Novel Bottomonium Spectral Results with Backus-Gilbert

Ben Page
Swansea University

and

Chris Allton (Swansea U.), Seyong Kim (Sejong U.)

FASTSUM

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Outline:

1. Spectral Reconstruction in Non-Relativistic (NR)QCD
2. The Backus-Gilbert Method
3. Laplace Shifting
4. Conditioning and Studies into the Covariance Matrix

Spectral Reconstruction in NRQCD

The spectral representation of the Euclidean correlation function $G_\Gamma(\tau, \boldsymbol{x}) = \langle O_\Gamma(\tau, \boldsymbol{x}) O_\Gamma(0, \mathbf{0}) \rangle$ is

$$G(\tau; T) = \int_0^\infty \rho(\omega; T) K(\omega, \tau) d\omega \tag{1}$$

where $K(\omega, \tau; T) = \cosh[\omega(\tau - 1/2T)] / \sinh(\omega/2T) \to e^{-\omega\tau}$ when $M \gg T$.

Aim: Invert the Laplace transform

$$G(\tau; T) = \int_0^\infty \rho(\omega; T) e^{-\omega\tau} \tag{2}$$

$\Longrightarrow$ This is ill-posed!

One method: Backus-Gilbert[1]

[1] Backus, Gilbert 1968, *Geophysical Journal International*

Backus-Gilbert: In a Nutshell

Construct sampling functions $A(\omega, \omega_0) = \sum_\tau c_\tau(\omega_0) K(\omega, \tau)$ which sample $\rho(\omega)$ at a particular value ω_0:

$$\begin{aligned} \hat{\rho}(\omega_0) &= \sum_\tau c_\tau(\omega_0) G(\tau) \\ &= \int A(\omega, \omega_0) \rho(\omega) d\omega \end{aligned} \tag{3}$$

$c_\tau(\omega_0)$ chosen to minimise the 'width' of $A(\omega, \omega_0)$ (i.e. maximise resolving power).
Our choice of width:[2]

$$J(\omega_0) = \int \left[A(\omega, \omega_0) - \delta(\omega - \omega_0)\right]^2 d\omega \tag{4}$$

[2] Oldenburg 1984, IEEE GE-22 vol. 6

Backus-Gilbert: In a Nutshell

Minimising $J(\omega_0)$ requires inversion of a near-singular matrix $\mathcal{K}$:

$$\mathcal{K}_{\tau\tau'} = \int K(\omega,\tau)K(\omega,\tau')\;d\omega = \int e^{-\omega(\tau+\tau')}\;d\omega \tag{5}$$

but $0 \leq \tau, \tau' < 128$ for our lattices $\Rightarrow$ This requires conditioning.

Our approach:

$$\mathcal{K}(\alpha) = \mathcal{K} + \alpha I \tag{6}$$

previously used[3]:

$$\mathcal{K}(\alpha) = \alpha\mathcal{K} + (1-\alpha)\text{Cov}[G] \tag{7}$$

[3] FASTSUM 2021, 2112.02075, 2112.04201

Laplace-Shifting

- The functions $A(\omega, \omega_0)$ replicate $\delta(\omega - \omega_0)$ *imperfectly* – $A(\omega, \omega_0)$ narrower for small ω_0
- $A(\omega, \omega_0)$ generated in absence of $G(\tau)$

Idea: Use Laplace shift transform to 'increase' resolution about a feature[4]:

$$e^{\Delta \cdot \tau} G(\tau) \xrightarrow{\mathcal{L}} \rho(\omega - \Delta) \tag{8}$$

Drawback: Statistical errors amplified

[4] FASTSUM 2021, 2112.02075, 2112.04201

Laplace-Shifting: An example

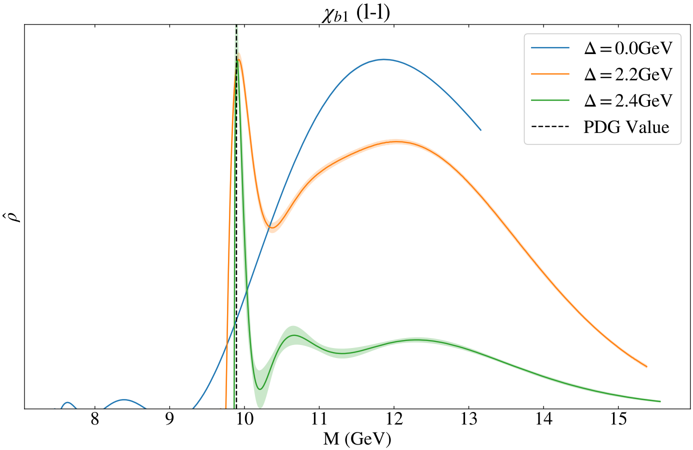
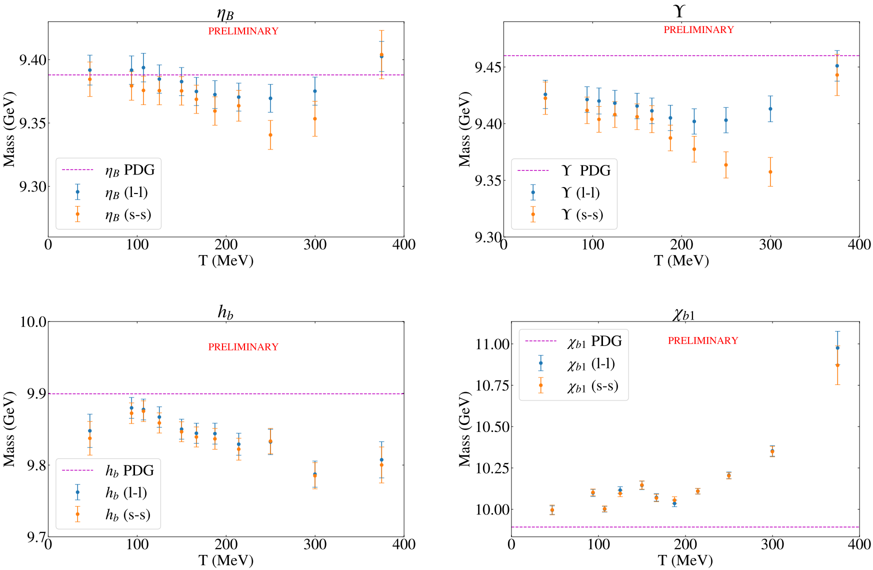


Figure: Laplace-shift in action using correlation functions for the χ_{b1}.

Results from the lattice - $b\bar{b}$ sector



Results from the lattice – Υ

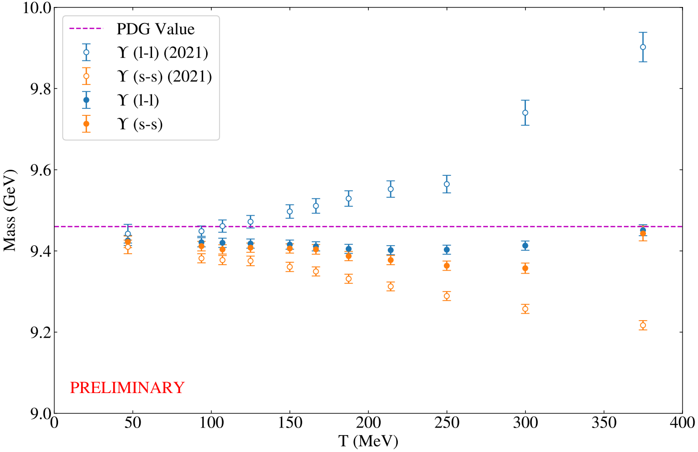


Figure: Υ mass comparison for local and smeared sources, 2021 v. 2022

Results from the lattice – Υ

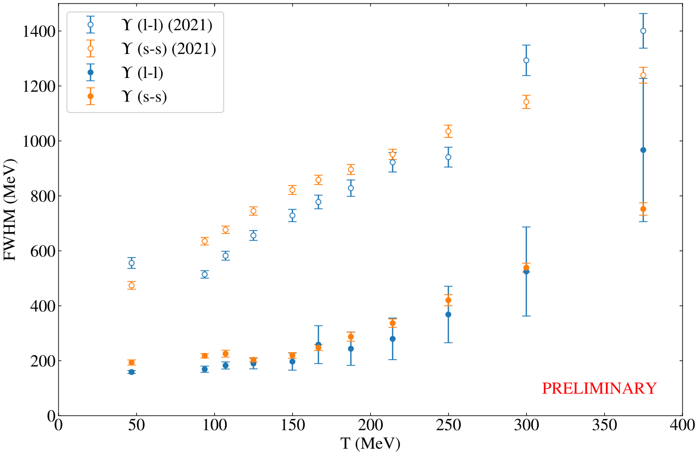


Figure: Υ FWHM comparison for local (l-l) and smeared (s-s) sources

Results from the lattice - Υ

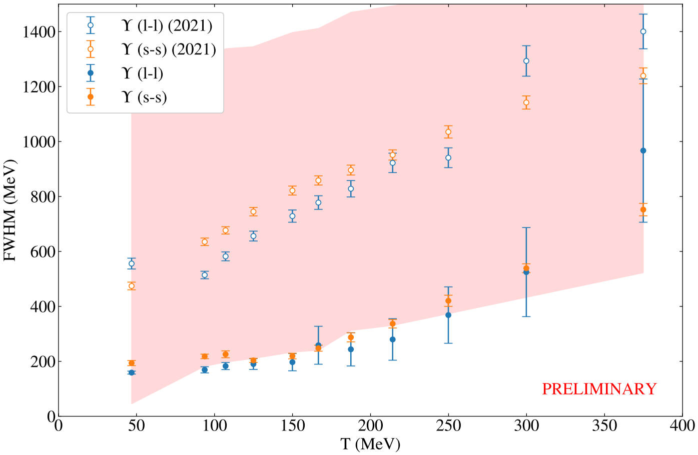


Figure: Υ FWHM comparison with resolution band

Studies into the covariance matrix

- $c_\tau(\omega_0)$ calculation requires inversion of an ill-conditioned matrix $\mathcal{K}$
- $\mathcal{K}$ may also be conditioned via Tikhonov-like approach with the covariance matrix Σ:

$$\mathcal{K}(\alpha) = \alpha\mathcal{K} + (1-\alpha)\mathrm{Cov}[G] \tag{9}$$

Question:

$\Longrightarrow$ How does the condition of $\mathrm{Cov}[G]$ change with Δ?

Studies into the covariance matrix

- Cov$[G]$ transforms as $e^{\Delta \cdot \tau}$Cov$[G]_{\tau\tau'} e^{\Delta \cdot \tau'}$ under Laplace shift
- For τ, τ' large, expect Cov$[G]$ scales as

$$\text{Cov}[G]_{\tau\tau'} \sim G(\tau) \cdot G(\tau') \sim e^{-M_{\text{ground}}(\tau+\tau')} \tag{10}$$

where M_{ground} is the mass of the ground state.

Principle:

$\implies$ When $\Delta = M_{\text{ground}}$, Cov$[G]_{\tau\tau'} \sim \mathcal{O}(1)$ and cond(Cov$[G]$) $\to \infty$

Studies into the covariance matrix

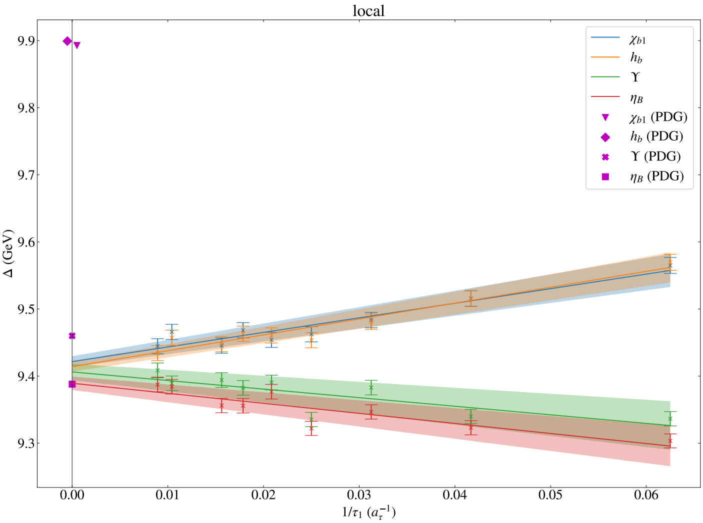


Figure: Covariance mass prediction from local sources

Studies into the covariance matrix

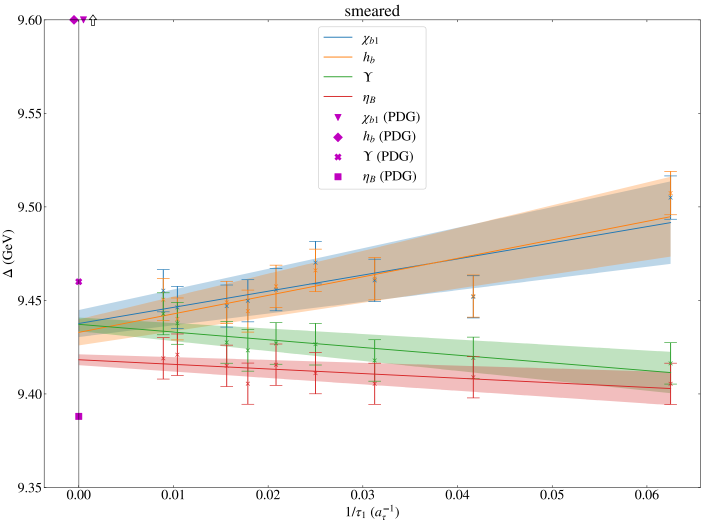


Figure: Covariance mass prediction from smeared sources

Studies into the covariance matrix

In general, $\Delta \neq M_{\text{ground}}$ but instead $\Delta \leq M_{\text{ground}}$

$\implies$ Parisi-Lepage scaling[5]

e.g. for Υ, Cov$[G]$ includes $\Upsilon\Upsilon$ correlation terms – conservation of angular momentum allows construction of $\eta_b\eta_b$ terms which scale with mass $M_{\eta_B} < M_{\Upsilon}$.

$\implies$ We can probe M_{η_b} from S/P-wave vector/pseudoscalar quantum numbers!

$\implies$ Laplace shift limited by smallest sector mass

[5] Parisi 1983, Lepage 1989

Conclusions and closing comments

- New results from Tikhonov-conditioned Backus-Gilbert for multiple heavy sector meson channels
- Laplace shifting gives improved results when paired with Backus-Gilbert
- Change in resolving power shows systematic uncertainties are still not controlled
- Studies into Laplace shifting Cov$[G]$ support Parisi-Lepage scaling for heavy sector
- Question: Does Laplace shifting offer improvements for other recon. techniques?

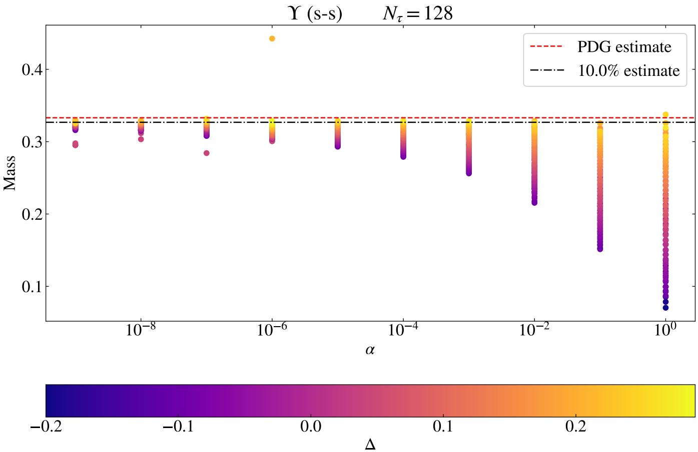
Thank you for your time.

Questions are welcome!

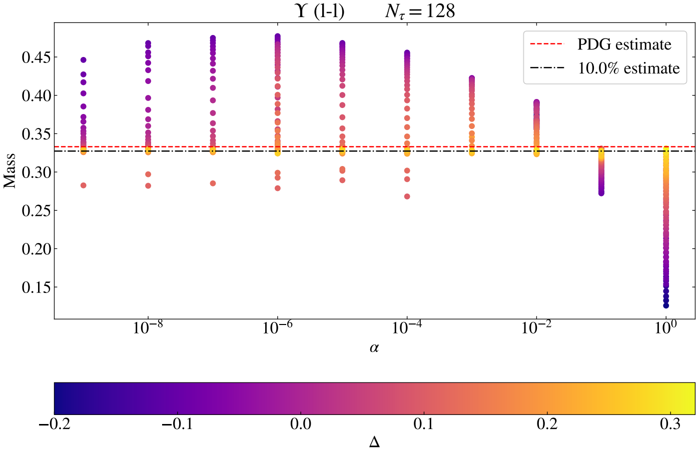
or contact me via:
Email: 900727@swansea.ac.uk

Technical Plots Overleaf

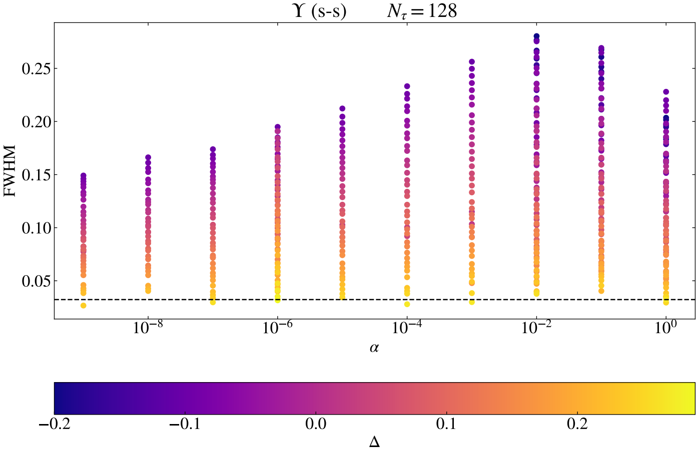
Fitting the mass



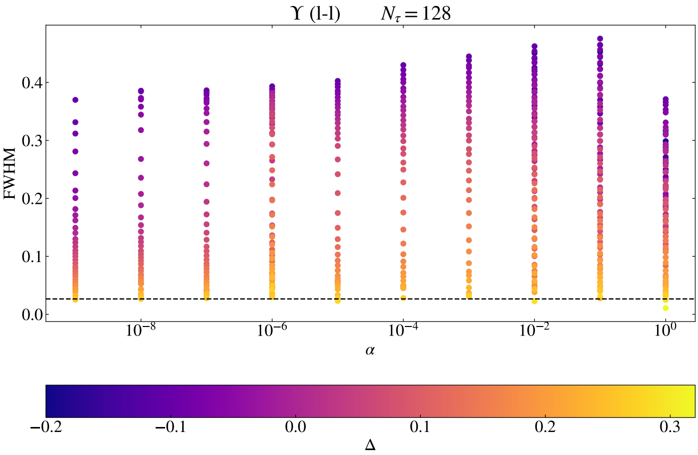
Fitting the mass



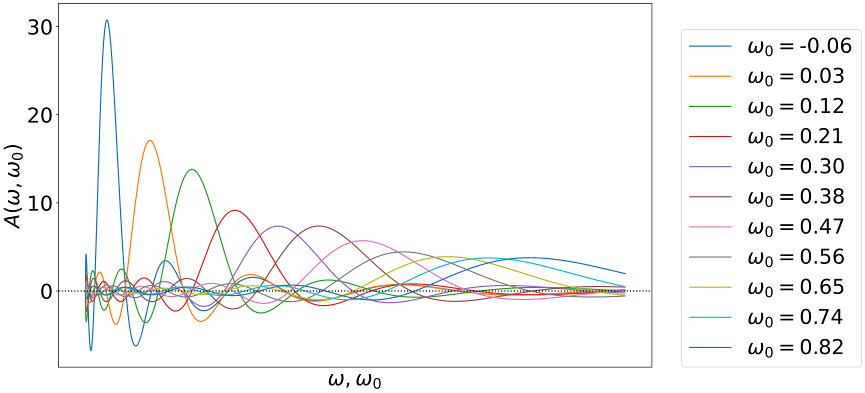
Fitting the width



Fitting the width



Averaging functions



χ_{b1} widths