SU(N) FRACTIONAL INSTANTONS

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YANG-MILLS FORMULATION ON $\mathbb{R} \times \mathbb{T}^3$ WITH TBC

FIBONACCI NUMBERS

Our particular set-up has the following considerations:

- SU(N) YM theory defined on an asymmetric torus of sizes $l_0 = sl$, $l_1 = l_2 = l/N$ and $l_3 = l$. The number of colours N is taken as the nth integer in the Fibonacci sequence: $\boxed{N = F_n}$.
- Twisted boundary conditions (TBC) on the three-torus with flux $\vec{m} = (0,0,m)$, where m is taken coprime with N as $\boxed{m = F_{n-2}}$.

Gauge Fixing is used as $A_0 = 0$...

TWISTED BOUNDARY CONDITIONS

$$A_i(x + l_j \hat{e}_j) = \Gamma_j A_i(x_0, \vec{x}) \Gamma_j^\dagger$$

CONSISTENCY RELATIONS

$$\Gamma_1 \Gamma_2 = e^{i\frac{2\pi m}{N}} \Gamma_2 \Gamma_1 \text{ and } \Gamma_3 \Gamma_i = \Gamma_i \Gamma_3 \quad i = 1,2$$

...but is not complete, allowing...

SINGULAR GAUGE TRANSFORMATIONS:

$$\Omega_{\vec{s}}(\vec{x} + l_i \hat{e}_i) = e^{i\frac{2\pi s_i}{N}} \Gamma_i \Omega_{\vec{s}}(\vec{x}) \Gamma_i^\dagger, s_i \in \mathbb{Z}_{\mathrm{N}}$$

THE WAVE FUNCTIONS TRANSFORM AS

$$\Psi_{\vec{e}}\left([\Omega_{\vec{s}}]A\right) = e^{i\frac{2\pi \vec{e}\cdot\vec{s}}{N}} \Psi_{\vec{e}}(A)$$

HOLONOMIES

The solutions can be interpreted as tunneling events interpolating between two pure gauge configurations characterised by its spatial Polyakov loop at $x_0 = \pm\infty$, defined as:

$$P_i(x_0, \vec{x}) \equiv \frac{1}{N}\mathrm{Tr}\left(\mathrm{P}\exp\left\{-i\int_0^{l_i} dx_i A_i(x_0, \vec{x})\right\}\Gamma_i\right)$$

IN OUR PARTICULAR SELECTION:

$$P(\gamma, w_1, w_2, w_3) = \frac{1}{N}\mathrm{Tr}\left(\Gamma_1^{w_1(\gamma)}\Gamma_2^{w_2(\gamma)}\Gamma_3^{w_3(\gamma)}\right)$$

TOPOLOGICAL CHARGE

Impose TBC to support fractional topological charge:

FIBONACCI NUMBERS

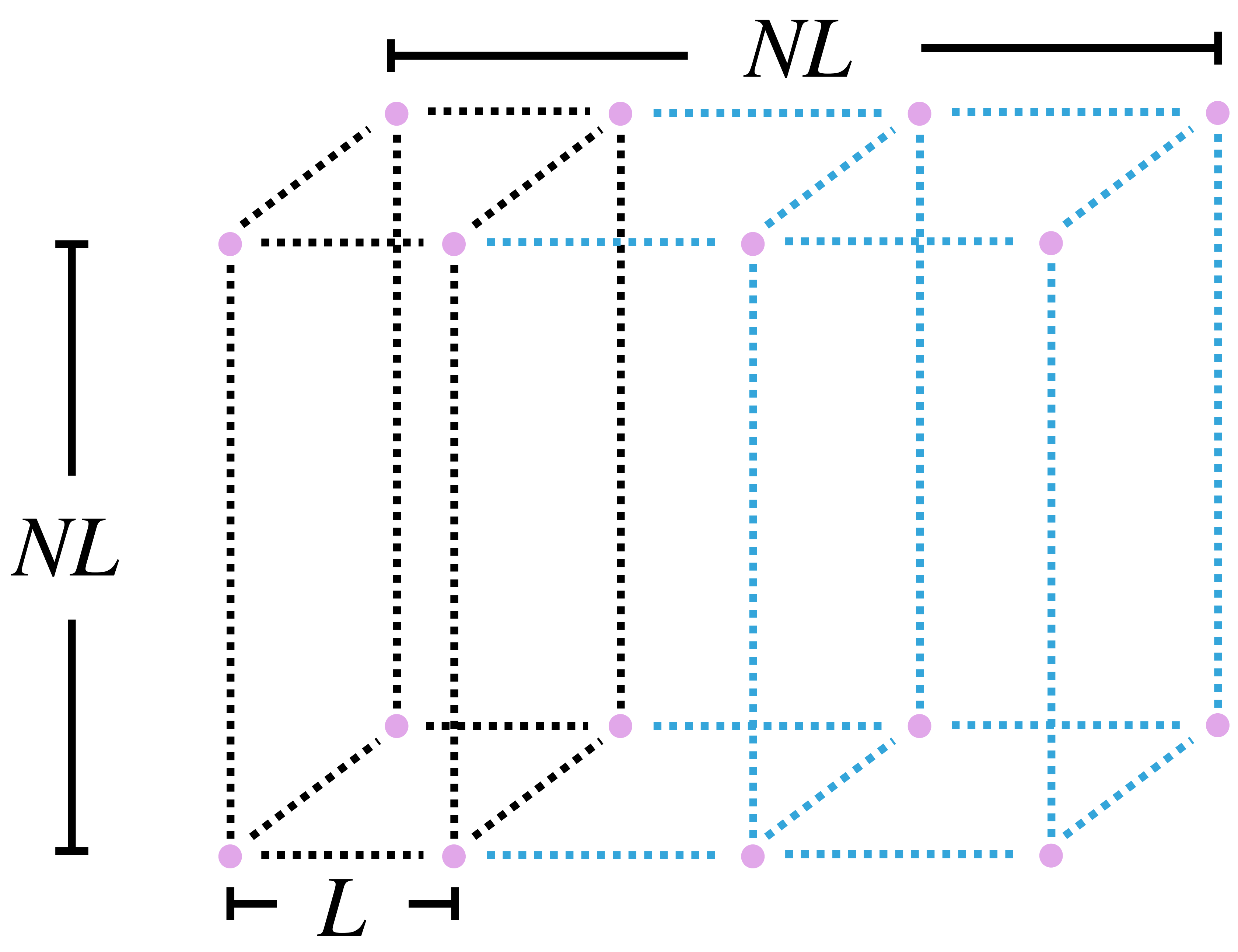
$$Q = \frac{1}{16\pi^2}\int d^4x\mathrm{Tr}\left(F_{\mu\nu}(x)\widetilde{F}_{\mu\nu}(x)\right) = \nu - \frac{\vec{k}\cdot\vec{m}}{N} = \boxed{(-1)^{n+1}F_{n-4} - \frac{(-1)^{n+1}F_{n-2}\times F_{n-2}}{F_n} = \frac{1}{F_n}}$$

WHY WE TAKE THIS GEOMETRY?

With TBC, colour and spatial DOF get entangled and the torus periods become effectively enlarged in the twisted planes by a factor of N.

THE EFFECTIVE LARGE N DYNAMICS IS THE ONE OF A SYMMETRIC TORUS OF SIZE l^4

[arXiv:1406.5655]



WHY THE FIBONACCI SEQUENCE?

The choice of m and N aims at avoiding large N phase transitions that would lead to $\mathbf{Z}_N \times \mathbf{Z}_N$ symmetry breaking.

THE OPTIMAL SEQUENCE TO APPROACH THE LARGE N LIMIT WITHOUT INSTABILITIES IS TO TAKE N AND m ALONG THE FIBONACCI SEQUENCE WITH $N = F_n$ AND $m = F_{n-2}$

[arXiv:1610.07972]

DENSITY PROFILES

We present a new type of SU(N) instanton configurations with fractional topological charge $Q = 1/N$ compatible with our choice of TBC.

IN THE LATTICE:
OBTAINED THROUGH GRADIENT FLOW ON A $sNL \times L^2 \times NL$ LATTICE, WITH $l = NLa$

Continuum limit

IN THE CONTINUUM:
SELF DUAL CONFIGURATIONS DEFINED ON A 4-DIMENSIONAL TORUS $sl \times (l/N)^2 \times l$

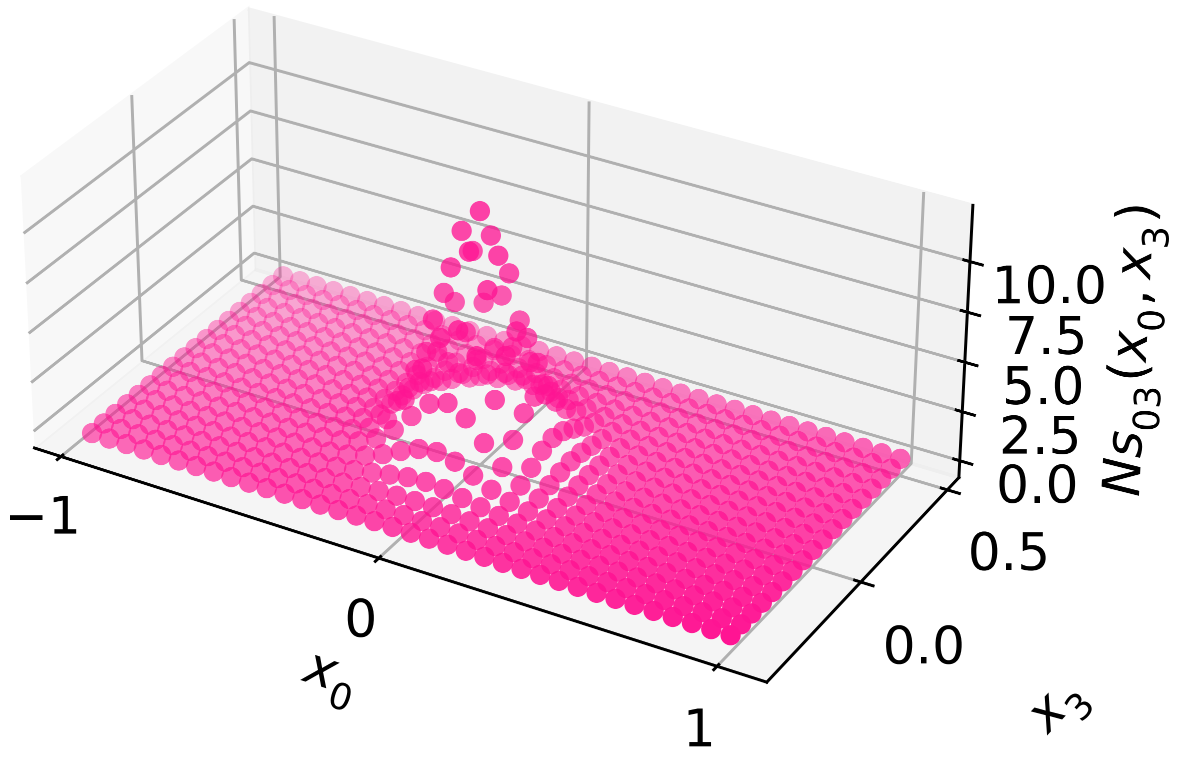
First, we have looked at action density profiles obtained by integrating the 4-dimensional density

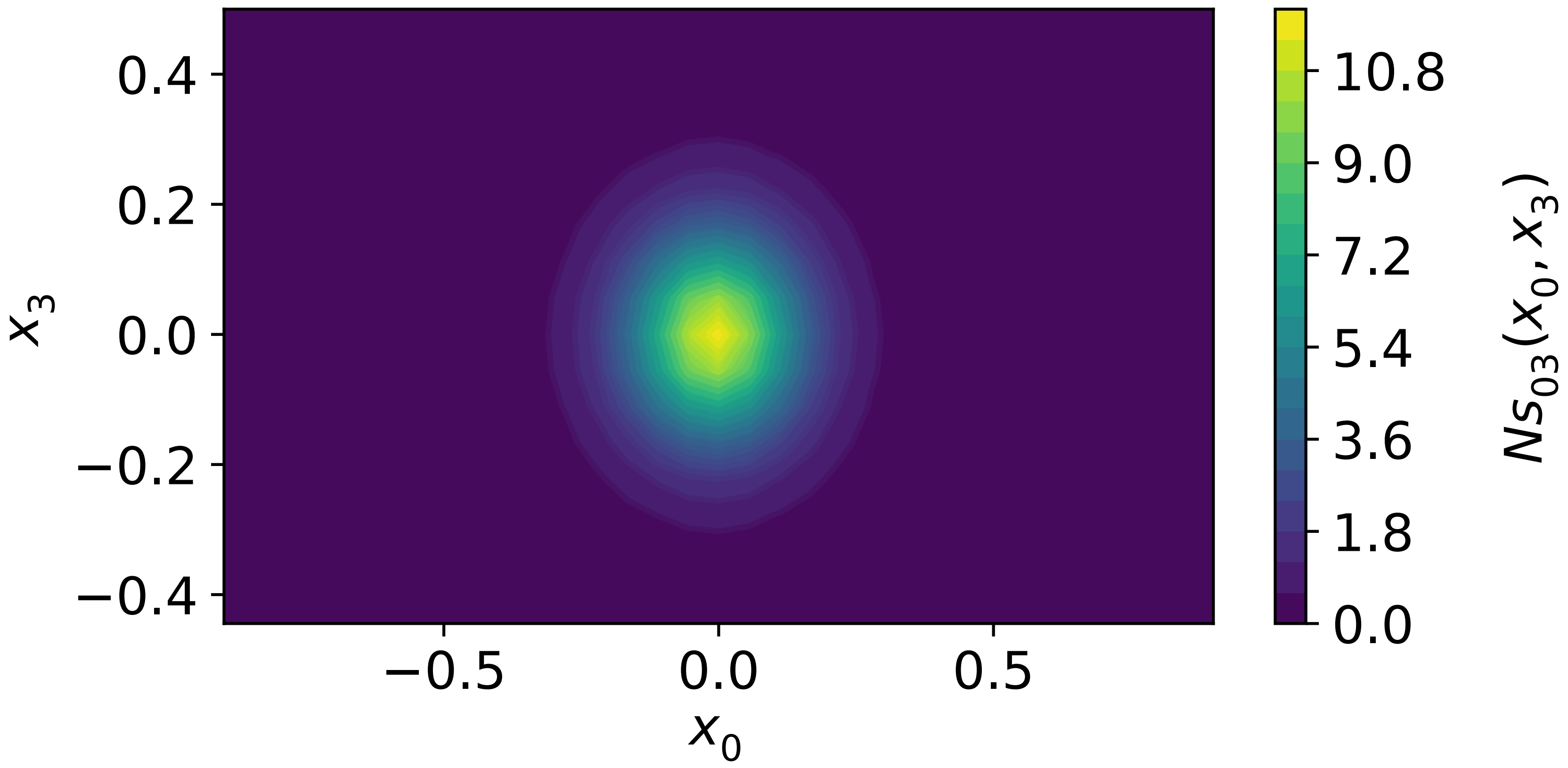
$$Ns_\mu(x_\mu) \equiv \left(\prod_{\rho \neq \mu} \int_0^{l_\rho} dx_\rho \right) Ns(x) \qquad Ns_{\mu\nu}(x_\mu, x_\nu) \equiv \left(\prod_{\rho \neq \mu,\nu} \int_0^{l_\rho} dx_\rho \right) Ns(x)$$

IN THE FIBONACCI FLAT CONSTRUCTION: $l^2 s_{12}(x_1, x_2)/N = 1$

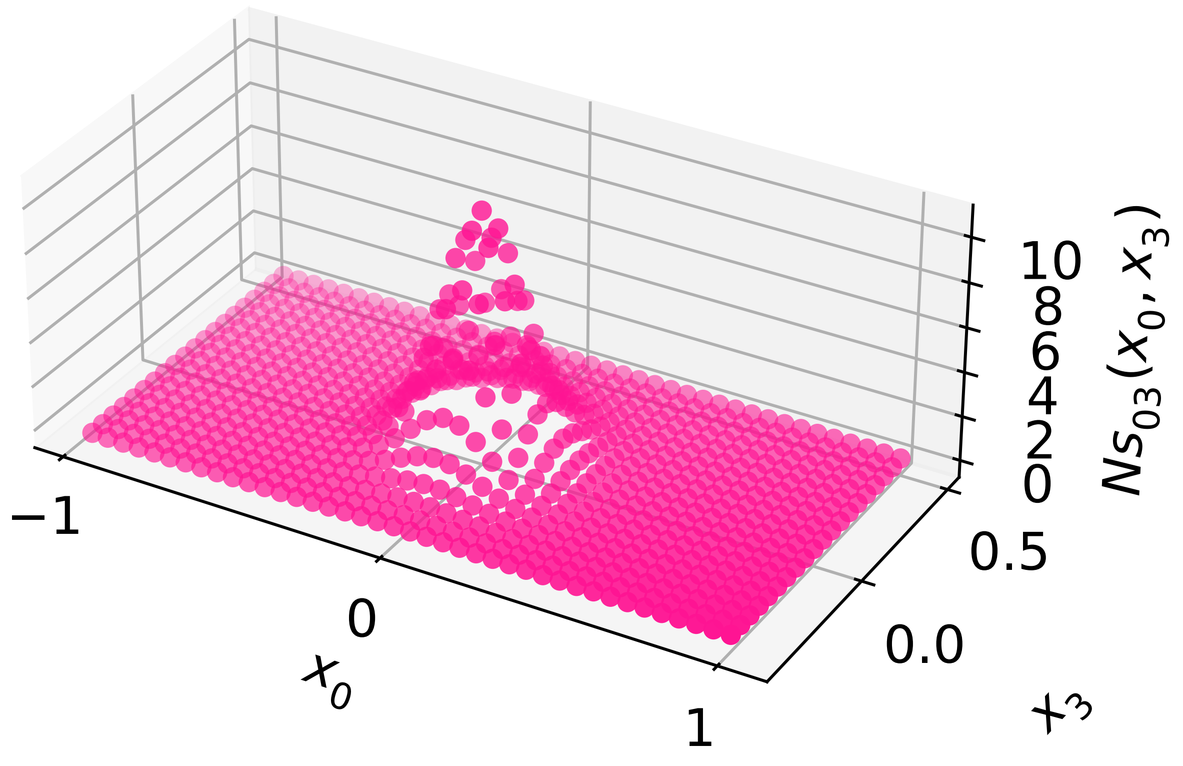

N=3
N=5
N=8
$Ns_{03}(x_0, x_3)$
x_0
x_3

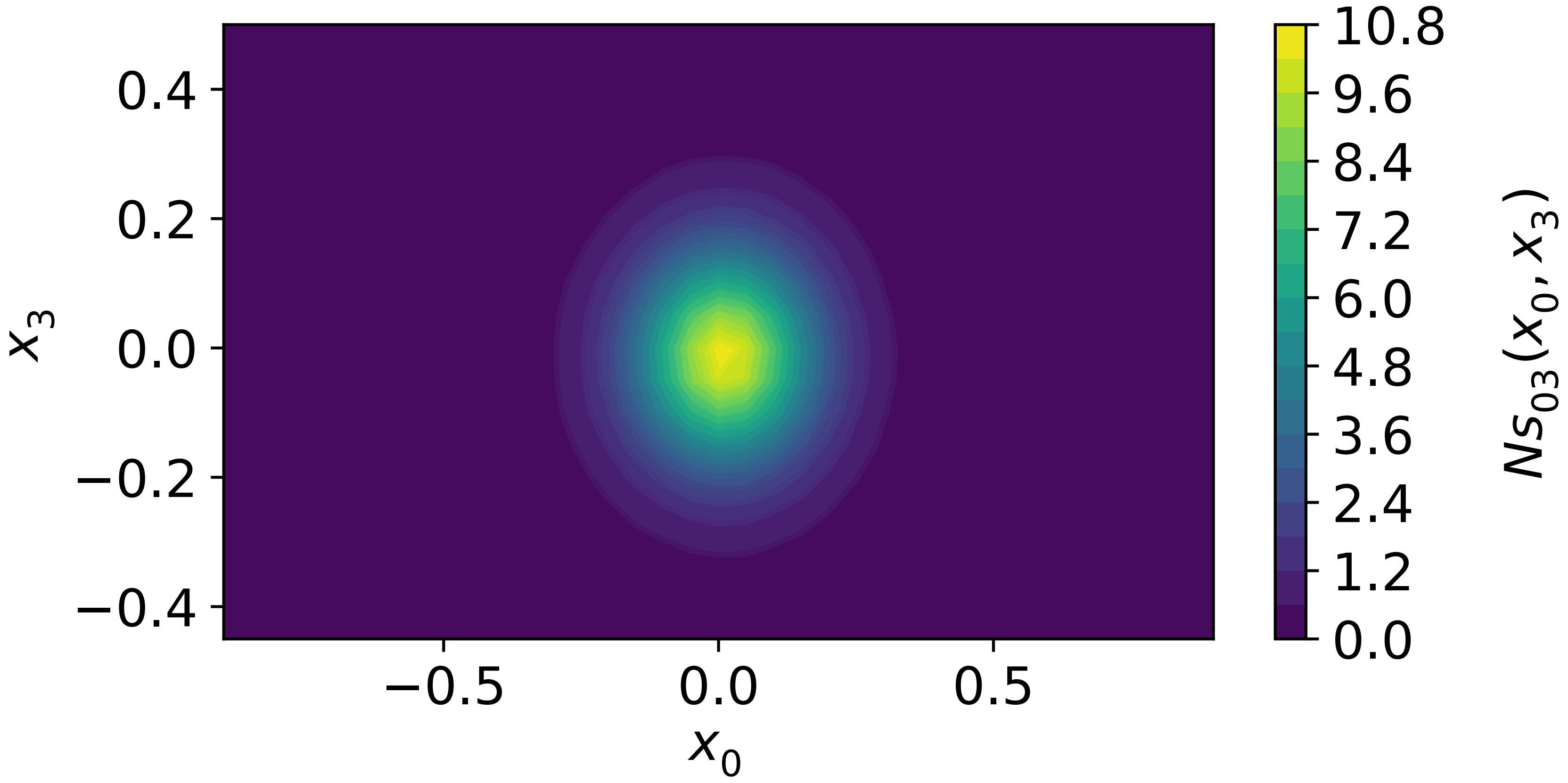
N=3


$Ns_{03}(x_0, x_3)$
10.0
7.5
5.0
2.5
0.0
−1
0
1
x_0
0.5
0.0
x_3

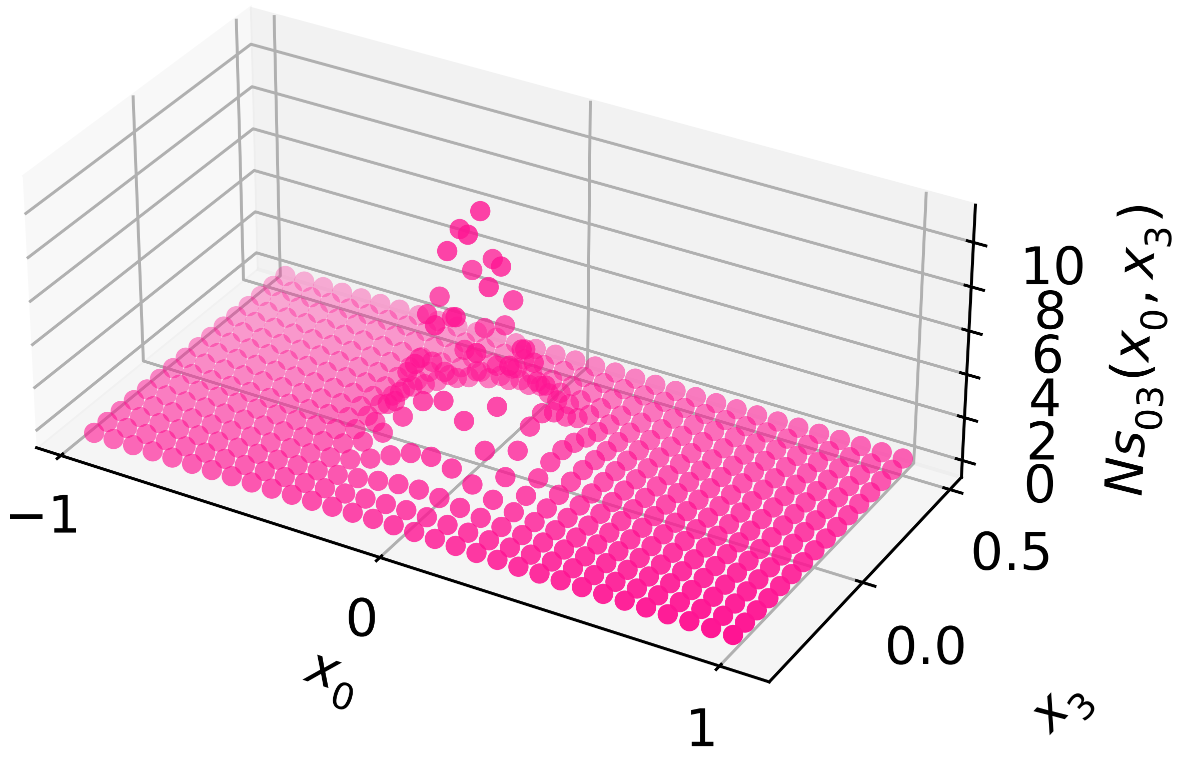

x_3
0.4
0.2
0.0
−0.2
−0.4
−0.5
0.0
0.5
x_0
10.8
9.0
7.2
5.4
3.6
1.8
0.0
$Ns_{03}(x_0, x_3)$

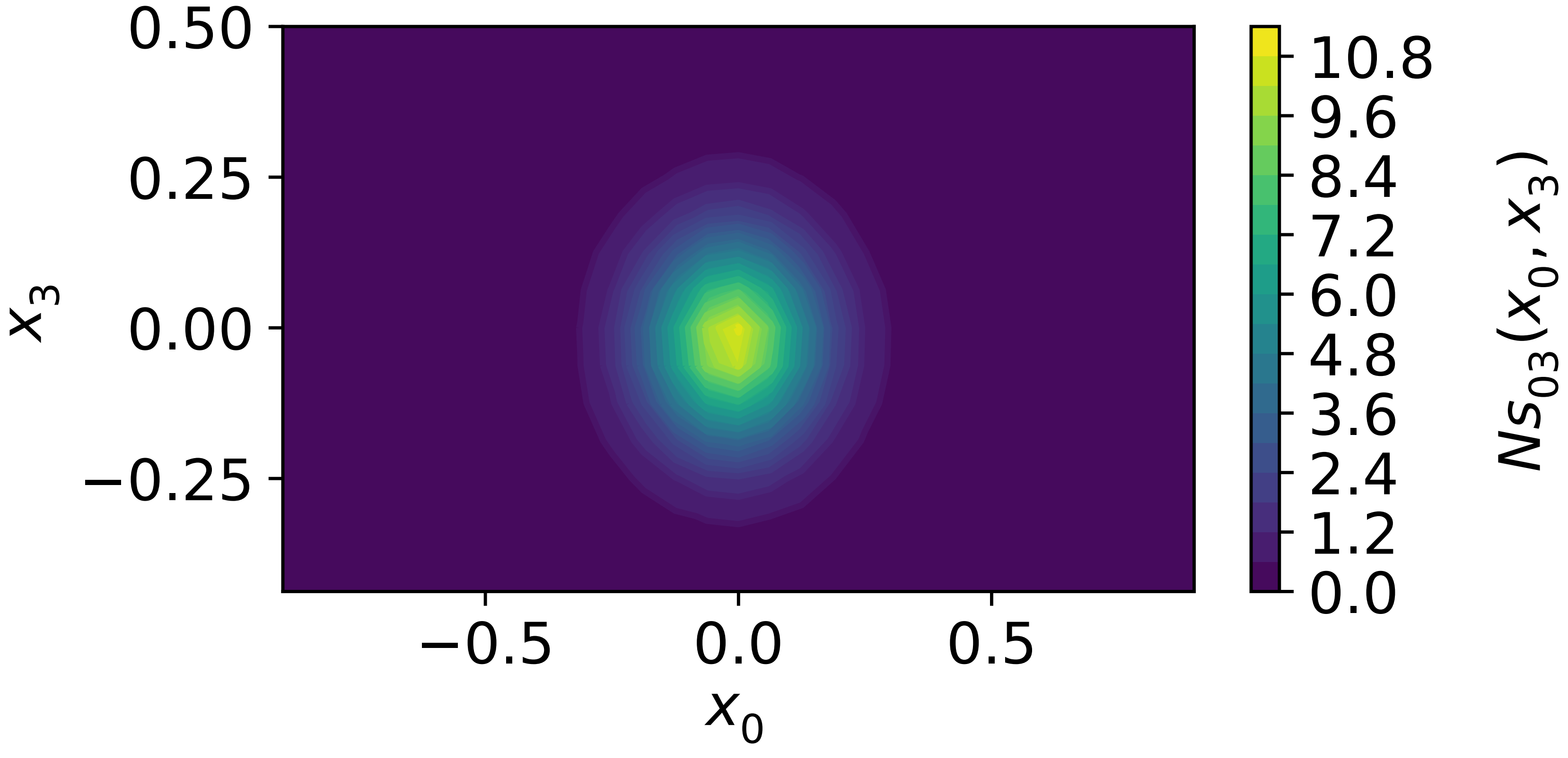
N=5

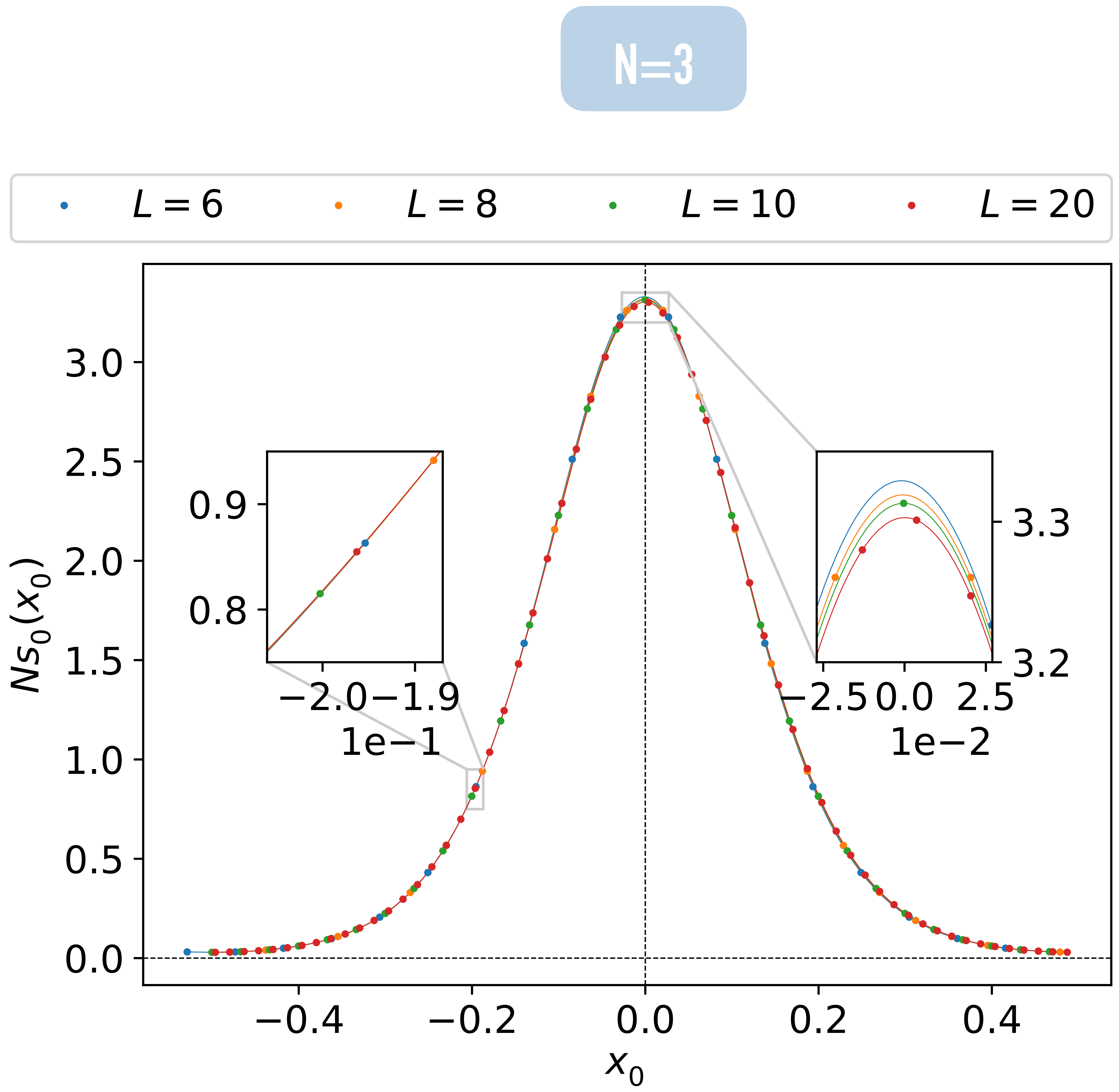

$Ns_{03}(x_0, x_3)$
10
8
6
4
2
0
−1
0
1
x_0
0.5
0.0
x_3

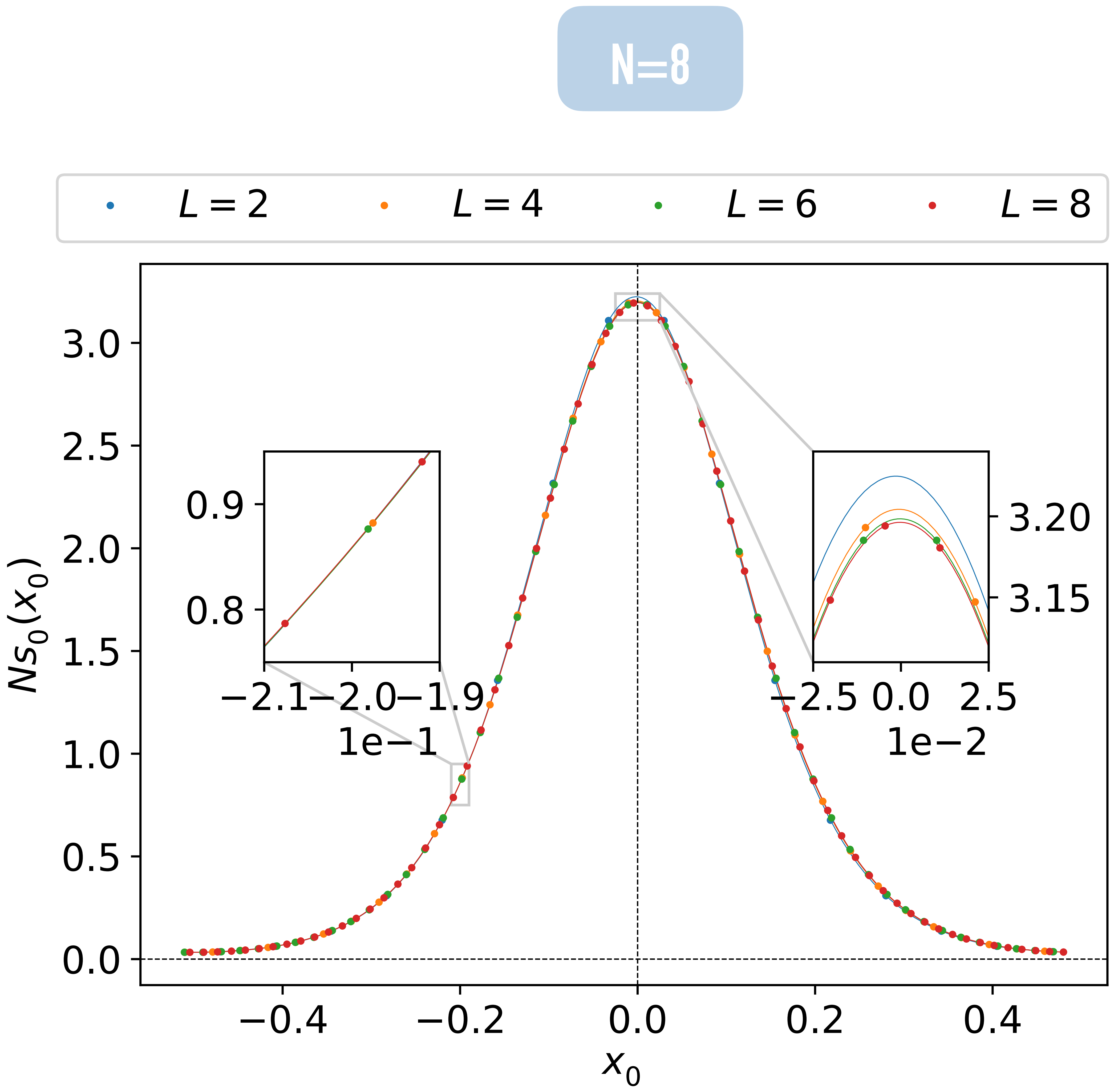

x_3
0.4
0.2
0.0
−0.2
−0.4
−0.5
0.0
0.5
x_0
10.8
9.6
8.4
7.2
6.0
4.8
3.6
2.4
1.2
0.0
$Ns_{03}(x_0, x_3)$

N=8


$Ns_{03}(x_0, x_3)$
10
8
6
4
2
0
−1
0
1
x_0
0.5
0.0
x_3

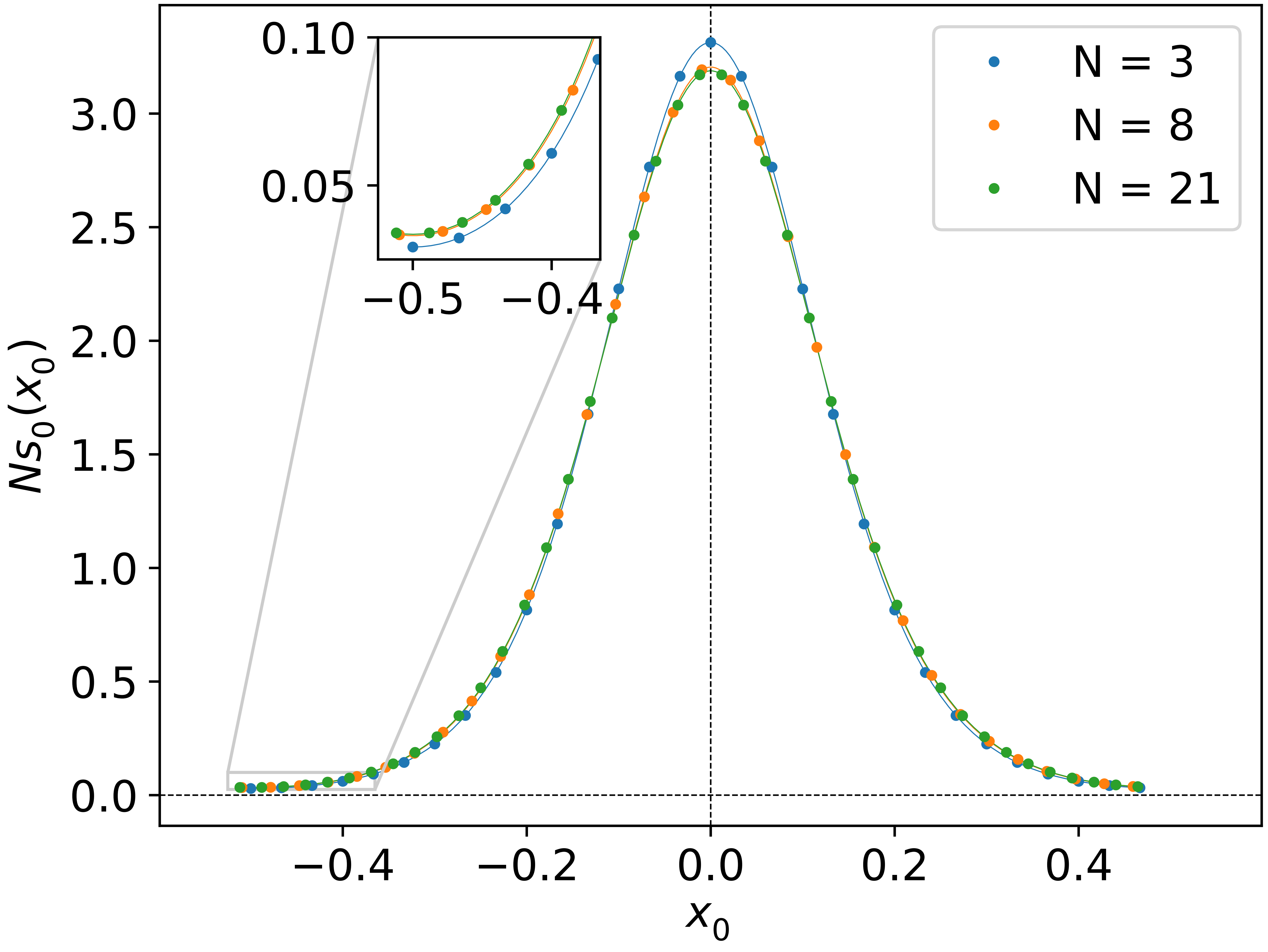

x_3
0.50
0.25
0.00
−0.25
−0.5
0.0
0.5
x_0
10.8
9.6
8.4
7.2
6.0
4.8
3.6
2.4
1.2
0.0
$Ns_{03}(x_0, x_3)$





We see very little dependence on the lattice spacing, indicating rather small lattice artefacts even for our smaller lattices. Discretization effects are controlled by LN.

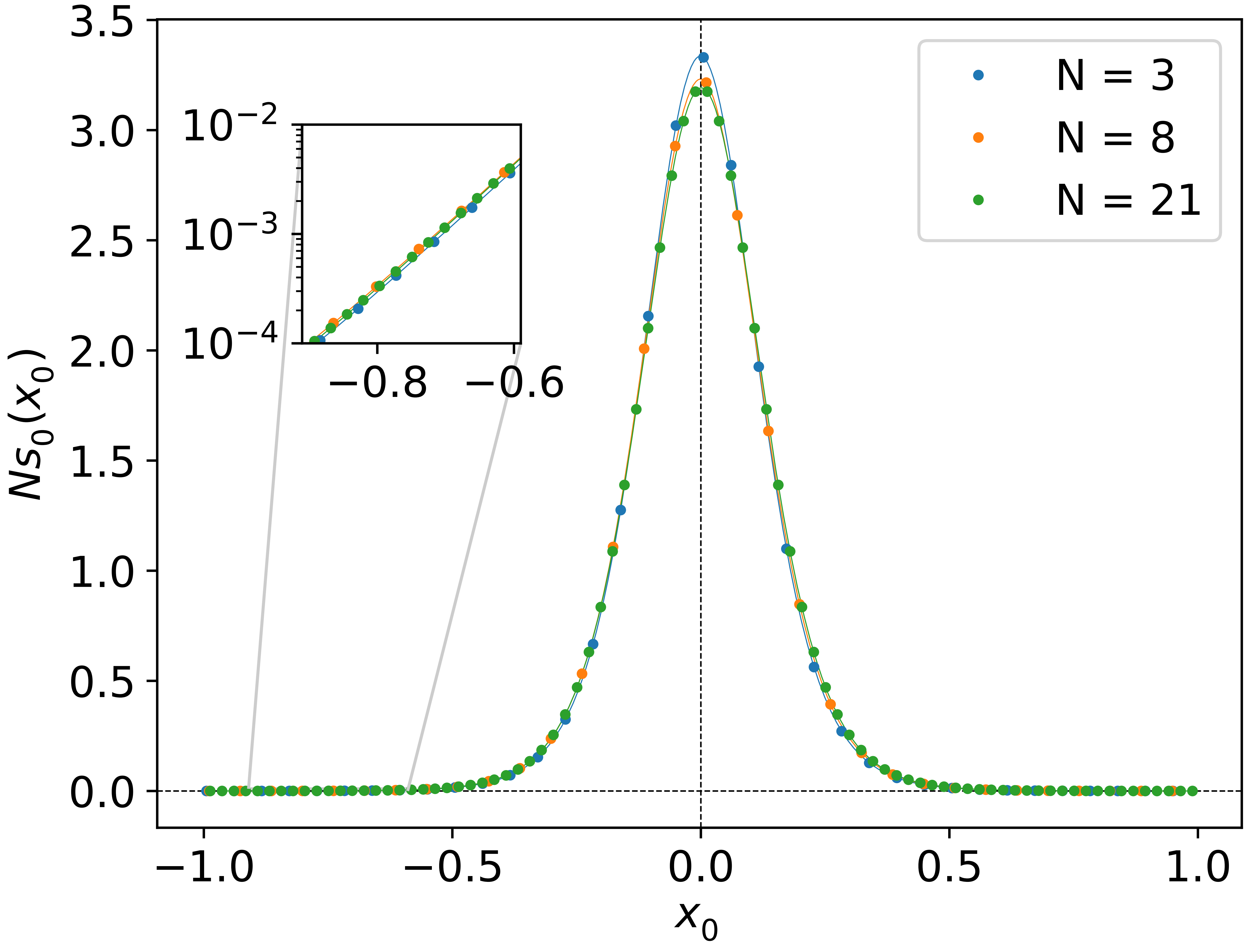
S=1



WIDTH AT HALF MAXIMUM

N	m	$\overline{m}$	L	w_1	w_2
3	1	1	6	0.268284	0.268456
5	2	2	12	0.283232	0.289695
8	3	-3	6	0.279391	0.285402
13	5	5	4	0.286707	0.287152
21	8	8	2	0.281166	0.286412

S=2

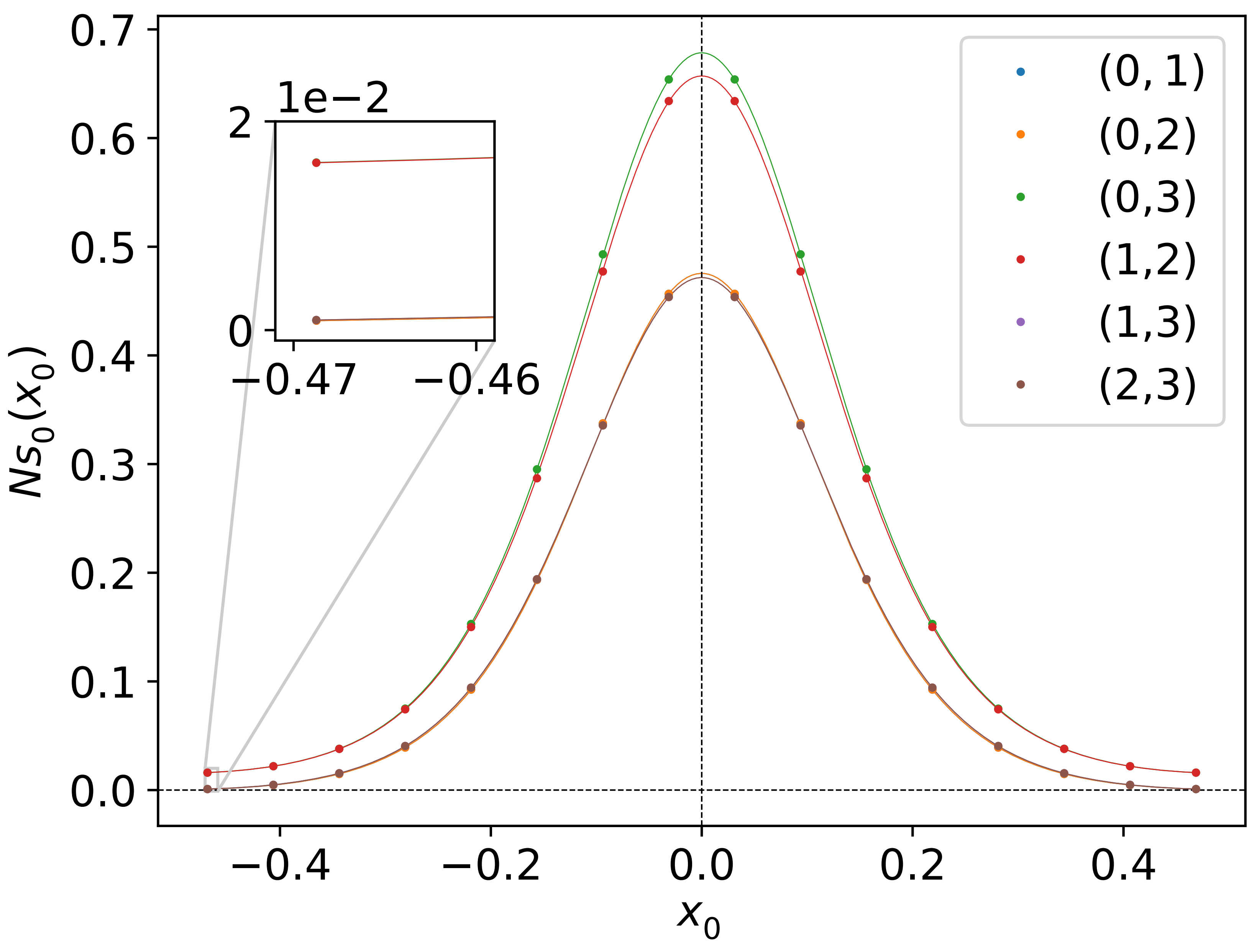


WIDTH AT HALF MAXIMUM

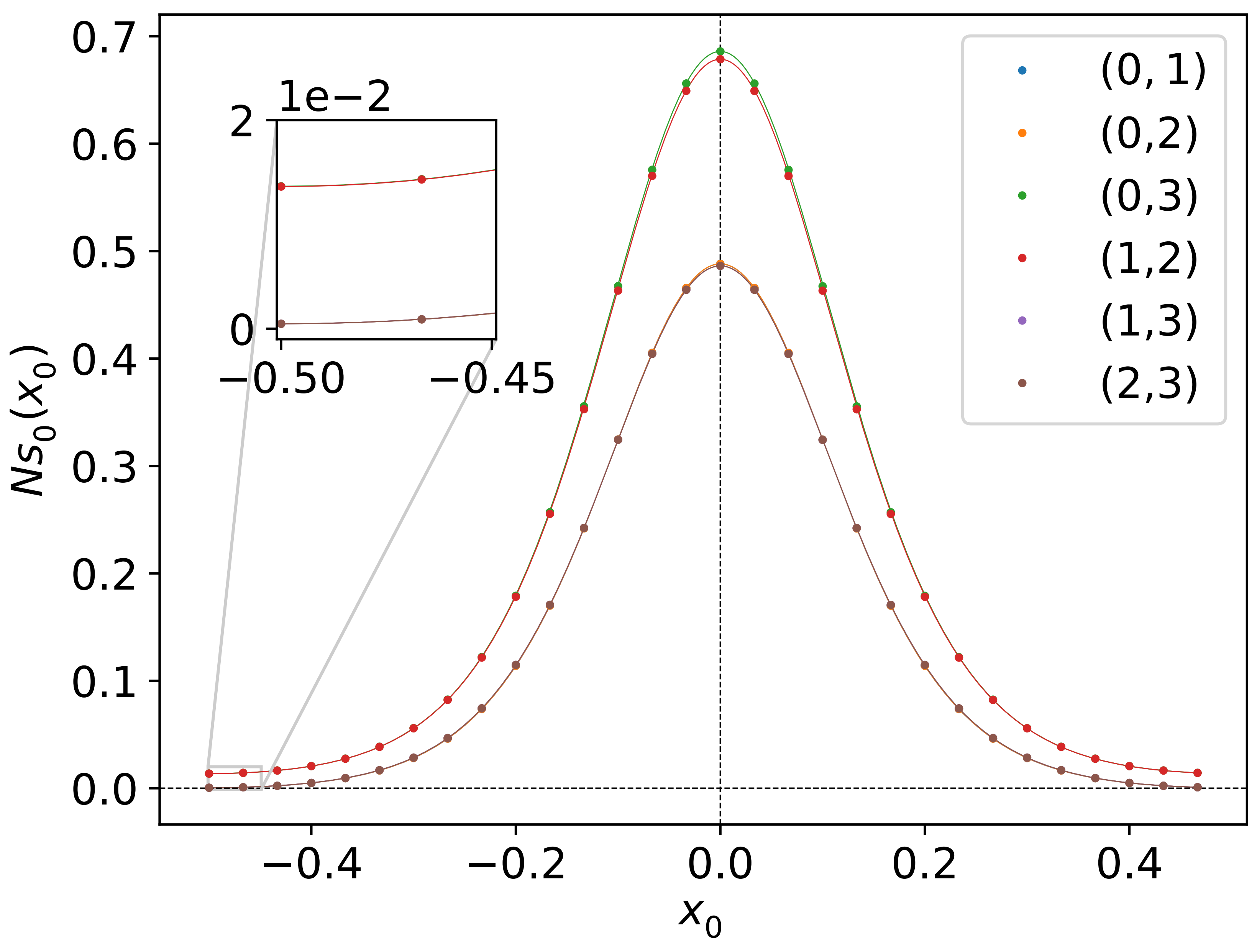
N	m	$\overline{m}$	L	w_1	w_2
3	1	1	6	0.268284	0.268456
5	2	2	12	0.283232	0.289695
8	3	-3	6	0.279391	0.285402
13	5	5	4	0.286707	0.287152
21	8	8	2	0.281166	0.286412

SELF-DUALITY

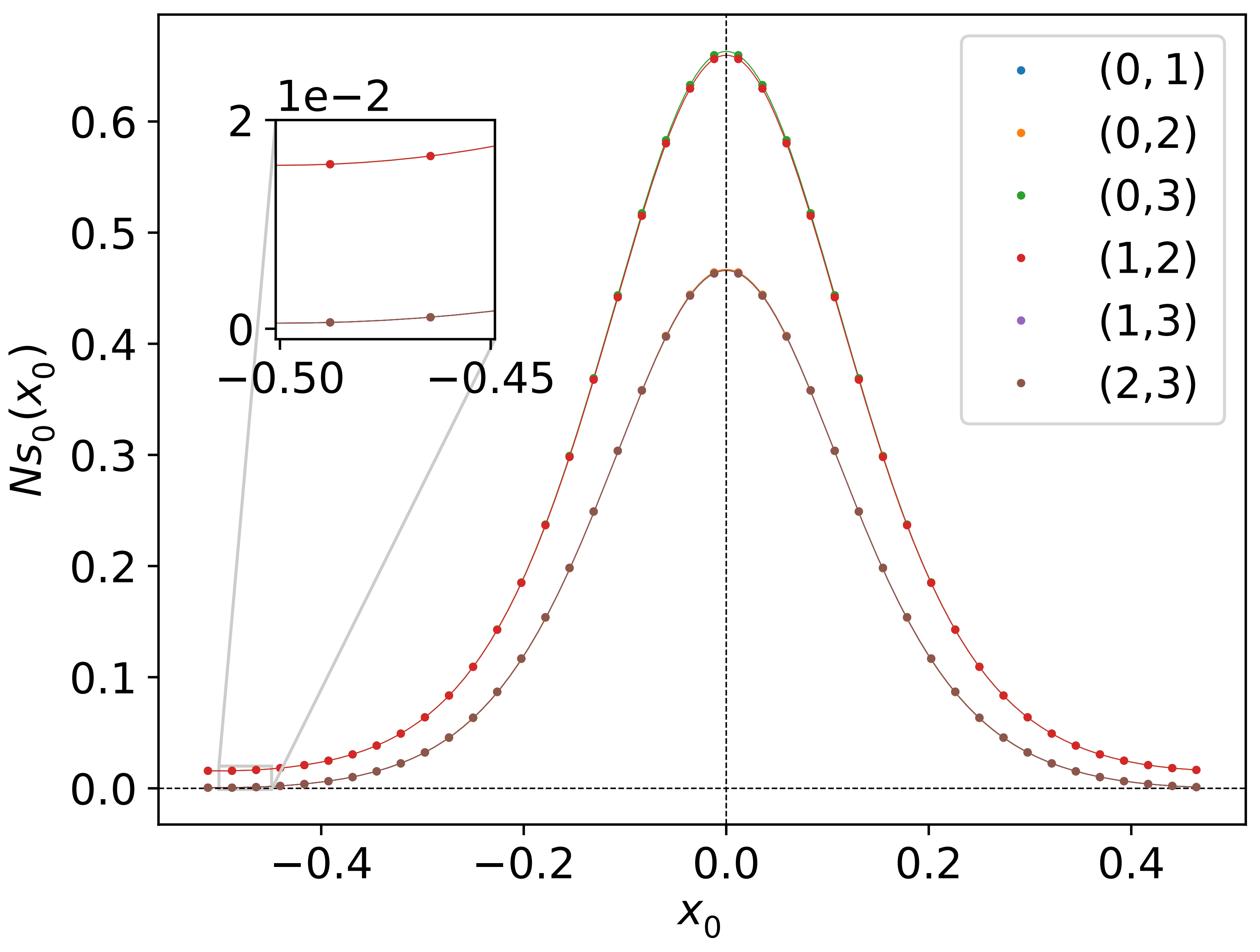
In order to test self-duality, we have looked separately at the different components of the electric and magnetic energies by computing the spatial integral of $\mathrm{ReTr}(F_{\mu\nu}^2)$, with μ and ν fixed.



N=3



N=8



N=21

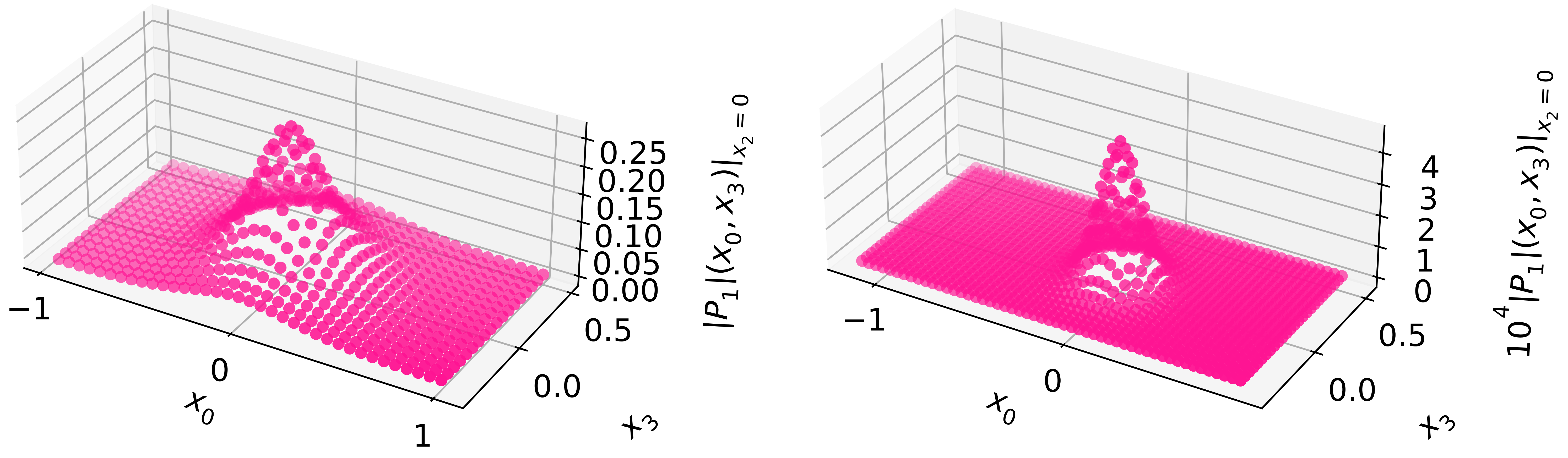
POLYAKOV LOOPS

We use $P_\mu(x)$ to denote ($1/N$ times) the trace of the Polyakov loop winding the torus once in direction μ, and will parameterize this quantity in terms of its modulus and phase as:

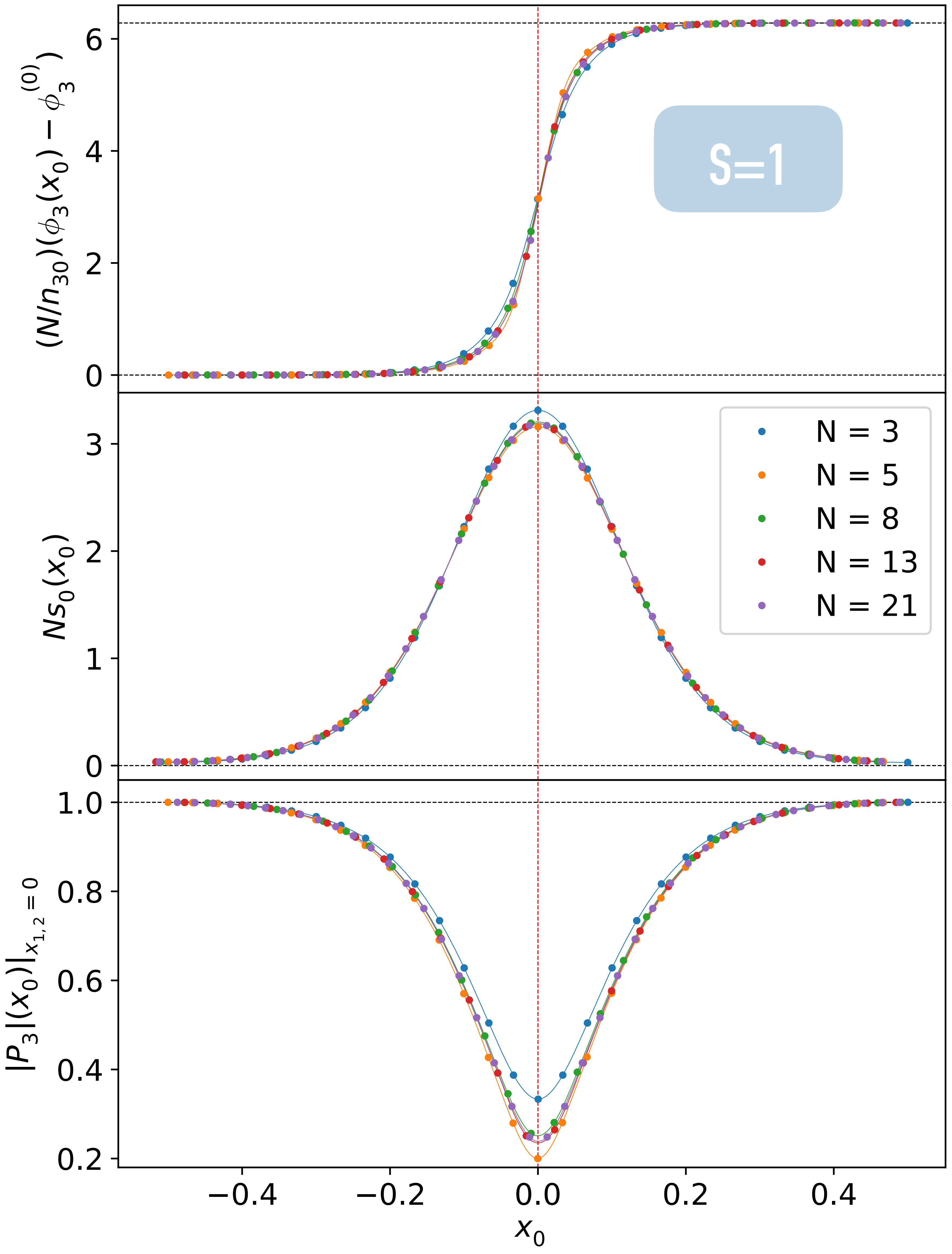
$$P_\mu(x) = \frac{1}{N}\mathrm{Tr}\left(\mathrm{P}\exp\left\{-i\int_0^{l_\mu} dx_\mu A_\mu(x)\right\}\Omega_\mu(x)\right) \equiv |P_\mu(x)|\, e^{i\phi_\mu(x)}$$

DUE TO TBC, POLYAKOV LOOPS SATISFY:

$$P_\mu(x + l_\nu \hat{e}_\nu) = e^{i\frac{2\pi n_{\mu\nu}}{N}} P_\mu(x)$$



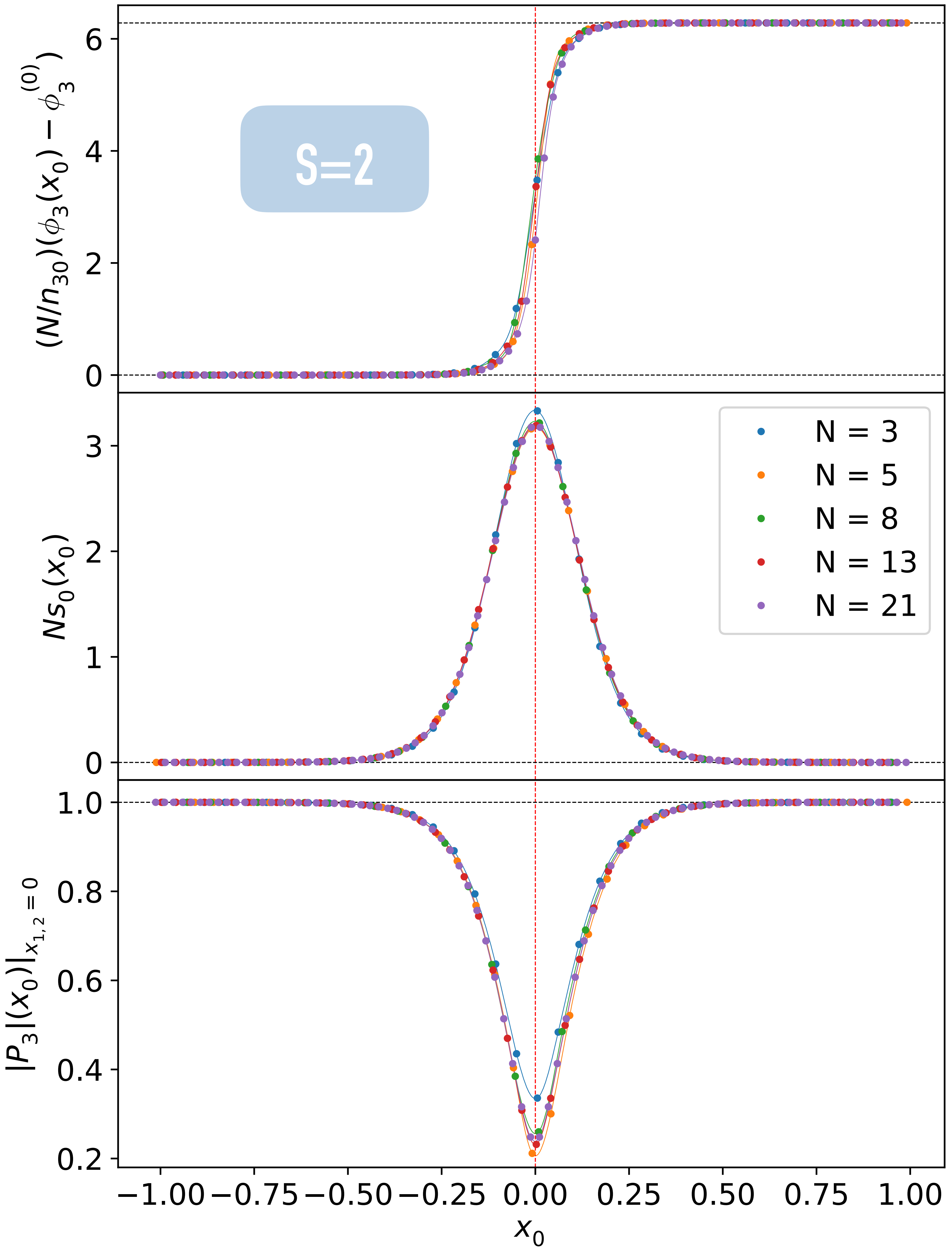
The value of $|P_1|$ remains everywhere very small and tends to zero at $x_0 \to \pm\infty$



DIFFERING BY $2\pi k/N$

ENERGY DENSITY PROFILE

MODULUS TENDING TO ONE



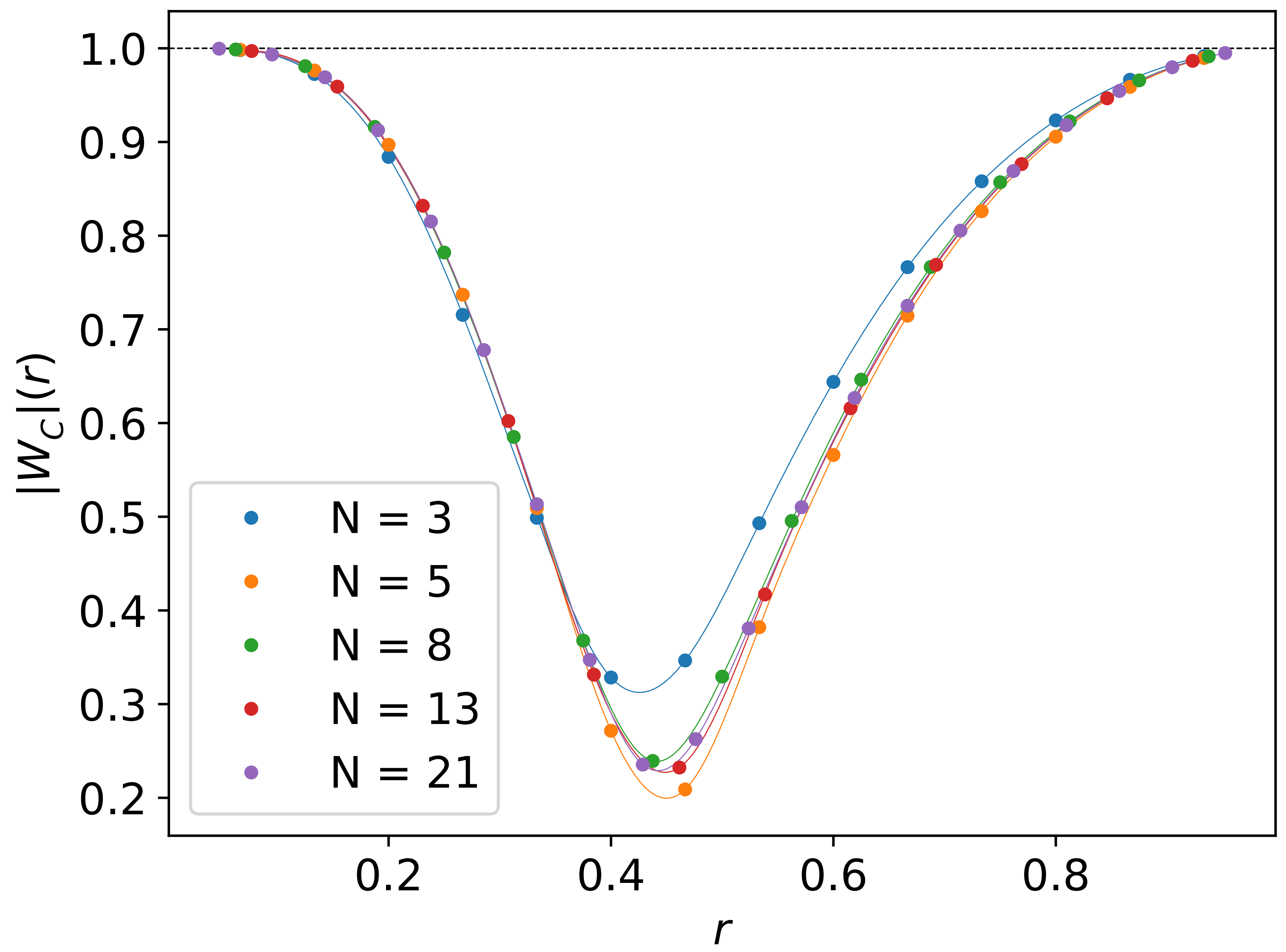
WILSON LOOPS

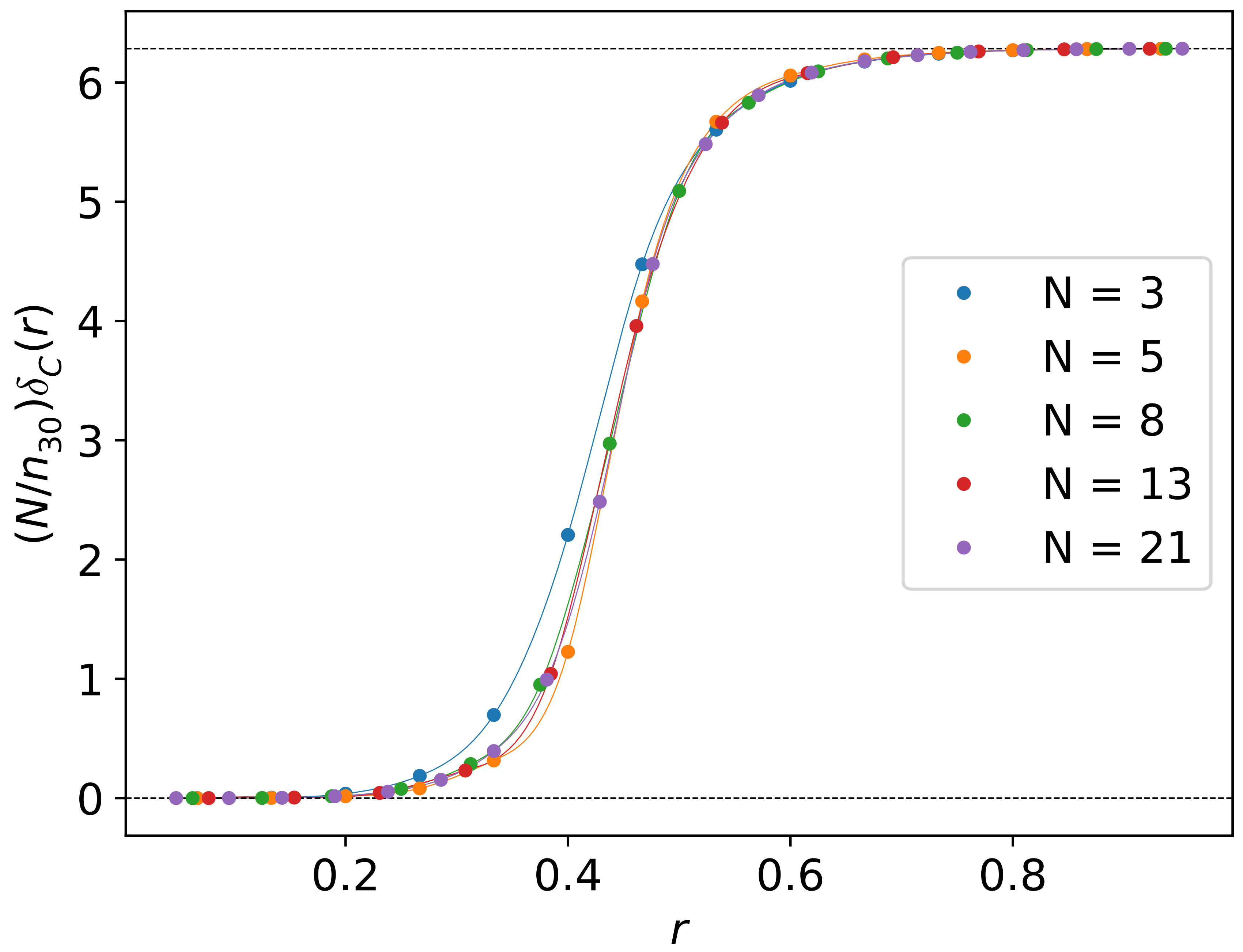
We will denote $W_C(r)$ the Wilson loop defined as

$$W_C(r) = \frac{1}{N}\mathrm{Tr}\Big(P\exp\Big\{-i\int_C dx_\mu A_\mu(x)\Big\}\Big) \equiv |W_C(r)|\, e^{i\delta_C(r)}$$

C A $T \times R$ SQUARE LOOP WITH

$T = R = r$ IN THE 03 PLANE





Wilson loop around the fractional instanton encloses a Z_N flux

- We have obtained numerical instanton-like solutions for gauge group SU(N) and **fractional topological charge** $Q = 1/N$. They have been obtained on a 4-torus with TBC and considering the number of colours and the magnetic flux N and m as the nth and $nth - 2$ integers in the **Fibonacci sequence**.
- The resulting configurations scale in the large N limit in agreement with the $\mathbf{R} \times \mathbf{T}^3$ Hamiltonian limit, representing **vacuum-to-vacuum tunneling** events. Action densities become independent of the twisted coordinates (x_1, x_2) and are localized in x_0
- The scaling of various other physical quantities in the large N limit has been analyzed, including **Polyakov loop operators** (showing how the fractional instantons interpolates between two flat connections) and **Wilson loop operators** (which is non-trivial and at large distance carries a $\mathbf{Z}_N$ flux)

THANK YOU FOR YOUR ATTENTION

QUESTIONS?

BACKUP SLIDES

PERTURBATIVE EXPANSION

The gauge fields satisfy TBC, fixing periodicity as $A_i(x + l_j \hat{e}_j) = \Gamma_j A_i(x_0, \vec{x}) \Gamma_j^\dagger$, and a Fourier expansion based on this boundary conditions is the following:

$$A_i(x_0, \vec{x}) = \frac{1}{\sqrt{l_1 l_2 l_3}} \sum_{\vec{p}}{}' \hat{A}_i(x_0, \vec{p}) e^{i\vec{p}\cdot\vec{x}} \widehat{\Gamma}(\vec{p})$$

THE NEW GENERATORS OF THE LIE ALGEBRA:

$$\widehat{\Gamma}(\vec{p}) = \frac{1}{\sqrt{2N}} e^{i\alpha(\vec{p})} \Gamma_1^{-\bar{m}n_2} \Gamma_2^{\bar{m}n_1}$$

In this expression, momenta is quantized in all three spatial directions as $p_i = 2\pi n_i / l$ and the prime in the sum indicates the exclusion of the cases where both n_1 and n_2 are equal to zero mod N.

The 3-dimensional box has the same effective size l in all three spatial directions.

SOLUTION ON T^4

We can contract solutions with constant curvature that become self-dual for certain values of the torus aspect ratios, an we can think our solutions as small perturbations of this construction.

DECOMPOSE THE THEORY INTO TWO SECTORS:

$$N = N_1 + N_2$$
$$n_{\mu\nu} = n^{(1)}_{\mu\nu} + n^{(2)}_{\mu\nu}$$

THE GAUGE FIELD AND FIELD STRENGTH ARE:

$$A_\mu(x) = \pi \frac{\Delta_{\nu\mu}}{N l_\mu l_\nu} x_\nu T \text{ AND } F_{\mu\nu}(x) = 2\pi \frac{\Delta_{\mu\nu}}{N l_\mu l_\nu} T$$

With our selection of TBC, this solutions become self-dual for the following torus aspect ratios:

$$l_0 l_3 = \frac{F_{n-m+1} F_{n-m}}{F_n^2 F_m F_{m-1}} \rightarrow l_0 l_3 = \frac{\varphi^{1-2m}}{F_m F_{m-1}} \text{ for } [n-m \rightarrow \infty]$$

WE HAVE SET SHORT DIRECTIONS TO SCALE AS

$$l_1 = l_2 = 1/F_n$$

SOLUTION ON $\mathbb{T}^4$ AND DEFORMATIONS