

Electromagnetic finite-volume effects to leptonic decays through order $1/L^3$ in QED_L

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Vetenskapsrådet

Motivation

- M. Di Carlo's talk: Our RBC/UKQCD result is

$$\delta R_{K\pi} = -0.0088(39)$$

total	(39)
total (w/o FVE)	(13)
statistical	(3)
FVE	(37)
fit	(11)
QED quenching	(5)
discretisation	(5)

Error budget:

- FVEs most important
 - One volume: Correct analytically
-
- Focus on the analytical volume-dependence [Phys. Rev. D 105, 074509, 2022]
 - Motivate our error
 - First some generalities...

Finite-size effects

- QED_L : Exclude photon zero-mode on each time-slice [Hayakawa, Uno 2008]

$$\text{QED}_L : \sum_{\mathbf{k}} \longrightarrow \sum'_{\mathbf{k}} = \sum_{\mathbf{k} \neq 0} \quad \text{Non-local}$$

- Massless photon + no zero-mode

$V = \mathbb{R} \times L^3$: \implies Finite-size effects (FSEs) in observable $\mathcal{O}(L)$:

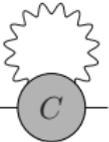
$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\text{IV}} = C_0 + C_{\log} \log m_P L + C_1 \frac{1}{m_P L} + C_2 \frac{1}{(m_P L)^2} + \dots$$

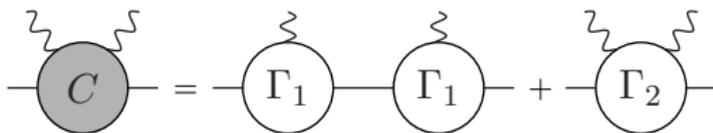
- **Scaling in L is observable-dependent:**
e.g. self-energy $C_0 = C_{\log} = 0$
- **Coefficients depend on physical particle properties:** masses, charges, structure (**form-factors**):
Point-like + **structure-dependent** \leftarrow Our work
- **At order $1/L^3$:** Non-locality of QED_L

Finite-size effects in QED_L

$$\begin{aligned}\Delta\mathcal{O}(L) &= \mathcal{O}(L) - \mathcal{O}_{\text{IV}} = \left(\frac{1}{L^3} \sum_{\mathbf{k}}' - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} f_{\mathcal{O}}(k_0, \mathbf{k}, \dots) \\ &= C_0 + C_{\log} \log m_P L + C_1 \frac{1}{m_P L} + C_2 \frac{1}{(m_P L)^2} + \dots\end{aligned}$$

- Method [Davoudi et al. 2014; Davoudi et al. 2019; Bijens et al. 2019]
- Singularities in $f_{\mathcal{O}}(k_0, \mathbf{k}, \dots)$: Expand $\mathbf{k} = 2\pi \mathbf{n}/L$
- Arbitrary order: Only need **sum-integral**: Numerical $c_j(\mathbf{v})$
- $f_{\mathcal{O}}(k_0, \mathbf{k}, \dots)$ including **structure-dependence**:

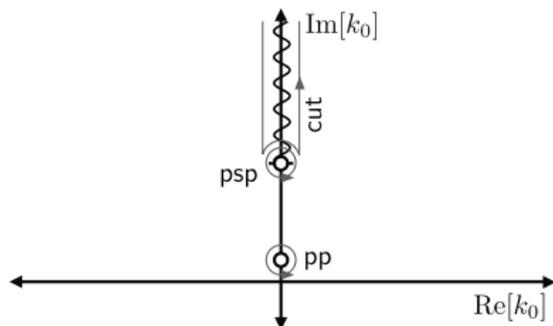
- 1 Define correlation function, e.g. 2-point function for mass 
- 2 Decompose into irreducible vertex functions: **Ward identities!**


$$\text{Diagram of } C = \text{Diagram of } \Gamma_1 \text{---} \Gamma_1 + \text{Diagram of } \Gamma_2$$

- 3 **Structure-dependence** from **form-factor decomposition**
- 4 The finite-volume expansion only contains **physical quantities**

Finite-size effects in QED_L

Mass:
Compton
scattering



$$\Delta\mathcal{O}(L) = \dots + C_2 \frac{1}{(m_P L)^2} + \left(C_3^{\text{pole}} + C_3^{\text{cut}} \right) \frac{1}{(m_P L)^3} + C_4 \frac{1}{(m_P L)^4} + \dots$$

- **Cut contribution:** From non-locality in QED_L : Integral to infinity (not in QED_C [Lucini, Patella, Ramos, Tantaló 2016])
- Interplay with off-shell effects in form factors: **Have to cancel!**
- **Structure-dependence** arises at order
 - $1/L^3$ for the **mass** (charge radius $\langle r_P^2 \rangle$)
 - $1/L^2$ for **leptonic decays** (axial vector form-factor F_A^P)

- Define the dimensionless FV function $Y^{(n)}(L)$ as

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\text{tree}} \left[1 + 2 \frac{\alpha}{4\pi} Y^{(n)}(L) \right] + \mathcal{O} \left(\frac{1}{L^{n+1}} \right)$$

$$Y^{(n)}(L) = Y_{\log} \log m_P L + Y_0 + \frac{1}{m_P L} Y_1 + \frac{1}{(m_P L)^2} Y_2 + \frac{1}{(m_P L)^3} Y_3 + \dots$$

- NB: $Y^{(1)}(L) = Y(L)$ of [RM-123/Soton, 2017] in different approach (universal)
- Given by Euclidean correlator for the decay $P^- \rightarrow \ell^- \nu_\ell$
- Structure-dependence in terms of form-factors from $1/L^2$ and on

Finite-size effects

- Diagrams give $Y^{(n)}(L)$ for $n = 2$ as

$$Y^{(2)}(L) = \frac{3}{4} + 4 \log \left(\frac{m_\ell}{m_W} \right) + \frac{c_3 - 2 c_3(\mathbf{v}_\ell)}{2\pi} - 2 A_1(\mathbf{v}_\ell) + 2 \log \left(\frac{m_W L}{4\pi} \right) \\ - 2 A_1(\mathbf{v}_\ell) \left[\log \left(\frac{m_P L}{4\pi} \right) + \log \left(\frac{m_\ell L}{4\pi} \right) \right] - \frac{1}{m_P L} \left[\frac{(1 + r_\ell^2)^2 c_2 - 4 r_\ell^2 c_2(\mathbf{v}_\ell)}{1 - r_\ell^4} \right] \\ + \frac{1}{(m_P L)^2} \left[- \frac{F_A^P}{f_P} \frac{4\pi m_P [(1 + r_\ell^2)^2 c_1 - 4 r_\ell^2 c_1(\mathbf{v}_\ell)]}{1 - r_\ell^4} + \frac{8\pi [(1 + r_\ell^2) c_1 - 2 c_1(\mathbf{v}_\ell)]}{(1 - r_\ell^4)} \right]$$

- Only F_A^P appears: Numerically negligible (1% effect on the point-like)
- Point-like agreement with RM-123/Soton: Different representations

$$c_3 = -\pi (4 + K_P - 4 \log 4\pi)$$

- The full $1/L^3$: Difficult so we did the point-like (see also [Tantalo et al. v2, 2016])

$$\frac{1}{(m_P L)^3} Y_3^{\text{pt}} = \frac{32\pi^2 c_0 (2 + r_\ell^2)}{(m_P L)^3 (1 + r_\ell^2)^3}$$

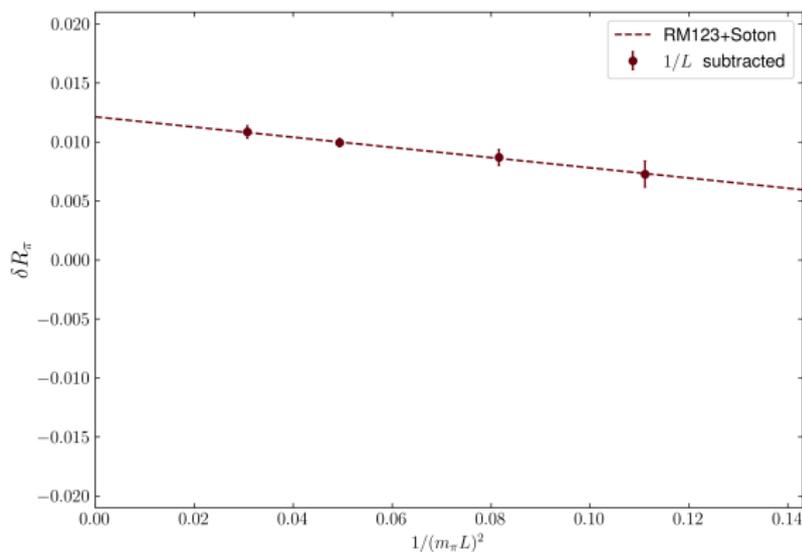
Comparison to lattice data from RM123/Soton

- $\delta R_{K\pi}$ calculated in [RM-123/Soton, 2019]

$$Y^{(n)}(L) = Y_{\log} \log m_P L + Y_0 + \sum_{j=1}^n \frac{1}{(m_P L)^j} Y_j + \mathcal{O} \left[\frac{1}{(m_P L)^{n+1}} \right]$$

- Fig. 9: Study volume-dependence in δR_K and δR_π by subtracting
 - * universal $Y^{(1)}(L)$
 - * point-like approximation $Y_{\text{pt}}^{(2)}(L)$
- Our result: Structure-dependence negligible, so $Y^{(2)}(L) \approx Y_{\text{pt}}^{(2)}(L)$
 \implies No $1/L^2$ -dependence in the data after subtracting $Y_{\text{pt}}^{(2)}(L)$
- Done at the unphysical masses $m_\pi \sim 320$ MeV and $m_K \sim 580$ MeV

Comparison to lattice data from RM123/Soton

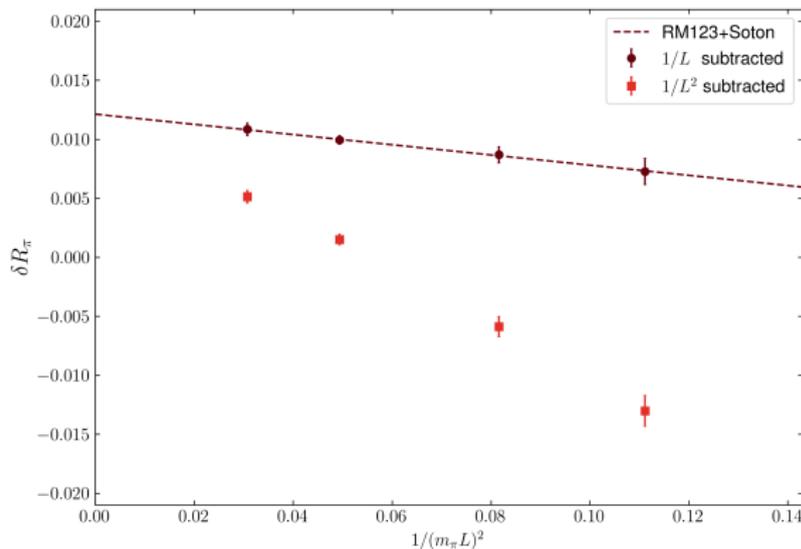


Subtracted **universal**
 $Y^{(1)}(L)$

Slope =
 $1/L^2$ -dependence

Plot from M. Di Carlo, Confinement 22

Comparison to lattice data from RM123/Soton



Plot from M. Di Carlo, Confinement 22

We now know **structure** negligible

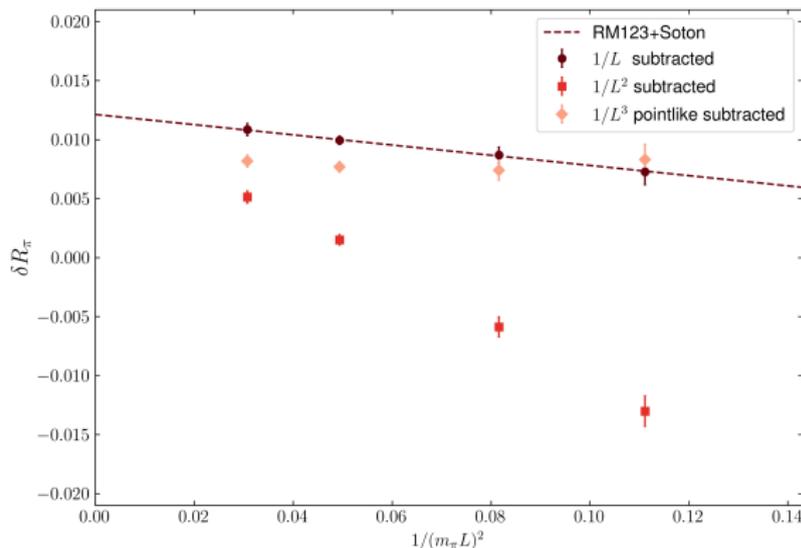
What is it?

2019: $Y_{SD}^{(2)}(L)$ not known

Subtracted **point-like**
 $Y_{pt}^{(2)}(L)$

Slope = $1/L^2 \implies$
structure-dependence?

Comparison to lattice data from RM123/Soton



Plot from M. Di Carlo, Confinement 22

- For pions of 320 MeV:

$$Y_\pi^{(2)}(L) : \frac{140}{(m_\pi L)^2}$$

$$Y_\pi^{(3), \text{pt}}(L) : \frac{-488}{(m_\pi L)^3}$$

Subtracted the
point-like $Y_{\text{pt}}^{(3)}(L)$

Big $1/L^3$ -dependence
looks like $1/L^2$

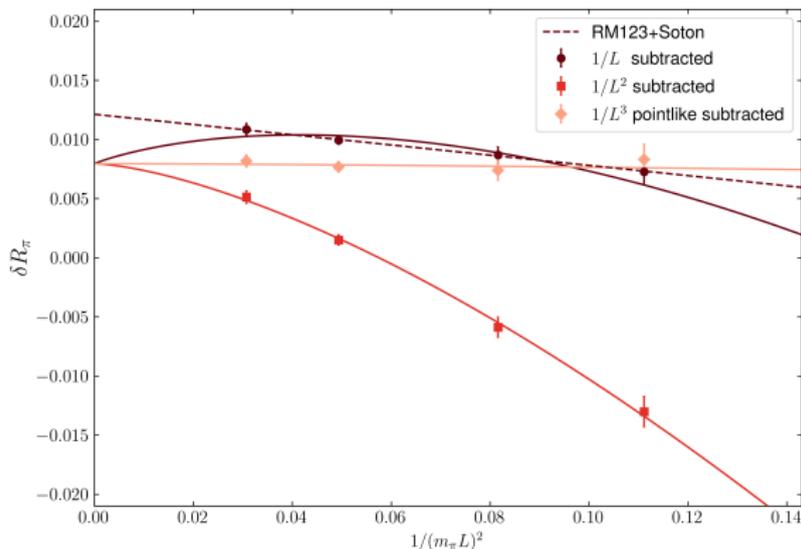
Comparison to lattice data from RM123/Soton

$$f^{(1)}(x = 1/(m_\pi L)) = a + b x^2 + (c^{\text{pt}} + c^{\text{SD+cut}}) x^3$$

$$f^{(2)}(x = 1/(m_\pi L)) = a + (c^{\text{pt}} + c^{\text{SD+cut}}) x^3$$

$$f^{(3)}(x = 1/(m_\pi L)) = a + c^{\text{SD+cut}} x^3$$

Plot from M. Di Carlo, Confinement 22



Analytical knowledge:
Fit a and $c^{\text{SD+cut}}$

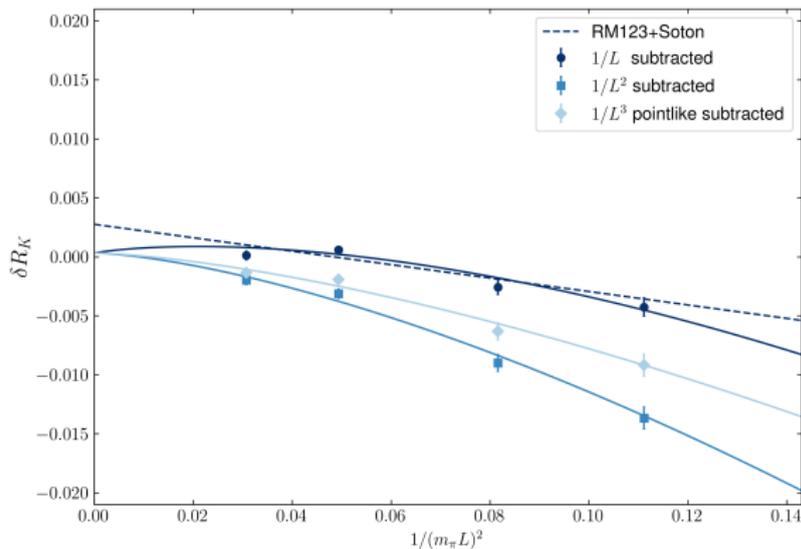
Add analytical c^{pt} and
 a, b for curves

Data well-described

Infinite-volume value
changes

Flatness: Cut and
structure-dependence
mild at $1/L^3$ for pions

Comparison to lattice data from RM123/Soton



Same for kaons

Plot from M. Di Carlo, Confinement 22

No flatness: Suggests cut and structure-dependence at $1/L^3$ for kaons

Conclusions and the way forward

- Leptonic decays: Sizeable point-like $1/L^3$ -coefficient
- Crucial to evaluate full $1/L^3$ -term or at least another volume
- We defined all necessary kernels to evaluate $1/L^3$, but difficult
- Analytical knowledge important for infinite-volume extrapolation
- The best we can do $\delta R_{K\pi}$: Size of point-like $1/L^3$ as systematic error

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total (w/o FVE)	(13)
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Back-up Slides

Decomposing vertex functions

- **Step 2:** Form-factor decomposition (**structure-dependence!**)

$$\Gamma_\mu(p, k) = (2p + k)_\mu F(k^2, (p + k)^2, p^2) + k_\mu G(k^2, (p + k)^2, p^2)$$

- Contains both on-shell and off-shell dependence

$$F^{(1,0,0)}(0, -m_P^2, -m_P^2) \equiv F'(0) = -\langle r_P^2 \rangle / 6$$

- $F^{(0,0,n)}(0, -m_P^2, -m_P^2)$: **Unphysical derivative!** \rightarrow **Must always cancel in the end!**
- How must they cancel, and what about $G(k^2, (p + k)^2, p^2)$?

Decomposing vertex functions

- **Step 3:** Use Ward identities, e.g.

$$k_\mu \Gamma^\mu(p, k) = D(p+k)^{-1} - D(p)^{-1}$$

- Define full propagator ($Z(p^2)$): z_n [BMW 2015; RM-123/Soton 2017]

$$D(p) = \frac{Z(p^2)}{p^2 + m_P^2}$$

- Ward identity yields G as a function of F and

$$F(0, p^2, -m_P^2) = F(0, -m_P^2, p^2) = Z(p^2)^{-1}$$

- Example relation: $z_1 = F^{(0,0,1)}(0, -m_P^2, -m_P^2)$

- **Unphysical derivative!** → **Must always cancel in the end!**

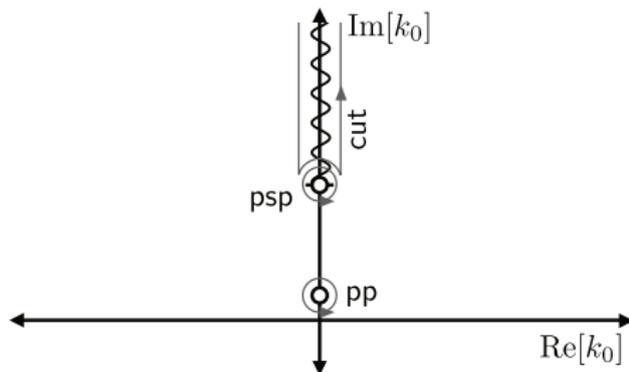
- **Equivalently:** We could put all non-physical quantities to zero directly

$$F(k^2, (p+k)^2, p^2) \rightarrow F(k^2) = 1 + k^2 F'(0) + \dots$$

$$Z(p^2) \rightarrow 1$$

The k_0 -integral

Mass:
Compton
scattering



- The poles are not enough! **Branch-cut** on the imaginary axis

$$\int \frac{dk_0}{2\pi} = \sum_{\text{poles}} + \int_{\text{cut}} \frac{dk_0}{2\pi}$$

- Smooth function on cut: add/subtract zero-mode $\mathbf{k} = 0$ in sum

$$\left(\frac{1}{L^3} \sum_{\mathbf{k}}' - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int_{\text{cut}} \frac{dk_0}{2\pi} \stackrel{\text{Poisson}}{=} - \frac{1}{L^3} \int_{\text{cut}} \frac{dk_0}{2\pi} \Big|_{\mathbf{k}=0} + \mathcal{O}(e^{-m_P L})$$

- **Branch-cut**: Specific $1/L^3$ term from QED_L prescription

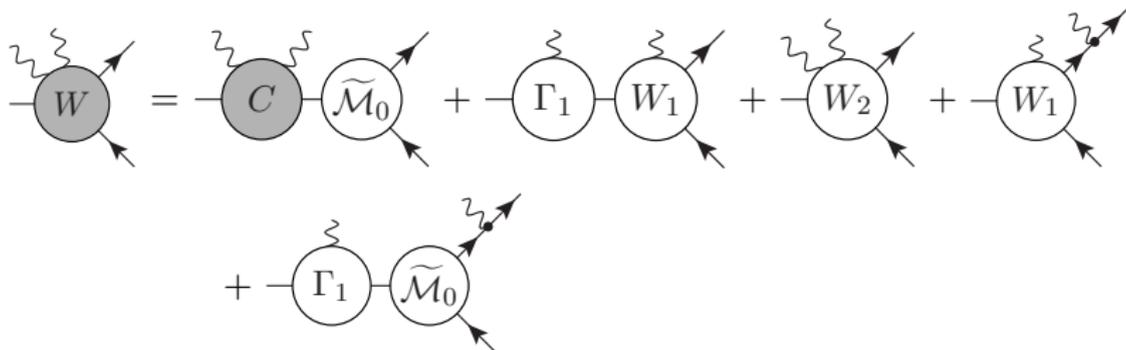
Finite-size effects in the mass

- Can use our knowledge of the Compton scattering amplitude decomposition to give $\Delta m_P^2(L)$ (c_j finite-size coefficients)

$$\Delta m_P^2(L) = e^2 m_{P,0}^2 \left\{ \frac{c_2}{4\pi^2 m_{P,0} L} + \frac{c_1}{2\pi (m_{P,0} L)^2} + \frac{\langle r_P^2 \rangle c_0}{3m_{P,0} L^3} + \frac{\mathcal{C}}{(m_{P,0} L)^3} + \mathcal{O} \left[\frac{1}{(m_{P,0} L)^4}, e^{-m_{P,0} L} \right] \right\}$$

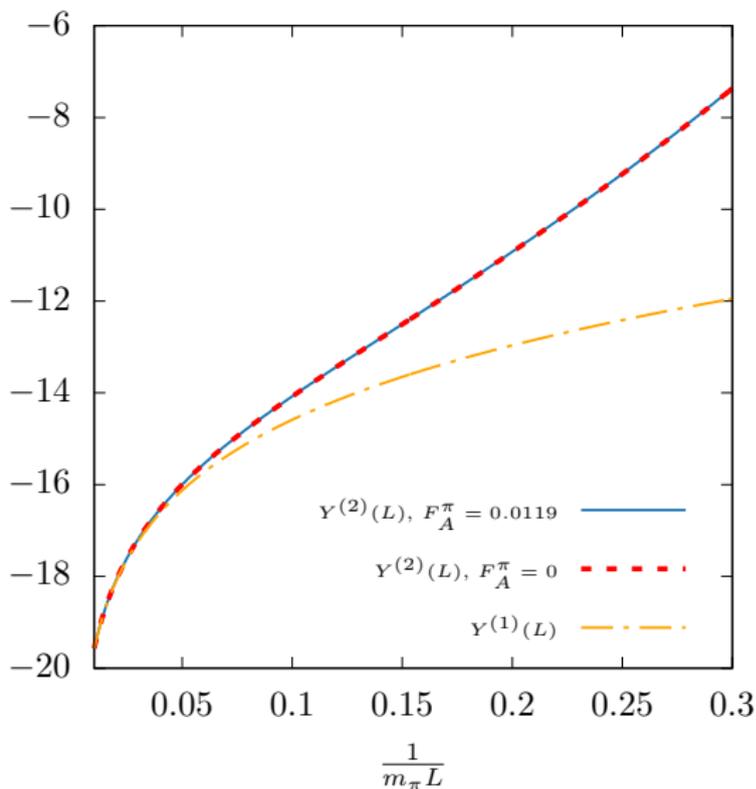
- Leading two terms: point-like ([Davoudi, Savage 2014; BMW 2015; RM-123/Soton 2017])
- Structure-dependence the same as in NRsQED! [Davoudi, Savage 2014]
- Branch-cut: Specific to QED_L (not in QED_C [Lucini, Patella, Ramos, Tantalo 2016])
- Need \mathcal{C} to make a prediction:
 - 1 Defined in terms of Compton tensor integrated to infinity
 - 2 Can cancel other contributions at order $1/L^3$
 - 3 For the mass: $\mathcal{C} > 0$

Leptonic decays



- W_1 and W_2 depend on unphysical off-shell derivatives of the decay constant:
 f_n [RM-123/Soton 2017]
- W_1 : $A_1(k^2, (p+k)^2)$, $V_1(k^2, (p+k)^2)$, $H_{1,2}(k^2, (p+k)^2)$: appear in $P^- \rightarrow \ell^- \nu_\ell \gamma^{(*)}$
 - On-shell: $F_A^P = A_1(0, -m_P^2)$ and $F_V^P = V_1(0, -m_P^2)$
 - Known from chiral perturbation theory [Bijnens, Ecker, Gasser 1992], lattice [RM-123/Soton 2020], experiment [...] (**Discrepancies** [RM-123/Soton 2020])
- W_2 : Structure-dependence starting at $1/L^4$ from $P^- \rightarrow \ell^- \nu_\ell \gamma^{(*)} \gamma^{(*)}$

Numerical results: Physical Pion



- The $1/L^2$ -correction is sizeable
- **NB:** Point-like $1/L^2$ completely dominates