Non-perturbative determination of couplings in Polyakov-loop effective theories

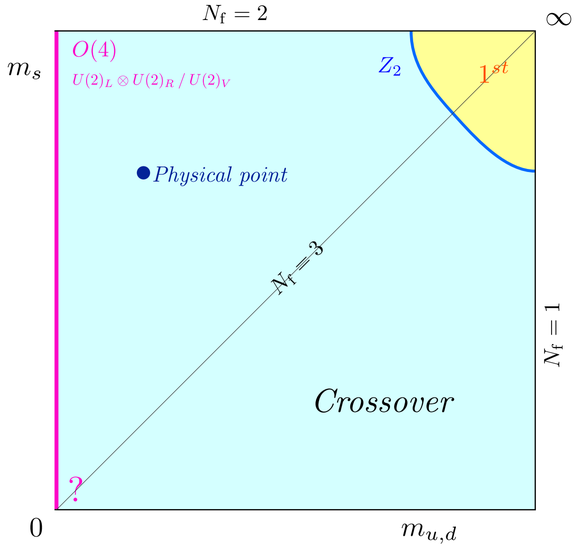
O. Philipsen, J. Scheunert, C. Winterowd

Outline

1. Motivation
2. Polyakov loop effective theories
3. Determination of couplings
4. Addition of heavy quarks
5. Finite-cluster method
6. Conclusion

Motivation

- Want to understand the phase-diagram of QCD
- The sign problem inhibits progress with direct simulations at nonzero μ
- Thimbles, Langevin, etc



Cuteri et al., JHEP (2021)

Alternatively…

- 3D effective theories derived in strong-coupling
- Milder sign problem at nonzero baryon chemical potential (many d.o.f. integrated out)
- Amenable to both analytical as well as numerical approaches
- BUT: Addition of light quarks hard and large number of effective couplings!

Background

- Direct application of strong-coupling expansion for free energy

$$S_W = \frac{\beta}{2} \sum_{\mathrm{p}} \Re \mathrm{Tr}\ U_p$$

$$Z = \int [dU][d\psi d\bar{\psi}] e^{-(S_G+S_F)} = \int [dU_0][dU_i] \det Q\ e^{-S_G}$$

$$S_F = \sum_n \left\{ \bar{\psi}(n)\psi(n) - \sum_\mu \left(\bar{\psi}(n)\kappa(1-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \bar{\psi}(n)(1+\gamma_\mu)U_\mu(n)\psi(n) \right) \right\}$$

- Resummation of gauge action accomplished by character expansion of gauge action

$$e^{-S_G} = c_0^{N_p} \prod_p \left(1 + \sum_{\mathbf{r}} d_{\mathbf{r}} a_{\mathbf{r}}(\beta) \chi_{\mathbf{r}}(U_p) \right)$$

$$c_{\mathbf{r}} = \int dU \chi^*_{\mathbf{r}}(U) e^{-S_G(U)}$$

expansion coefficients

Langelage, Münster, Philipsen, JHEP (2008)

Langelage and Philipsen, JHEP (2010)

- Subtraction of zero-temperature ($N_\tau \to \infty$) graphs
- Accurate determination of deconfinement transition in $SU(2)$ and $SU(3)$ theory

CRC-TR 211

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Polyakov Loop Effective Theories

- Originally both spatial and temporal gauge links integrated out

$$Z = c_0^{N_p} \sum_G \Phi(G)$$

where

$$\Phi(G) = \int [dU] \prod_{p \in G} d_{\mathbf{r}_p} a_{\mathbf{r}_p} \chi_{\mathbf{r}_p}(U) = \prod_i \Phi(X_i)$$

disjoint "polymers"

- Apply moment-cumulant formalism when computing the free energy: $f = -\frac{1}{V} \log Z$

- Integrate over just spatial links and obtain a dimensionally-reduced effective theory solely in terms of Polyakov loops

$$-S_{\text{eff}} = \ln \int [dU_i] \prod_p \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right]$$

$$\boxed{\lambda_1 S_1 + \lambda_2 S_2 + \ldots}$$

effective couplings depend on β and N_τ!

Polyakov Loop Effective Theories

- Effective couplings λ_i represent couplings between Polyakov loops in the effective theory

$$S_1 = \lambda_1 \sum_{\langle \mathbf{x},\mathbf{y} \rangle} \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.} \right)$$

nearest-neighbors coupling

$$S_2 = \lambda_2 \sum_{[\mathbf{x},\mathbf{y}]} \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.} \right)$$

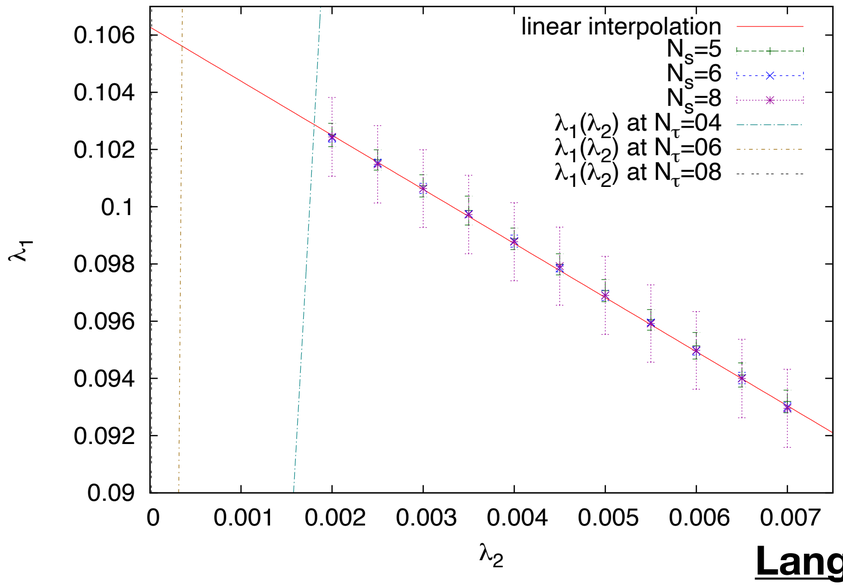
next-nearest-neighbors coupling

resummation!

$$S_1 \to \sum_{\langle \mathbf{x},\mathbf{y} \rangle} \log \left(1 + \lambda_1 \left(L_{f,\mathbf{x}} L^*_{f,\mathbf{y}} + \text{c.c.} \right) \right)$$

log-action

- Simulations of two-coupling model give good agreement with YM simulations



Map back to β_c for fixed N_τ using strong-coupling expressions for couplings

$$\lambda_1 = u^{N_\tau} e^{N_\tau P(u;N_\tau)}$$

$$\lambda_2 = N_\tau \left(N_\tau - 1 \right) u^{2N_\tau + 2}$$

- Effects of longer-range interactions and interactions of "higher" representations are suppressed (small β)
- Near β_c, more couplings become important!

Langelage et al., JHEP 2010

Including heavy quarks

- Terms generated by gauge action invariant under global $Z(3)$ center symmetry

$$S_{\text{symm}} = \sum_{\mathbf{x},\mathbf{r}} \sum_{n} \sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}' c^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_{i}^{n} \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i))$$

terms $Z(3)$ and cubic symmetry

- Introduction of quarks explicitly breaks this symmetry

$$S_{\text{eff}} = S_{\text{symm}} + S'$$

explicit breaking terms

$$S' = \sum_{f=1}^{N_f} \sum_{i} S'_i[W, W^\dagger]$$

- Hopping parameter expansion of quark determinant

$$\det Q = \det Q_{\text{stat}} \det Q_{\text{kin}}$$

"static" **"kinetic"**

$$\det Q_{\text{kin}} = e^{\text{Tr} \log Q_{\text{kin}}}$$

expanded as series in κ^2

- One hopes that low-orders are sufficient

CRC-TR 211

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Determining couplings non-perturbatively

- Effective action can be expanded and powers of characters at given site can be reexpressed as linear combination of characters

in practice, number of terms truncated

$$e^{-S_{\text{symm}}} = \tilde{\mathcal{N}} \left(1 + \sum_{\mathbf{x},\mathbf{r}} \sum_{n} \sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}' \tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i)) \right)$$

no correlation at distances larger than largest separation of terms in effective action

- Better representation which includes long-range correlations is log action

$$e^{-S_{\text{symm}}} = \mathcal{N}_0 \prod_{\mathbf{x},\mathbf{r},n} \prod_{\{\mathbf{r}_i,\mathbf{x}_i\}}' \left[1 + \lambda^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \left(\chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i)) + \text{c.c.} \right) \right]$$

Bergner et al., JHEP 2015

- Observable calculated in full effective theory (no truncation) should match full QCD

$$\tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \propto \langle \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i)) \rangle_{\text{eff}} = \langle \chi_{\mathbf{r}}(W(\mathbf{x})) \prod_i^n \chi_{\mathbf{r}_i}(W(\mathbf{x}+\mathbf{x}_i)) \rangle_{\text{QCD}}$$

- Express correlators in QCD as a perturbative series in couplings of log-action

"Inverse" Monte Carlo method **Wozar et al., PRD (2007)**

Determining couplings non-perturbatively

- Expansion guided by knowledge of couplings from strong-coupling
- Expressions for correlators can be inverted to obtain $\lambda_i(\beta, \kappa, N_\tau), h_i(\beta, \kappa, N_\tau)$

$$\lambda_3 \equiv \lambda^f_{(2,0,0),\bar{f}} \propto u^{2N_\tau+6}$$

$$\lambda_4 \equiv \lambda^f_{(1,1,1),\bar{f}} \propto u^{3N_\tau+4}$$

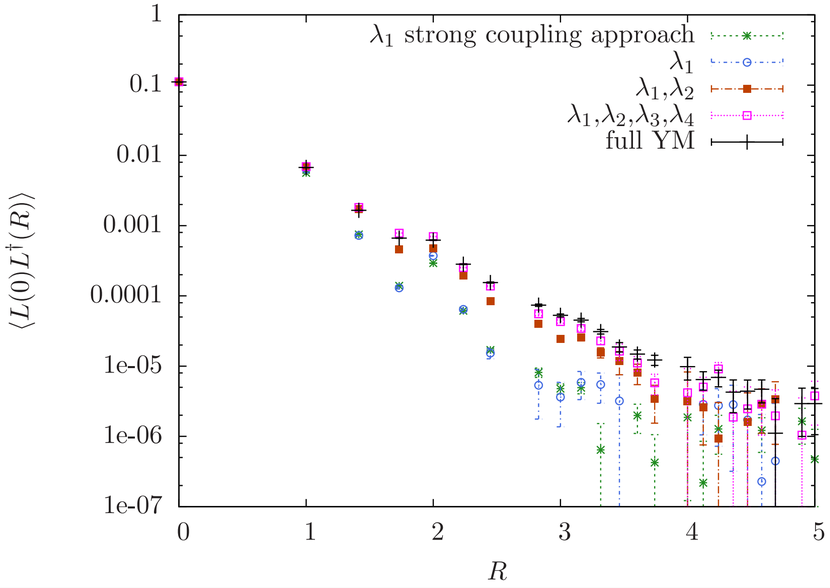
$$\lambda_5 \equiv \lambda^f_{(3,0,0),\bar{f}} \propto u^{3N_\tau+4}$$

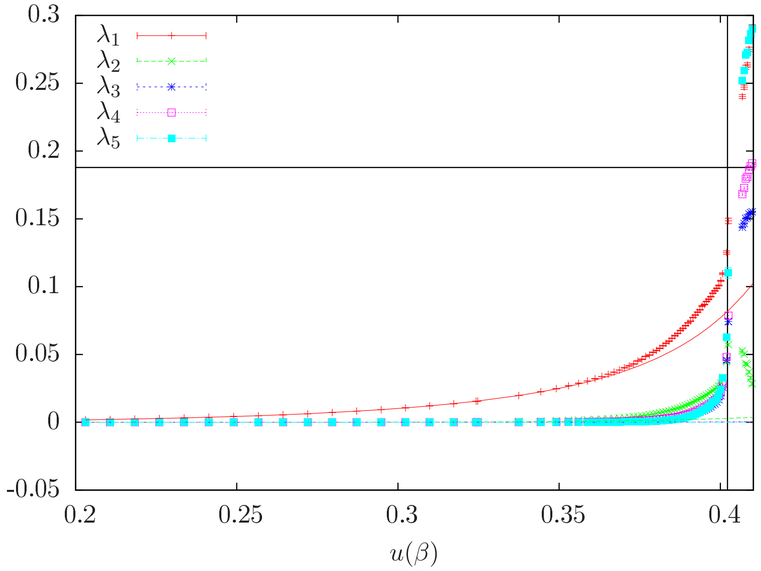
$$\lambda_{\text{adj}} = \lambda^{(1,1)}_{(1,0,0),(1,1)} \propto v^{N_\tau} \propto u^{2N_\tau}$$

$$\lambda_{\text{sextet}} = \lambda^{(2,0)}_{(1,0,0),(0,2)} \propto w^{N_\tau} \propto u^{2N_\tau}$$

$$\tilde{\lambda}_a = \sum_{\{n_i\}} \sum_{\{m_i,\bar{m}_i\}} c^{(a)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}$$

$$\tilde{h}_a = \sum_{\{n_i\}} \sum_{\{m_i,\bar{m}_i\}} d^{(a)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}$$





Bergner et al., JHEP 2015

couplings become large at β_c

Pure YM, $N_\tau = 4,6$; good agreement

Evaluation of log-action

- Given a set of couplings $\{\lambda_i, h_i\}$, how to efficiently evaluate Z (and it's derivatives)?

$$\tilde{Z}(G) = \frac{1}{Z_0(G)} \int \prod_{v \in V(G)} dL_v \det Q_{\text{stat},v} \prod_{i=\{\text{NN},\ldots 5\text{NN}\}} \prod_{l \in E_i(G)} \prod_{\mathbf{r}(l)} \left[1 + \lambda_{i'} (L_{\mathbf{r}(l),v_1(l)} L_{\bar{\mathbf{r}}(l),v_2(l)} + \text{c.c.})\right] \prod_j \Delta_i^{(j)}(l,\kappa)$$

$$\tilde{Z}(G) = 1 + \sum_{g \in G \setminus \emptyset} \tilde{\phi}(g)$$

cast as sum over subgraphs

- Give a subgraph g, weight can be determined by performing site integrals

$$I_{n,m} = \int dL\, L^n\, (L^*)^m$$

$$\tilde{\phi}(g) = \frac{1}{z_0^{|V(g)|}} \int \prod_{v \in V(g)} dL_v \det Q_{\text{stat},v} \prod_{l \in E(g)} \prod_{\text{r}(l)} \lambda_i(l) \left(L_{\mathbf{r}(l),v_1(l)} L_{\bar{\mathbf{r}}(l),v_2(l)} + \text{c.c.}\right) \prod_j \Delta_i^{(j)}(l,\kappa)$$

terms generated by hopping expansion

$$z_0 = \int dL \det Q_{\text{stat}}[L, L*]$$

- Brute-force evaluation on thermodynamically large system difficult (disconnected graphs)

Finite-cluster method

- Each graph weight $\tilde{\phi}(g)$ depends only on the topology of g and NOT the underlying lattice

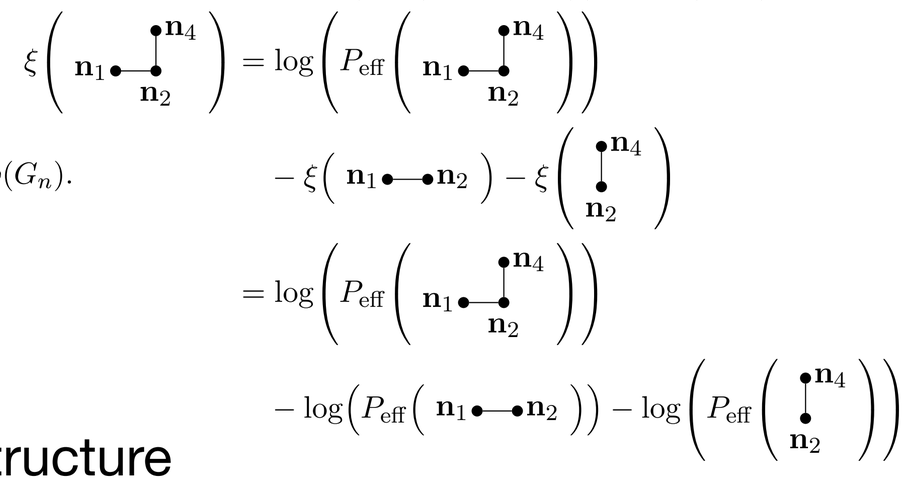
- Derivation of effective action or evaluation of $\log Z$ could have worked on arbitrary embedding graph

$$\xi(G) = \log \tilde{Z}(G) - \sum_{g \in \mathcal{G}_c(G) \setminus G} \xi(g)$$

Scheunert, 2021

$$-S_{\text{eff}} = \log \det Q_{\text{stat}} + \log \left[1 + \sum_{G \in \mathcal{G}(G_{\Lambda_s})} \phi(g) \right]$$

$$P_{\text{eff}}(\mathcal{G}_c(G_{\Lambda_s})) := 1 + \sum_{n=1}^{|\mathcal{G}_c(G_{\Lambda_s})|} \sum_{\{G_1,\ldots,G_n\} \in \mathcal{D}_n(\mathcal{G}_c(G_{\Lambda_s}))} \varphi(G_1) \cdots \varphi(G_n).$$



- Direct evaluation of weights on small clusters

- Avoids embedding of disconnected graphs and preserves log-structure

- Ideal for evaluation of series expansion for correlators in the effective theory

Finite-cluster method

- Direct application to $\log Z$ for effective theory

$$\frac{\log \tilde{Z}}{V} = \sum_{l=1}^{N_{\max,\mathrm{MD}}} \sum_{g\in\{\mathcal{G}_c(l)\}} \sum_{p\in\mathcal{P}} \frac{W(G;p)}{S(G)} \xi(G^{(p)}_{\Lambda_s})$$

- Completely generalizable to arb. reps and n-point correlation functions

$$\xi(G) \to \xi(G; J_{\mathbf{r},\mathbf{x}})$$

$$\boxed{\xi^{(n)} \equiv \frac{\partial^n \xi(G)}{\partial J_{\mathbf{r},\mathbf{x}} \ldots \partial J_{\mathbf{r}_{n-1},\mathbf{x}_{n-1}}}\Big|_{J=0}}$$

introduces rooted/colored vertices at canonical positions

$$\langle L_{\mathbf{r}}(\mathbf{x}) \ldots L_{\mathbf{r}_{n-1}}(\mathbf{x}_{n-1})\rangle = \sum_{l=1}^{N_{\max,\mathrm{MD}}} \sum_{g\in\{\mathcal{G}_{c,n}(l)\}} \sum_{p\in\mathcal{P}} \frac{W^{(n)}(G;p)}{S(G)} \xi^{(n)}(G^{(p,n)}_{\Lambda_s})$$

- Main cost in computing generalized weak embedding numbers of colored graphs
- Publically available software for graph isomorphism (canoncalization) problem: Nauty
- Incorporation of higher-order in κ^2 terms relatively straightforward

modification of $\tilde{\phi}(g)$

CRC-TR 211

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Conclusion and Outlook

- Matching between correlators in QCD and PEFT
- Can in principle do better then strong-couplings expressions for couplings
- Efficient method for calculating series expression for arbitrary correlators in PEFT
- $N_f = 2$ dynamical Wilson simulations in order to obtain both gauge and fermion couplings
- Nonzero μ dependence of fermion couplings known analytically
- Ultimate goal: exploring chiral region with PEFT

Backup

$O\left(\kappa^2\right)$ **contribution to log-action**

$$\prod_{\langle \mathbf{n},\mathbf{m}\rangle}\left(1+2\frac{\kappa^2 N_\tau}{N_c}(W_{1100}(\mathbf{n})-W_{0011}(\mathbf{n}))(W_{1100}(\mathbf{n})-W_{0011}(\mathbf{n}))\right)$$

$$W^{(f)}_{n_1 m_1 n_2 m_2}(\mathbf{n}) := \mathrm{tr}\left(\frac{\left(h_1^{(f)}W(\mathbf{n})\right)^{m_1}}{\left(1+h_1^{(f)}W(\mathbf{n})\right)^{n_1}}\frac{\left(\bar{h}_1^{(f)}W(\mathbf{n})^\dagger\right)^{m_2}}{\left(1+\bar{h}_1^{(f)}W(\mathbf{n})^\dagger\right)^{n_2}}\right).$$