Topological susceptibility, scale setting and universality from $Sp(N_c)$ gauge theories

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$$\mathrm{Sp}(N_c) = \left\{ U \in \mathrm{SU}(N_c) \mid \Omega U \Omega^T = U^* \right\}, \qquad \Omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} \tag{1}$$

- BSM: Attractive Composite Higgs models based on $\mathrm{Sp}(N_c)$ gauge symmetry.

  Bennett et al. 2018; Ferretti and Karateev 2014

- Large-N_c: Non-trivial alternative to the $\mathrm{SU}(N_c)$ and $\mathrm{SO}(N_c)$ families of gauge groups

  Lovelace 1982; 't Hooft 1974

- SIMP: Strongly Interacting Dark Matter

  Hochberg et al. 2015; Kulkarni et al. 2022

- Topological structure of the vacuum?

The topological structure of the $\mathrm{Sp}(N_c)$ vacuum

Solutions with finite action describe semiclassical barrier penetration between different sectors. The "true" vacua are linear combinations of Q-vacua

$$|\theta\rangle = \sum_Q e^{iQ\theta}|Q\rangle, \qquad Q = \int \mathrm{d}^4x\; q(x)\;, \qquad q(x) = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}F^{\rho\sigma}$$

- We have that $Q \in \pi_3(\mathrm{Sp}(N_c)) = \mathbb{Z}$.
- We can define

$$\tilde{S} = -\frac{N_c}{2\lambda}\int \mathrm{d}^4x\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - N_c\frac{\theta}{N_c}\int \mathrm{d}^4x\; q(x), \qquad \lambda = g^2 N_c \tag{2}$$

The θ dependence in gauge theories has attracted a lot of interest especially at large-N_c.

- The $U(1)_A$ problem and the Witten-Veneziano formula: topological susceptibility.

  'T Hooft 1976; Veneziano 1979; Witten 1979
- The strong-CP problem
- θ dependence of observables like glueballs masses...

Bonanno et al. 2022

Theta dependence in gauge theories

General arguments dictate that the free energy $F(\theta)$ should be 2π periodic, even and have a minimum at $\theta = 0$.

$$F(\theta) = f_G \min_k h\left(\frac{\theta + 2\pi k}{N_c}\right) \qquad k = 0, ..., N_c - 1 \tag{3}$$

Witten 1980, 1998

In the neighbourhood of $\theta = 0$,

$$F(\theta) - F(0) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots) \tag{4}$$

Bonati et al. 2016

where

$$\chi = \left.\frac{\partial^2 F(\theta)}{\partial\theta^2}\right|_{\theta=0} = \int \mathrm{d}^4x \, \langle q(x)q(0)\rangle \tag{5}$$

is the topological susceptibility.

- The evaluation of χ is difficult on the lattice: UV fluctuations, additive renormalizations
- The gradient flow will allow us to both obtain χ and set the scale

The Gradient Flow

The gradient flow $B_\mu(x,\,t)$ is defined by

$$\frac{\mathrm{d}B_\mu(x,\,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,\,t), \qquad G_{\mu\nu}(t) = [D_\mu,\, D_\nu]\,, \qquad D_\mu \equiv \partial_\mu + [B_\mu,\,\cdot\,] \tag{6}$$

where the independent variable t is known as *flow time*, and $B_\mu(x,\,0) = A_\mu(x)$.

Lüscher 2010, 2014

- $B_\mu(x,t)$ is a renormalized field, dimensional physical quantities can be computed at $t > 0$. For example,

$$E(t) = \frac{1}{4}\mathrm{Tr}\,G_{\mu\nu}(t)G_{\mu\nu}(t) \propto \frac{\alpha(\mu)}{t^2}, \tag{7}$$

  where $\alpha(\mu)$ is the renormalized coupling at scale $\mu = 1/\sqrt{8t}$.

- $B_\mu(x,\,t)$ is is a smoothening of $A_\mu(x)$ and drives it towards the classical minima. Then

$$q(x,\,t) = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,G_{\mu\nu}(t)G_{\rho\sigma}(t), \tag{8}$$

  will be free of the UV-fluctuations that make the computation of Q difficult.

Scale Setting from the Wilson Flow

$$\mathcal{E}(t) = t^2 E(t), \qquad \mathcal{W}(t) = t\frac{d}{dt}\left\{t^2\langle E(t)\rangle\right\} , \tag{9}$$

Borsanyi et al. 2012

We will use two different scales t_0 and w_0, defined as

$$\mathcal{E}(t)|_{t=t_0} = \mathcal{E}_0\,, \qquad \mathcal{W}(t)|_{t=w_0^2} = \mathcal{W}_0\,, \tag{10}$$

and $\mathcal{E}_0$, $\mathcal{W}_0$ are reference values, chosen at convenience.

Note that:

- t_0 captures physics at scales $< \sqrt{t_0}$, w_0 captures physics at scales $\sim \sqrt{t_0}$.
- At leading order in $\lambda = 4\pi N_c \alpha$,

$$\mathcal{E}(t) = \frac{3\lambda}{64\pi^2} C_2(F), \quad C_2(F) \text{ quadratic Casimir of F representation} \tag{11}$$

this suggests a scaling law relating $\mathcal{E}(t)$ in different gauge groups.

The numerical setup

Ensembles of configurations of $Sp(N_c)$ pure gauge theories were collected for:

- $N_c = 2,\ 4,\ 6,\ 8.$
- Heat Bath + Over Relaxation updates à la Cabibbo-Marinari.
- $(\beta,\ V)$, $V = (La)^4$, chosen to avoid Finite Size Effects.

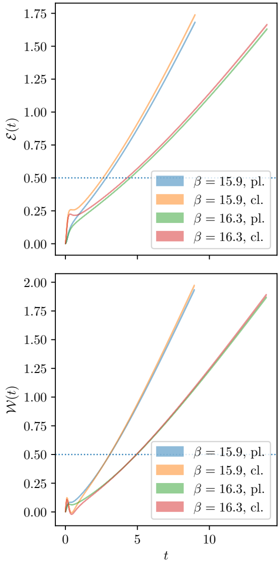
Then, for each ensemble,

- Each configuration was set as the initial condition for the numerical integration of the Wilson Flow
- $\mathcal{E}(t)$, $\mathcal{W}(t)$, $q_L(t)$ were computed on the interval $0 < t < L^2/32$.
- Clover expression was used for $q_L(t)$, both clover and tree level were used for $\mathcal{E}(t)$ and $\mathcal{W}(t)$.

N_c	L/a	β	$\tilde{\lambda}$ (tadpole imp.)
2	20 – 32	2.55 – 2.70	~ 2.6
4	20 – 24	7.7 – 8.2	~ 2.8
6	16 – 20	15.75 – 16.3	~ 2.9
8	16	26.5 – 27.2	~ 3.0

$$\tilde{\lambda} \equiv \frac{d_G}{\beta} \left\langle \frac{\Re \mathrm{Tr}\, \mathcal{P}_{\mu\nu}}{2N} \right\rangle , \quad d_G = n(2n+1) \quad (12)$$

The Wilson Flow – $N_c = 6$



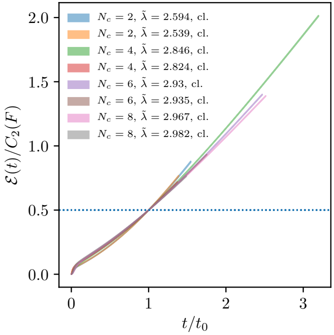
For each value of the coupling, we integrate the flow equations numerically with a third-order Runge-Kutta.

- The quantitites $\mathcal{E}(t)$ and $\mathcal{W}(t)$ are obtained using two different discretizations for$G_{\mu\nu}$: plaquette, clover, ...
- The scales can be set from

$$\mathcal{E}(t)|_{t=t_0} = \mathcal{E}_0, \qquad \mathcal{W}(t)|_{t=w_0^2} = \mathcal{W}_0, \tag{13}$$

- The reference values $\mathcal{E}_0$ and $\mathcal{W}_0$ are a priory arbitrary.

Scaling of the flow as $N_c \to \infty$



Perturbatively,

$$\mathcal{E}(t) = \frac{3\lambda}{64\pi^2} C_2(F), \tag{14}$$

where

$$C_2(F) = \frac{N_c + 1}{4} \tag{15}$$

for $\mathrm{Sp}(N_c)$.

- It is natural to scale the reference values as follows,

$$\mathcal{E}(t) = c_e C_2(F), \qquad \mathcal{W}(t) = c_w C_2(F) \tag{16}$$

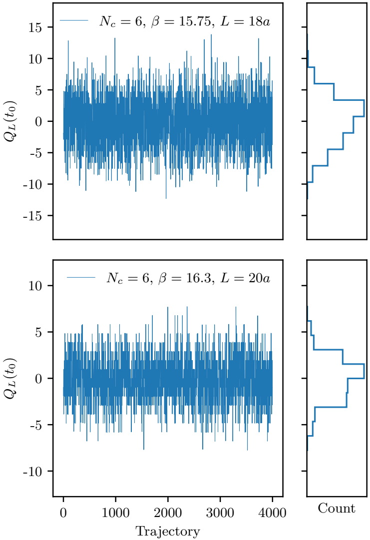
  and this suggests to rescale the flow time accordongly,

$$t \longrightarrow t/t_0 \tag{17}$$

- This allows us to take $N_c \to \infty$ at fixed λ.

- Effects from higher order terms...?

The lattice topological charge



- The topological charge is defined as

$$Q_L(t) = \sum_x q_L(x,\, t) \tag{18}$$

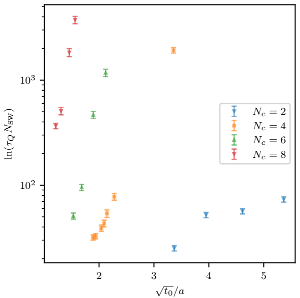
- We use the integer α-rounding to obtain quasi-integer values for Q_L,

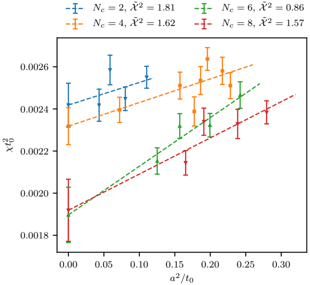
$$\tilde{Q}_L(t) \equiv \mathrm{round}\left(\tilde{\alpha} \sum_x q_L(x,\, t)\right), \tag{19}$$

where $\tilde{\alpha}$ is determined by minimising

$$\Delta(\tilde{\alpha}) = \left\langle [\tilde{\alpha} Q_L - \mathrm{round}\,(\tilde{\alpha} Q_L)]^2 \right\rangle . \tag{20}$$

The topological susceptibility in the continuum limit





The topological susceptibility can be obtained as

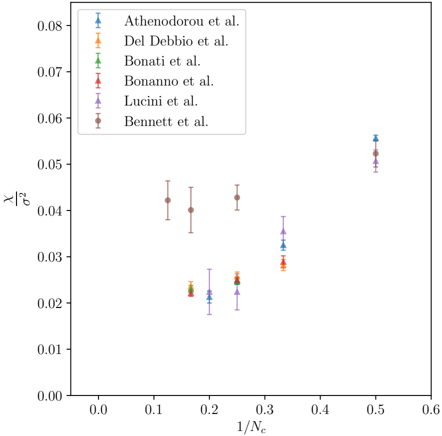
$$\chi_L a^4 = \frac{\langle Q_L^2 \rangle}{L^4} \tag{21}$$

- χ_L was computed for every ensemble in units of the lattice spacing.
- As observed for $SU(N)$ gauge theories, τ_Q diverges exponentially as $a \to 0$.
- For each value of N_c, we obtain the continuum limit extrapolation of χt_0^2 from

$$\chi\, t_0^2(a) = \chi\, t_0^2(a=0) + c_t \frac{a^2}{t_0} \tag{22}$$

N_c	χt_0^2	χ/σ^2
2	0.00242(10)	0.0523(29)
4	0.002318(87)	0.0428(27)
6	0.00190(13)	0.0401(49)
8	0.00192(15)	0.0424(42)

A comparison with $SU(N)$ gauge theories

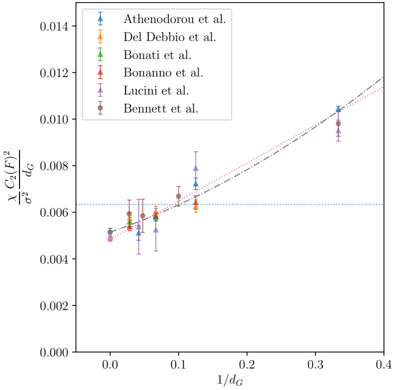


- The topological susceptibility has been computed for SU(N_c) by various collaborations over more than 20 years.
- We provide the first estimate of this quantity for Sp(N_c) gauge groups with $N_c = 2, 4, 6, 8$.

Note that:

- The susceptibilities coincide for Sp(2) $\simeq SU(2)$.
- For $N_c \geq 4$ they seem to tend to different limits.

A new universal ratio



- Once the data are rescaled with $C_2(F)/d_G$, they overlap. Define
$$\eta_\chi \equiv \frac{\chi}{d_G}\frac{C_2(F)^2}{\sigma^2},$$
- NDA yields $\eta_\chi = O\left(1/(4\pi)^2\right) \simeq 0.0065$ at $N_c \to \infty$
- As the data seem to lie on a straight line, we fit with
$$\eta_\chi(d_G) = \eta_\chi^\infty + \frac{c}{d_G} \tag{23}$$
and that yields
$$\eta_\chi^\infty = 0.004841(76), \quad \mathcal{X}_r^2 = 1.56 \tag{24}$$
- The same fit with only the $SU(N_c)$ data yields $\mathcal{X}_r^2 = 1.83$.
- Fitting with $1/d_G^2$ corrections does not change the extrapolation apreciably.

Thus our final estimate is
$$\lim_{N_c\to\infty} \eta_\chi = (48.41 \pm 0.76 \pm 3.05) \times 10^{-4} \tag{25}$$

Conclusion

- The Scale setting was performed for an interval of the inverse coupling for $N_c = 2,\ 4,\ 6,\ 8$.
- The scaling properties of the Wilson flow were analyzed and found to agree with the perturbative prediction.
- The first estimate of the continuum topological susceptibility was obtained for $N_c = 2,\ 4,\ 6,\ 8$
- A universal large-N_c limit was proposed for the topological susceptibility in Yang-Mills theories.

Thank you!

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Discretizations for $E(t)$ and $q_L(x, t)$

- For $E(t)$: Plaquette (pl.) or Clover (cl.) expressions

$$E(t) = \frac{1}{4}\text{Tr}\ V_{\mu\nu}(t)V_{\mu\nu}(t), \qquad E(t) = \frac{1}{4}\text{Tr}\ \mathcal{C}_{\mu\nu}(t)\mathcal{C}_{\mu\nu}(t), \tag{26}$$

Comparing the results obtained with these will allow use to estimate the magnitude of discretization effects.

- For $q_L(x, t)$: The Clover (cl.) expression

$$q_L(x, t) = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\text{Tr}\,\mathcal{C}_{\mu\nu}(t)\mathcal{C}_{\rho\sigma}(t) \tag{27}$$

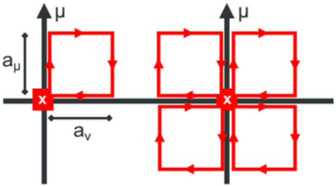

Figure: Left: Plaquette, Right: Clover, courtesy of Rothkopf 2021

We define the theory on a hypercubic euclidean space-time lattice, with Wilson action

$$S_{\mathrm{W}}[U_\mu] \equiv \beta \sum_x \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \Re \mathrm{Tr} \mathcal{P}_{\mu\nu}\right) , \qquad \beta \equiv \frac{2N_c}{g_0^2} \tag{28}$$

where

$$\mathcal{P}_{\mu\nu}(x) \equiv U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) , \qquad U_\mu(x) \equiv \exp\left(i \int_x^{x+\hat{\mu}} \mathrm{d}\lambda^\mu \tau^A A_\mu^A(\lambda)\right) , \tag{29}$$

We define the *Wilson* flow,

$$\frac{\partial V_\mu(x,\, t)}{\partial t} = -g_0^2 \left\{\partial_{x,\,\mu} S_{\mathrm{W}}\left[V_\mu\right]\right\} V_\mu(x,\, t) , \tag{30}$$

and the quantities $E(t)$ and $q_L(t)$ from their lattice discretizations.

A new universal ratio

It is believed that the theory is confining and gapped even at $\theta \neq 0$, and that

$$F(\theta) = f_G \min_k h\left(\frac{\theta + 2\pi k}{N_c}\right), \qquad k = 0, ..., N_c - 1 \tag{31}$$

Now:

- Each of the d_G gauge fields contributes equally to $F(\theta)$,

$$f_G \propto d_G \sim N_c^2 \tag{32}$$

- From perturbative arguments

$$\sigma \propto C_2(F) \tag{33}$$

  where the proportionality factor must also depend on N_c.

As a result, the following quantity should encode some universal feature and have a finite large-N_c limit,

$$\eta_\chi \equiv \frac{\chi}{d_G}\frac{C_2(F)^2}{\sigma^2}, \qquad \lim_{N_c\to\infty} \eta_\chi = b\frac{\chi_\infty}{\sigma_\infty} < \infty \tag{34}$$