Chiral Symmetry Breaking in QED induced by an External Magnetic Field

D. K. Sinclair and J. B. Kogut

- Introduction
- Simulations and Results
- Discussion and Conclusions
- Appendix: Lattice QED in an external Magnetic Field

Introduction

We simulate lattice QED in an external magnetic field using the methods developed for lattice QCD (RHMC) to study non-perturbative effects, in particular, chiral symmetry breaking. (See my talk at Lattice 2021.)

We restrict ourselves to considering a constant (in space and time) magnetic field B in the z (3) direction.

Classically electrons in a magnetic field traverse helical orbits around magnetic field lines. The motion parallel to the magnetic field is free while that orthogonal to the magnetic field is bound (circular).

Quantum mechanics restricts the motion perpendicular to the magnetic field to a discrete set of transverse energy levels – the Landau levels.

As B increases the radii of such orbits decreases and their spacing increases until all electrons occupy the lowest level, effectively leading to an effective reduction from $3+1$ to $1+1$

dimensions.

Without external fields QED has a $U(1) \times U(1)$ chiral symmetry despite the fact that $U(1)_{axial}$ is anomalous, because 4-dimensional QED does not have instantons.

It is unclear whether the addition of an external magnetic field breaks this symmetry explicitly.

(Free electrons in an external magnetic field preserve chiral symmetry.)

Approximate calculations using Schwinger-Dyson equations predict that QED in an external magnetic field breaks chiral symmetry giving the electrons a dynamical mass $\propto \sqrt{|eB|}$ and a chiral condensate $\propto |eB|^{3/2}$ [see for example the review article of Miransky and Shovkovy and references].

We present preliminary evidence that QED in an external magnetic field does exhibit such dynamical chiral symmetry breaking, based on our lattice QED simulations.

Simulations and Results

Our first QED simulations were performed on 36^4 lattices with $m = 0.1$ and $m = 0.2$ with $\alpha = 1/137$, for a set of B values in the range $0 \leq |eB| \leq 2\pi \times 140/36^2 = 0.6787...$, which covers the range over which we can expect reasonable agreement with continuum results.

At each m and B we ran for 12500 trajectories storing a configuration every 100 trajectories for further analysis.

The expected value of the dynamical electron mass at $\alpha = 1/137$ is much smaller than the smallest lattice electron mass we are able to simulate:

$$m_{dynamical} \sim 10^{-35}\sqrt{2|eB|}$$

and thus its influence on $\langle\bar{\psi}\psi\rangle$ will be too small to measure.

We are therefore simulating at a stronger coupling $\alpha = 1/5$, which is still in the range where the $eB = 0$ theory is expected to be perturbative. Here, we expect:

$$m_{dynamical} \sim 10^{-4}\sqrt{2|eB|}$$

which should be manageable.

First we consider $eB = 0$ where chiral symmetry should be unbroken for $m \to 0$, $\langle\bar{\psi}\psi\rangle \propto m\Lambda^2$ up to log terms. This means that the chiral condensate is controlled by the ultra-violet and should be insensitive to the finite lattice-size. So we should get good results, even if mL is not much greater than 1 as would normally be required.

Figure 1 shows the chiral condensate at $\alpha = 1/5$, $eB = 0$ for a range of mass values $0.001 \leq m \leq 0.2$ on a 36^4 lattice. At $m = 0.001$ we also include the results from a 48^4 lattice. As expected, there is almost no lattice size dependence. The condensate does appear to vanish in the $m \to 0$ limit.

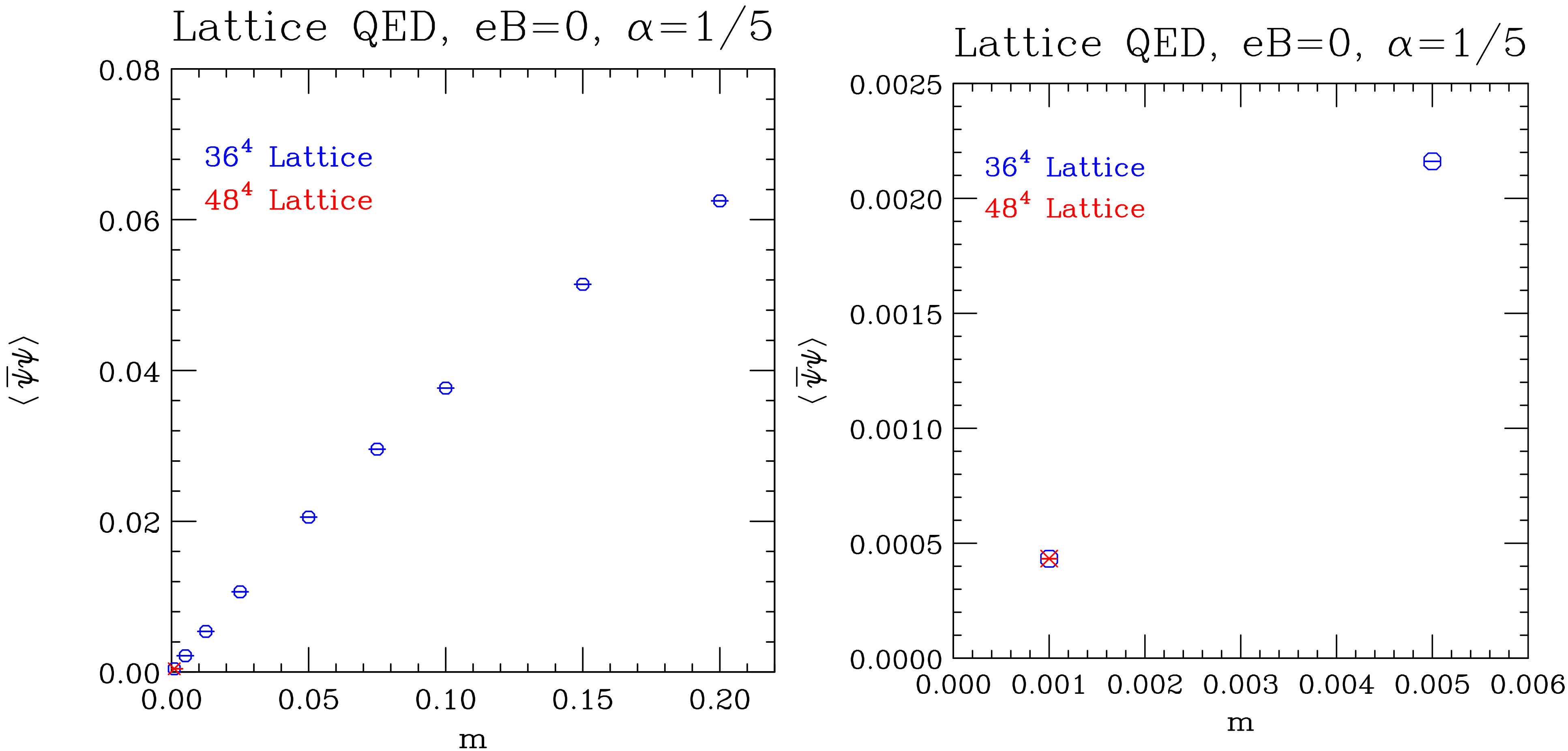

Figure 1: $\langle\bar{\psi}\psi\rangle$ as a function of mass at $eB = 0$, showing lattice size dependence.

Next we consider $eB \neq 0$ in particular we choose $eB = 2\pi \times 100/36^2 = 0.4848...$, relatively large, but not too close to 0.63, much above which the lattice is not expected to give reliable results.

Here we expect chiral symmetry to be broken and $\langle\bar{\psi}\psi\rangle$ to remain finite as $m \to 0$. When this happens, the chiral condensate should be dominated by the infrared or the dynamical mass scale and be sensitive to the lattice size. However, since the electrons are restricted to the lowest Landau level, this already restricts their domain in the x, y plane so the size sensitivity only applies to z and t directions. Thus we fix the lattice extent in the x and y directions, and measure the finite size dependence on that in the z and t directions.

Figure 2 shows the mass dependence of $\langle\bar{\psi}\psi\rangle$ on the input electron mass m on 36^4, $36^2 \times 64^2$ and $36^2 \times 96^2$ lattices. We note that for $m \geq 0.025$ a 36^4 lattice is adequate. For $m = 0.0125$, while a 36^4 lattice shows significant finite size effects, a $36^2 \times 64^2$ lattice is probably adequate. At $m = 0.005$

a 36^4 lattice shows large finite size effects, a $36^2 \times 64^2$ lattice shows small but significant finite size effects while a $36^2 \times 96^2$ lattice is probably adequate. For $m = 0.001$ a 36^4 lattice has very large finite size effects, and we suspect that a $36^2 \times 64^2$ also has large finite size effects. Simulations on larger lattices ($36^2 \times 96^2$ or $36^2 \times 128^2$) are needed.

Even without any larger lattice simulations at $m = 0.001$, the current results strongly suggest that the presence of a relatively large external magnetic field does produce a non-zero chiral condensate, which breaks the chiral symmetry at $m \to 0$.

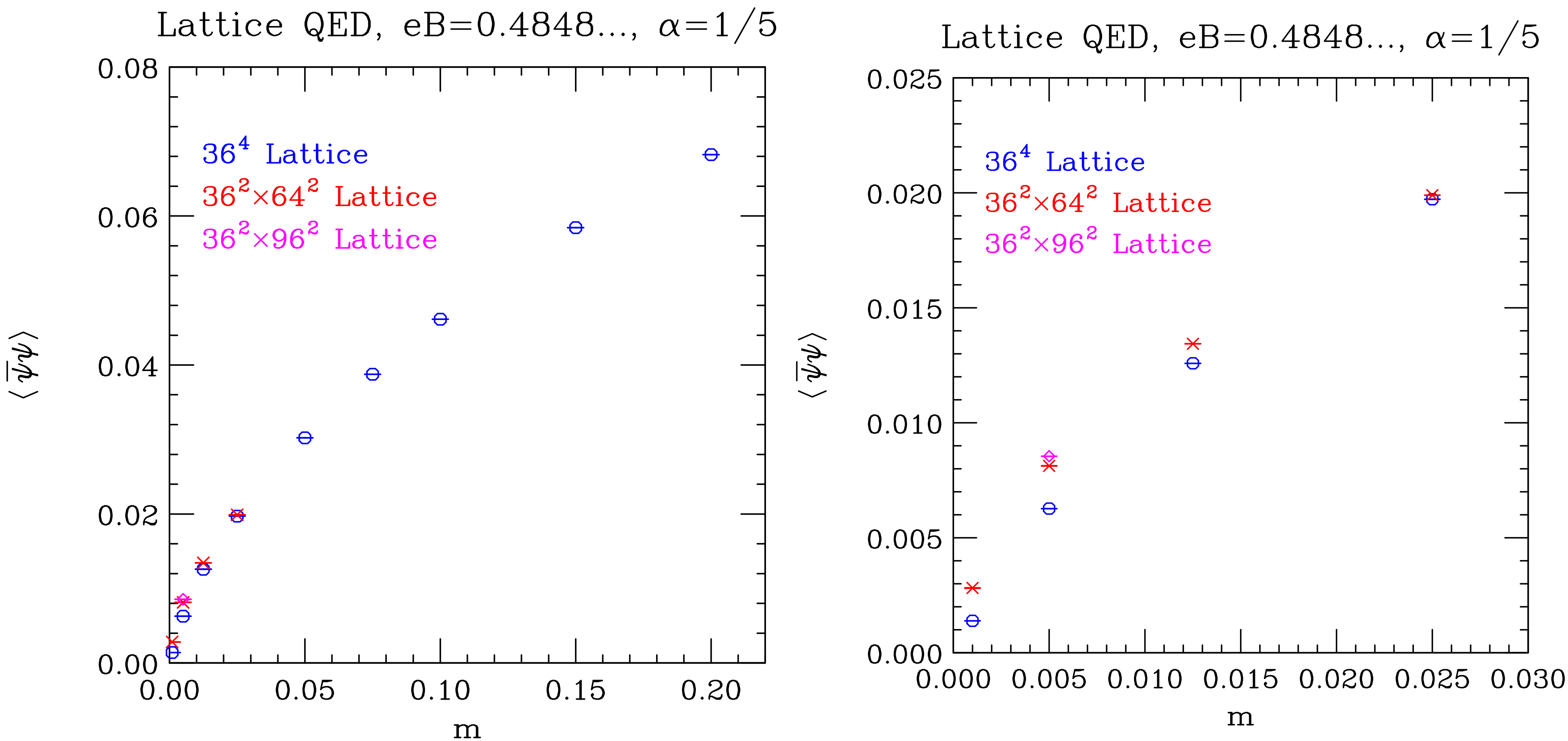

Figure 2: $\langle\bar{\psi}\psi\rangle$ as a function of mass at $eB = 2\pi \times 100/36^2$,showing dependence on lattice size in the z and t directions.

The size of the $m = 0$ condensate appears to be at least an order of magnitude greater than is predicted by the Schwinger-Dyson approach. However, we have ignored the effects of renormalization. While this was valid at the physical fine structure constant $\alpha \approx 1/137$ and $m = 0.1, 0.2$ where these effects are small, it is not valid at $\alpha_{lattice} = 1/5$.

The improved rainbow approximation used by Gusynin-Miransky-Shovkovy in their Schwinger-Dyson calculations relies on selection of a gauge where loop corrections to the electron photon vertex are suppressed by positive powers of α. This implies that as α is increased, the approximation of the vertex by its bare value becomes less valid. We suggest that $\alpha = 1/5$ is too large for the improved rainbow approximation to be valid.

Finally one needs to know how large an effect the staggered fermion symmetry breaking has on these lattice results.

Discussion and Conclusions

- We simulate lattice QED in a constant (in time and space) external magnetic field B using methods (RHMC) developed for lattice QCD.
- Approximate calculations using Schwinger-Dyson methods indicate that the magnetic field 'catalyses' chiral symmetry breaking in the $m \to 0$ limit giving the electron a dynamical mass $\propto \sqrt{eB}$ and a chiral condensate $\propto (eB)^{3/2}$.
- Our lattice simulations, using $\alpha = 1/5$ to enhance the signal, appear to indicate that dynamical chiral symmetry breaking does occur for our chosen $eB = 0.4848...$.
- However, to put this observation on firmer ground, we will need to extend our simulations at the lowest mass $m = 0.001$ to larger lattices. Because the magnetic field restricts the electron's motion in the plane orthogonal to the magnetic field, we need to only increase the lattice extents in the direction of the magnetic field and the (euclidean) time direction. (Planned sim-

ulations are on $36^2 \times 96^2$ and/or $36^2 \times 128^2$ lattices.)

- Our simulations indicate condensates at least an order of magnitude larger than do Schwinger-Dyson methods.
- We plan to make other measurements on stored configurations. In particular, we will measure the effects of QED in an external magnetic field on the coulomb field of a static charged particle placed in said magnetic field.
- We should redo these simulations at other values of eB to check that the condensate does scale like $|eB|^{3/2}$.
- Since it is possible that the chiral symmetry breaking for QED in an external magnetic field is explicit, we are contemplating simulating QED with more than 1 electron 'flavour', since the flavour chiral symmetry is not broken explicitly by the external magnetic field. Hence, any flavour chiral symmetry breaking must be spontaneous, with its associated Goldstone bosons.
- We contemplate using lattice methods to study the physics of QED in an external electric field. However, the action of QED in

an external electric field is complex (Sauter-Schwinger effect), so such studies will be less straight-forward.

These simulations were performed on the Bebop Cluster at ANL, Cori at NERSC using an ERCAP(DOE) allocation, Perlmutter at NERSC using early-user access and using XSEDE(NSF) allocations on Expanse at UCSD, Bridges-2 at PSC and Stampede-2 at TACC.

One of us (DKS) would like to thank G. T. Bodwin for helpful discussions, while JBK would like to acknowledge conversations with A. Shovkovy and V. Yakimenko.

Appendix: Lattice QED in an external Magnetic Field

We simulate using the non-compact gauge action

$$S(A) = \frac{\beta}{2}\sum{}_{n,\mu<\nu}[A_\nu(n+\hat{\mu}) - A_\nu(n) - A_\mu(n+\hat{\nu}) + A_\mu(n)]^2$$

where n is summed over the lattice sites and μ and ν run from 1 to 4 subject to the restriction. $\beta = 1/e^2$. The functional integral to calculate the expectation value for an observable $\mathcal{O}(\mathcal{A})$ is then

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int_{-\infty}^{\infty}\Pi_{n,\mu}dA_\mu(n)e^{-S(A)}[\det\mathcal{M}(A+A_{ext})]^{1/8}\mathcal{O}(\mathcal{A})$$

where $\mathcal{M} = M^\dagger M$ and M is the staggered fermion action in the presence of the dynamic photon field A and external photon field A_{ext} describing the magnetic field B (or rather eB). M is defined by

$$M(A + A_{ext}) = \sum{}_\mu D_\mu(A + A_{ext}) + m$$

where the operator D_μ is defined by

$$[D_\mu(A+A_{ext})\psi](n) = \frac{1}{2}\eta_\mu(n)\{e^{i(A_\mu(n)+A_{ext,\mu}(n))}\psi(n+\hat{\mu}) - e^{-i(A_\mu(n-\hat{\mu})+A_{ext,\mu}(n-\hat{\mu}))}\psi(n-\hat{\mu})\}$$

and η_μ are the staggered phases. Note that this treatment of the gauge-field–fermion interactions is compact and so has period 2π in the gauge fields.

We implement the RHMC simulation method of Clark and Kennedy, using a $(12,12)$ $[(15,15)]$ rational approximation to $\mathcal{M}^{-1/8}$ and $(20,20)$ $[(25,25)]$ rational approximations $\mathcal{M}^{\pm 1/16}$. To account for the range of normal modes of the non-compact gauge action we vary the trajectory lengths τ over the range,

$$\frac{\pi}{2\sqrt{\beta}} \leq \tau \leq \frac{4\pi}{\sqrt{2\beta(4-\sum_\mu \cos(2\pi/N_\mu))}},$$

of the periods of the modes of this gauge action.

A_{ext} are chosen in the symmetric gauge as

$$
\begin{aligned}
A_{ext,1}(i,j,k,l) &= -\frac{eB}{2} \quad (j-1) \qquad i \neq N_1 \\
A_{ext,1}(i,j,k,l) &= -\frac{eB}{2}(N_1+1) \quad (j-1) \qquad i = N_1 \\
A_{ext,2}(i,j,k,l) &= +\frac{eB}{2} \quad (i-1) \qquad j \neq N_2 \\
A_{ext,2}(i,j,k,l) &= +\frac{eB}{2}(N_2+1) \quad (i-1) \qquad j = N_2
\end{aligned}
$$

while $A_{ext,3}(n) = A_{ext,4}(n) = 0$. In practice we subtract the average values of A_{ext},μ from these definitions. This choice produces a magnetic field eB in the $+z$ direction on every $1,2$ plaquette except that with $i = N_1$, $j = N_2$, which has the magnetic field $eB(1 - N_1N_2)$. Because of the compact nature of the interaction, requiring $eBN_1N_2 = 2\pi n$ for some integer $n = 0, 1, ...N_1N_2/2$ makes the value of this plaquette indistinguishable from eB. Hence $eB = 2\pi n/(N_1N_2)$ lies in the interval

$[0, \pi]$.

One of the observables we calculate is the electron contribution to the effective gauge action per site $\frac{-1}{8V}\mathbf{trace}[\ln(\mathcal{M})]$. For $\ln(\mathcal{M})$ we use a $(30, 30)$ rational approximation to the logarithm. Here we use the Chebyshev method of Kelisky and Rivlin. While this has worse errors than a Remez approach, it preserves some of the properties of the logarithm itself, and is applicable on the whole complex plane cut along the negative real axis.