{\text{eff}}^{(2)}/N_c = \frac{\beta}{2\lambda}\int d^2q\,|\delta\phi_j(\mathbf{q})|^2 \;+\frac{\beta}{2}\sum_{i,j}\delta\phi_i^*(\mathbf{q}))\delta\phi_j(\mathbf{q})\Gamma^F_{\phi_i\phi_j}(\mathbf{q}^2))$$

$$\Gamma^F_{\phi_i\phi_j} = \int \mathrm{d}^3p\,\mathrm{tr}\left(c_i\tilde{\bar{Q}}^{-1}(p+q)c_j\tilde{\bar{Q}}^{-1}(p)\right)$$



- $\Gamma_{\phi_i\phi_j}(\mathbf{q})$ gives fermionic contribution to curvature

$\Rightarrow$ Strategy: Identify $L_2(\mathbf{q})$, as found for the GN model, for more complex models

- Potential terms & kinetic terms for ϕ_j will not change the monotonic behavior of $\Gamma^{(2)}$

First example: Gross-Neveu-Yukawa model

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$$S[\bar{\psi},\psi,\varphi] = \int \mathrm{d}^3x \Big(\frac{1}{2g^2}\varphi^2 + \frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4}\varphi^4 + \bar{\psi}_a \left(\gamma_\nu\partial_\nu + \gamma^0\mu + h\varphi\right)\psi_a\Big)$$

- By redefinitions of fields and couplings

$$\frac{S_{\text{eff}}[\sigma]}{N_c} = \int \mathrm{d}^3x \left(\frac{m_0^2}{2}\sigma^2 + \frac{\gamma}{2}(\partial_\nu\sigma)(\partial_\nu\sigma) + \frac{\kappa}{4}\sigma^4\right) - \ln\left(\mathrm{Det}(\not\partial + \gamma^0\mu + \sigma)\right)$$

- The two point function yields

$$\Gamma^{(2)} = m_0^2 - \ell_1 + \underbrace{L_2(\mathbf{q}^2,\bar{\sigma},\mu)}_{\text{known from GN model!}} + \frac{\gamma}{2}\mathbf{q}^2 + \frac{3\kappa}{2}\bar{\sigma}^2$$

- No inhomogeneous phases - as long as m_0^2, γ, κ are chosen such that action is bounded
- General statement for bosonized FF and corresponding Yukawa model

$U(2N_c)$ chiral symmetry + Parity breaking

- Study full chiral (flavor) symmetry group $U(2N_c)$ + possibility of parity breaking via η_{45}

$$S = \int \mathrm{d}^3x \Big(N_c \frac{\sigma^2 + \eta_4^2 + \eta_5^2 + \eta_{45}^2}{2\lambda} + \bar{\psi}_a \left(\not{\partial} + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_4 \eta_4 + \mathrm{i}\gamma_5 \eta_5 + \mathrm{i}\gamma_{45}\eta_{45} \right) \psi_a \Big)$$

- Two-point function as $\bar{\eta}_{45} = 0$ ($\bar{\eta}_4 = \bar{\eta}_5 = 0$ through chiral rotation)

$$S_{\mathrm{eff}}^{(2)}/N_c = \frac{\beta}{2} \int \frac{d^2q}{(2\pi)^2} \sum_{\phi \in \{\sigma, \eta_4, \eta_5, \eta_{45}\}} |\delta\tilde{\phi}(\mathbf{q})|^2 \Gamma_\phi^{(2)}(\mathbf{q}^2)$$

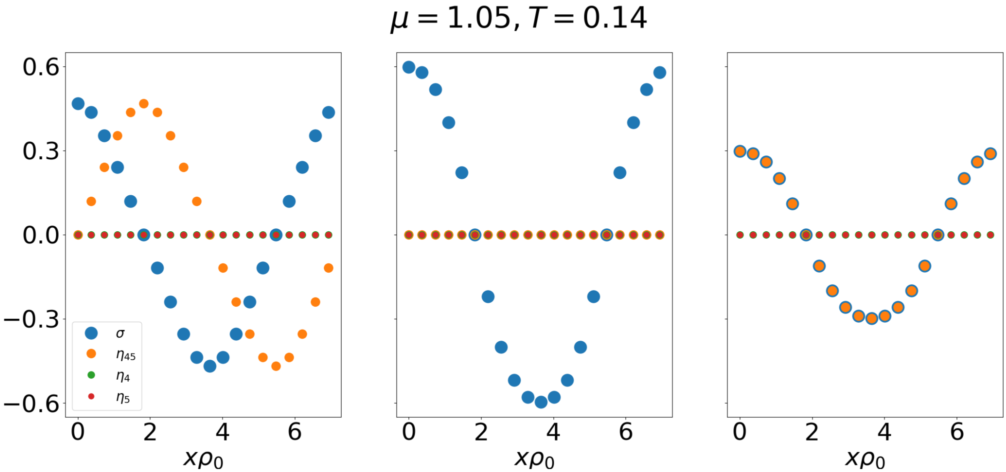
$$\Gamma_{\eta_{45}}^{(2)} = \Gamma_\sigma^{(2)} = \frac{1}{\lambda} - \ell_1 + L_2(\mathbf{q}^2, \bar{\sigma}, \mu)$$

$$\Gamma_{\eta_4}^{(2)} = \Gamma_{\eta_5}^{(2)} = \frac{1}{\lambda} - \ell_1 - \underbrace{\frac{1}{2}\mathbf{q}^2 \ell_2(\mathbf{q}^2, \bar{\sigma}, \mu^2)}_{L_2 \text{ without a } \bar{\sigma}^2 \text{ prefactor}}$$

- Again, monotonically increasing with q

Minimization on the lattice: $U(2N_c)$ chiral symmetry

- Exemplatory: IP on the lattice corresponds to instability region (vanishes in continuum limit)
- CDW with (σ, η_{45}) and "GN-like" phase in the $U(2N_c)$ model within instability region



'NJL' model in 2+1 dimensions

- Study higher symmetry groups via introduction of isospin

$$S=\int \mathrm{d}^3x\Big(N_c\frac{\sigma^2+\vec{\pi}_4^2+\vec{\pi}_5^2}{2\lambda}+\bar{\psi}_{a,f}\left(\not{\partial}+\gamma_0\mu+\sigma++\mathrm{i}\gamma_4\vec{\tau}_{fg}\,\vec{\pi}_4+\mathrm{i}\gamma_5\vec{\tau}_{fg}\,\vec{\pi}_5\right)\psi_{a,g}\Big)$$

- Second order corrections turn out to be diagonal again

$$S^{(2)}_{\text{eff}}/N_c=\frac{\beta}{2}\int\frac{d^2q}{(2\pi)^2}\sum_{\phi\in\{\sigma,\vec{\pi}_4,\vec{\pi}_5\}}|\delta\tilde{\phi}(\mathbf{q})|^2\Gamma^{(2)}_{\phi}(\mathbf{q}^2)$$

$$\Gamma^{(2)}_{\sigma}=\frac{1}{\lambda}-\ell_1+L_2(\mathbf{q},\bar{\sigma},\mu)$$

$$\Gamma^{(2)}_{\vec{\pi}_5}=\Gamma^{(2)}_{\vec{\pi}_4}=\frac{1}{\lambda}-\ell_1-\frac{1}{2}\mathbf{q}^2\ell_2(\mathbf{q}^2,\bar{\sigma},\mu^2)$$

- Again, montonically increasing with $q\Rightarrow$ No inhomogeneous phase

Including different chemical potentials

- GN$_P$ with chiral imbalance

$$S = \int \mathrm{d}^3x \left(N_c \frac{\sigma^2 + \eta_{45}^2}{2\lambda} + \bar{\psi}_a \left(\not{\partial} + \gamma_0 \mu + \gamma_0 \gamma_{45} \mu_{45} + \sigma + \gamma_{45} \eta_{45} \right) \psi_a \right)$$

- Can be transformed to

$$S = \int \mathrm{d}^3x \left(N_c \frac{\phi_L^2 + \phi_R^2}{4\lambda} + \bar{\psi} \left[\not{\partial} + \gamma_0 \left(P_L \mu_L + P_R \mu_R \right) + P_L \phi_L + P_R \phi_R \right] \psi \right)$$

$$P_{L/R} = (\mathbb{1} \pm \gamma_{45}), \quad \mu_{L/R} = (\mu \pm \mu_{45}), \quad \phi_{L/R} = (\sigma \pm \eta)$$

- Decomposition to two independent GN models (Q_2 are in irreducible 2×2 representation)

$$Q[\mu, \mu_{45}, \sigma, \eta_{45}] = \begin{pmatrix} Q_2[\mu_L, \phi_L] & 0 \\ 0 & \tilde{Q}_2[\mu_R, \phi_R] \end{pmatrix}$$

GN$_P$ with chiral imbalance

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$$\Gamma_{\phi_L}^{(2)} = \frac{1}{2\lambda} - \tfrac{1}{2}\ell_1 + \tfrac{1}{2} L_2(\mathbf{q}, \bar{\phi}_L, \mu_L)$$

$$\Gamma_{\phi_R}^{(2)} = \frac{1}{2\lambda} - \tfrac{1}{2}\ell_1 + \tfrac{1}{2} L_2(\mathbf{q}, \bar{\phi}_R, \mu_R)$$

- Setting $\eta_{45} = 0$ yields $\phi_L = \phi_R$, i.e. GN model with chiral imbalance (compare[10])

[10] L. Pannullo, M. Wagner, M. Winstel, *Symmetry* **2022**, *14*, 265.

The full model

- $c_j \in \{\mathbb{1}, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45}\}$ with corresponding scalars ϕ_j

$$S = \int \mathrm{d}^3x \left[N_c \sum_i \frac{\phi_i^2}{2\lambda} + \bar{\psi}_f \left(\partial\!\!\!/ + \gamma_0\mu + \sum_j c_j \phi_j \right) \psi_f \right]$$

- 16×16 matrix in field space has to be diagonalized to compute $S_{\mathrm{eff}}^{(2)}$
- In principle: Possible, but roots of high order polynomials occur $\Rightarrow$ Not solvable
- However, we do not expect an IP to be generated by the interplay of all channels

Conclusions - $2+1$-dimensional FF and Yukawa models

- No inhomogeneous condensation via stability analysis in the renormalized limit
- Strong regularization scheme dependence at finite regulator values
- Inhomogeneous condensates with energy barrier towards homogeneous ground state still possible
  - No evidence found in exemplary model studies with minimization on the lattice

Ongoing studies regarding inhomogeneous phases

- Regularization scheme dependence in $3+1$ dimensions (talk by Laurin Panullo yesterday)
- Scalar lattice field theory - negative wave function renormalization & inhomogeneous order parameters with bosonic fluctuations

Appendix

- Chiral GN model with only η_5

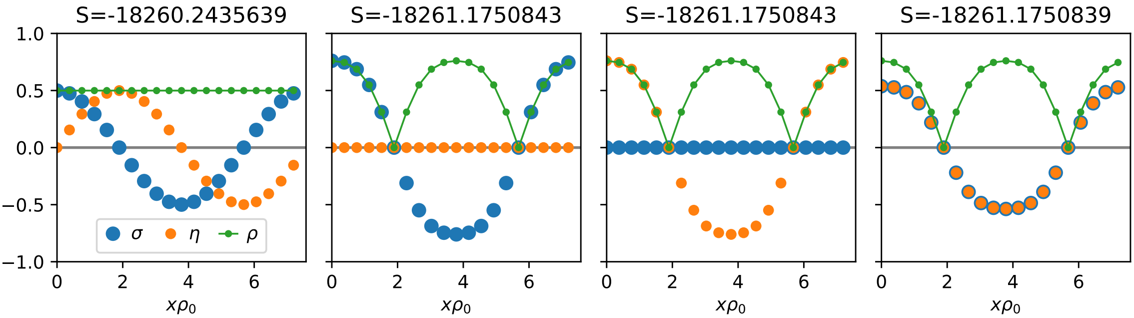
$$S[\bar{\psi}_f, \psi_f, \sigma, \eta] = \int \mathrm{d}^3x \left[\frac{N_f}{2\lambda}(\sigma^2 + \eta_5^2) + \bar{\psi}_f \left(\gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma + \mathrm{i}\gamma_5 \eta_5\right) \psi_f \right],$$

- Invariant under $U(1) \times U_{\gamma_{45}}(1) \times U_{\gamma_5}(1)$

Numerical results of the $2+1$ chiral Gross-Neveu model

- 2D minimization at $(\mu, T)/\rho_0 = (1.00, 0.13)$ with naive fermions
- CDW solution is energetically disfavored
- global minimum is a 1D GN-like oscillation without a phase shift between σ and η
  $\Rightarrow \eta(x)$ can be rotated in $\sigma(x)$ via chiral rotation $\Rightarrow$ model reduces to $2+1$-GN model

$\Rightarrow$ Expect and observe same phase diagram with three different lattice regularizations



Numerical results with different regularizations

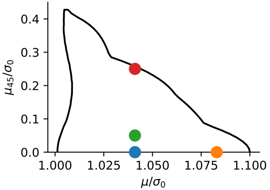
$\Rightarrow$ Expect and observe same phase diagram with three different lattice regularizations

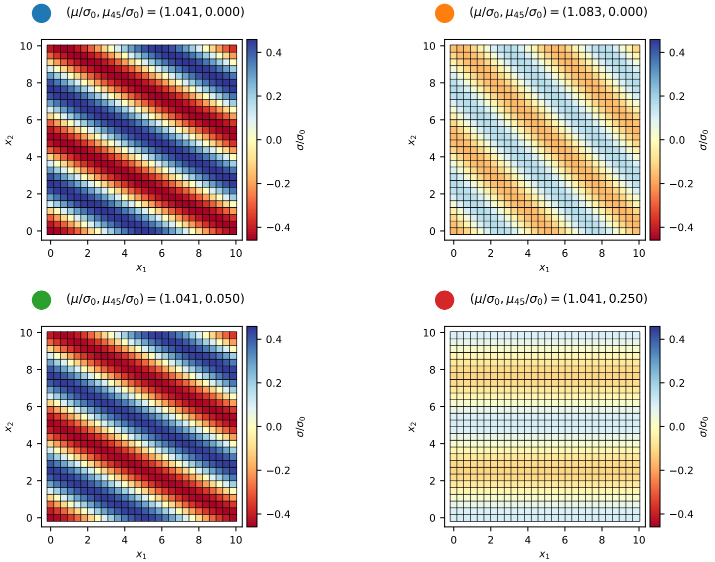
- Naive fermions:
  - Inhomogeneous phase appears as a lattice artifact for one of two different regularizations at finite a
  - For the other regularization: No inhomogeneous phase at same a
  - Difference between both regularizations: Different function in interaction term to suppresses doubler-mixing interactions
- SLAC fermions:
  - No inhomogeneous phase at finite a

- Analogous results to the 2+1 GN model with $\mathbb{Z}_2$ symmetry

Within instability region

$$a\sigma_0 \approx 0.3649,$$
$$L\sigma_0 = 10.22,$$
$$T/\sigma_0 = 0.114$$





Symmetries and chiral imbalance

- Discrete chiral symmetries (4×4 Representation of Euclidean Dirac algebra)

$$\psi_f \to \gamma_4 \psi_f \,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_4 \,, \quad \sigma \to -\sigma,$$
$$\psi_f \to \gamma_5 \psi_f \,, \quad \bar{\psi}_f \to -\bar{\psi}_f \gamma_5 \,, \quad \sigma \to -\sigma$$

- $\gamma_{45} = \mathrm{i}\gamma_4\gamma_5$ generates continuous (chiral) symmetry
- Dirac-Operator is block-diagonal

$$Q[\mu, \mu_{45}, \sigma] = \begin{pmatrix} Q^{(2,+)}[\mu + \mu_{45}, \sigma] & 0 \\ 0 & Q^{(2,-)}[\mu - \mu_{45}, \sigma] \end{pmatrix}$$

- Dirac operators build from irreducible fermion representation

$$Q^{(2,\pm)}[\mu, \sigma] = \pm\tau_2(\partial_0 + \mu) \pm \tau_3\partial_1 \pm \tau_1\partial_2 + \sigma$$

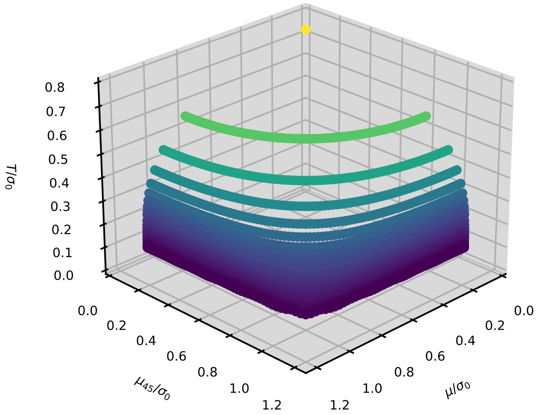
- $\mu \neq 0, 0 \leqslant \mu_{45} \leqslant \mu$ increases chiral imbalance, i.e. difference between $\mu_L = \mu + \mu_{45}$ for upper 2 comp. and $\mu_R = \mu - \mu_{45}$ for lower 2 comp.
- What are the effects on the respective (in-)homogeneous phases?

$\Rightarrow$ Study with two different lattice regularizations using naive fermions and different coupling to σ

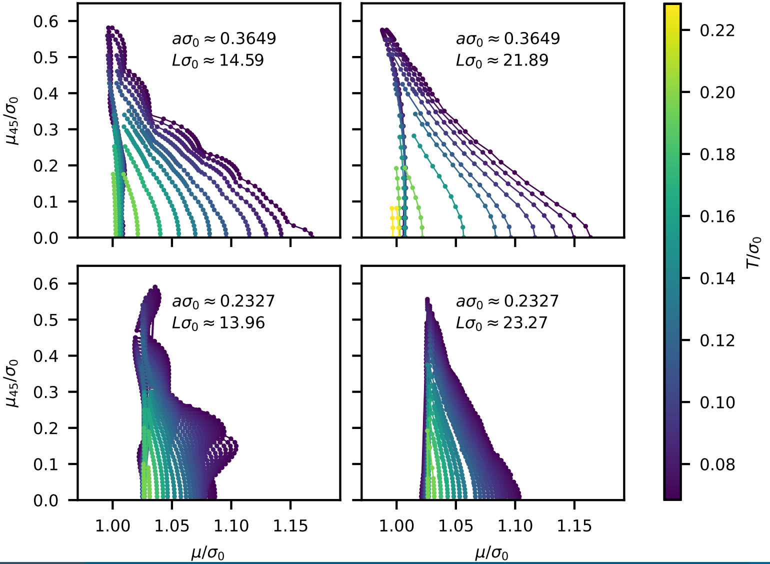
Homogeneous phase diagram, $\sigma(\mathbf{x}) = \text{const.}$

- $\sigma(\mathbf{x}) = \bar{\sigma} = \text{const.}$, Minimization of lattice action, identical for both discretization
- Theoretically observed symmetry $\mu_{45} \leftrightarrow \mu$ & $\mu \to -\mu$ & $\mu_{45} \to -\mu_{45}$

$$a\sigma_0 = 0.2327, L\sigma_0 = 27.92$$



Stability analysis on multiple lattices



Minimization with respect to $\sigma = \sigma(\mathbf{x})$

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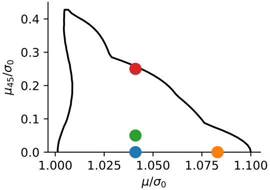

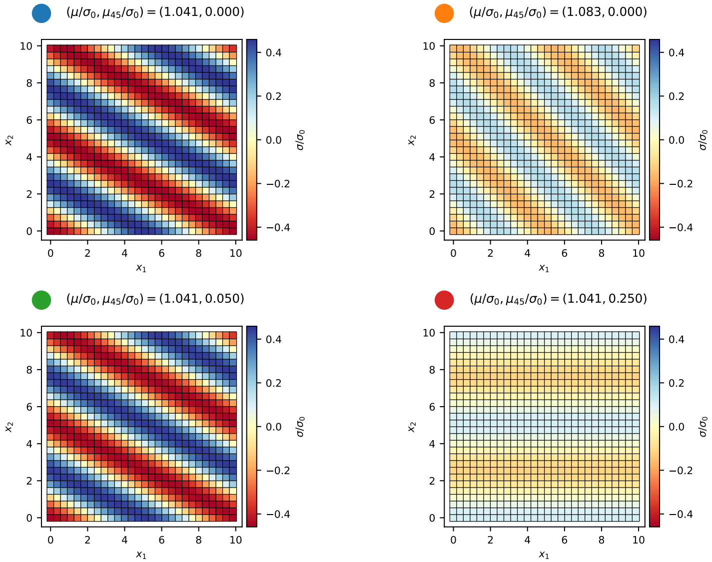

Chiral transformations

- Symmetries of one free massless spinor in 2+1 dimensions (Generalization to N_c via rotations in this diagonal space)

$$\psi_f \to e^{i\theta\Gamma}\psi_f \qquad \Gamma \in \{\mathbb{1}, \gamma_{45}, \gamma_4, \gamma_5\}$$

- For the Gross-Neveu model only a subgroup is realized

$$\psi_f \to \gamma_5\psi_f,\ \bar{\psi}_f \to -\bar{\psi}_f\gamma_5 \tag{2}$$

$$\psi_f \to \gamma_4\psi_f,\ \bar{\psi} \to -\bar{\psi}_f\gamma_4 \tag{3}$$

- Together with this discrete transformation we have continuous symmetries

$$\psi_f \to e^{i\phi\gamma_{45}}\psi_f,\ \bar{\psi}_f \to \bar{\psi}_f e^{-i\phi\gamma_{45}} \tag{4}$$

$$\psi_f \to e^{i\alpha}\psi_f,\ \bar{\psi}_f \to \bar{\psi}_f e^{-i\alpha} \tag{5}$$

- Combination of (4) with (2) reproduces (3)

Symmetries of NJL

$$S = \int \mathrm{d}^3x \left(N_c \frac{\sigma^2 + \vec{\pi}_4^2 + \vec{\pi}_5^2}{2\lambda} + \bar{\psi}_{a,f} \left(\not{\partial} + \gamma_0 \mu + \sigma + + \mathrm{i}\gamma_4 \vec{\tau}_{fg} \, \vec{\pi}_4 + \mathrm{i}\gamma_5 \vec{\tau}_{fg} \, \vec{\pi}_5 \right) \psi_{a,g} \right)$$

- Invariant under $U(1) \times U_{\gamma_{45}}(1) \times SU_L(2) \times SU_R(2)$
- Corresponding projectors can be defined either as $P^5_{L/R} = \frac{1}{\sqrt{2}} \left(1 \pm \gamma_5\right)$ or as $P^4_{L/R} = \frac{1}{\sqrt{2}} \left(1 \pm \gamma_4\right)$
- Left- and right-handed fermions can be rotated independently