# Inclusive semi-leptonic $B_{(s)}$ mesons decay at the physical $b$ quark mass

**Alessandro Barone**

in collaboration with

Andreas Jüttner, Shoji Hashimoto,
Takashi Kaneko, Ryan Kellermann

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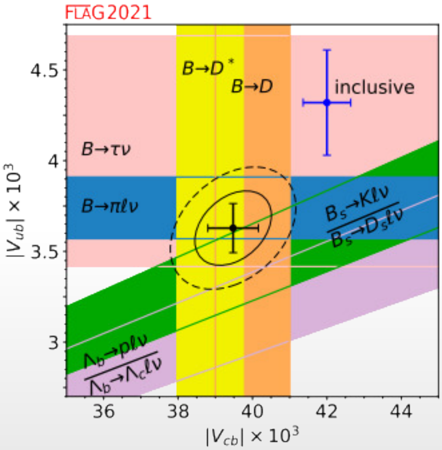
# Introduction and motivations




[Aoki et al. (2021)[1]]

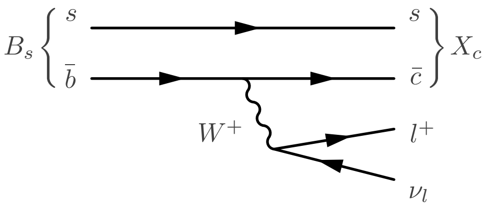
- $\sim 3\sigma$ discrepancy (in the plot) between inclusive/exclusive determination;
- lattice QFT represents a fully non-perturbative theoretical approach to QCD;
- no current predictions from lattice QCD for the inclusive decays.

This talk: Pilot study $B_s \to X_c$

- improve existing strategies for inclusive decays on the lattice;
- compare two different methods for the analysis.

$\rightarrow$ see also **Ryan Kellermann**'s talk

# Differential decay rate




**Decay rate:** $$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\mathrm{d}q_0\mathrm{d}E_l} = \frac{G_F^2|V_{cb}|^2}{8\pi^3}\boxed{L_{\mu\nu}}\boxed{W^{\mu\nu}},$$

Leptonic tensor

Hadronic tensor

$$\boxed{W^{\mu\nu}} = \sum_{X_c}(2\pi)^3\delta^{(4)}(p-q-r)\frac{1}{2E_B}\langle B_s(\boldsymbol{p})|\, J^{\mu\dagger}(-\boldsymbol{q})\,|X_c(\boldsymbol{r})\rangle\,\langle X_c(\boldsymbol{r})|\, J^{\nu}(\boldsymbol{q})\,|B_s(\boldsymbol{p})\rangle\,.$$

contains all the **non-perturbative** QCD

# Total decay rate

$$\boxed{\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \, \sqrt{\boldsymbol{q}^2} \, \bar{X}}\,,$$

kinematics

$$\boxed{\bar{X}} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, \boxed{k_{\mu\nu}} \times \boxed{W^{\mu\nu}}\,, \qquad \omega = E_{X_c}\,.$$

portal to compute the $\Gamma/|V_{cb}|^2$

from lattice?

# Inclusive decays on the lattice

[Hashimoto (2017)[2], Gambino and Hashimoto (2020)[3]]

We need the non-perturbative calculation of the hadronic tensor

$$W^{\mu\nu}(\boldsymbol{q},\omega) \sim \sum_{X_c} \langle B_s| J^{\mu\dagger} |X_c\rangle \langle X_c| J^{\nu} |B_s\rangle \,.$$

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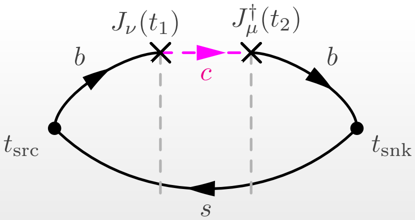
On the lattice, this is achieved with a **4pt correlation function**:




- $t_{\rm src}$, $t_2$, $t_{\rm snk}$ fixed
- $t_{\rm src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$
- $t$ small $\rightarrow$ signal-to-noise ratio deteriorate with t

$$\boxed{C^{\mu\nu}(t) \quad \leftrightarrow \quad \langle B_s|\tilde{J}^{\mu\dagger}(-\boldsymbol{q},0)\, e^{-t\hat{H}}\, \tilde{J}^{\nu}(\boldsymbol{q},0)|B_s\rangle} \, .$$

# Lattice data (Euclidean)

$$0 \quad 1 \quad 2 \quad \cdots \quad T \longrightarrow t/a$$

finite/discrete number of time-slices $t = -i\tau$

lattice data (correlation function)

$$\boxed{C(t)} = \int_0^\infty \mathrm{d}\omega \, \boxed{\rho(\omega)} \, e^{-\omega t}$$

spectral function: $\sum_j \langle 0 | \mathcal{O}^\dagger | j \rangle \langle j | \mathcal{O} | 0 \rangle \, \delta(\omega - E_j)$

$$\rho(\omega) \underset{\text{ill-posed problem}}{\overset{\text{trivial}}{\rightleftarrows}} C(t)$$

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$$
\begin{array}{ccccc}
0 & 1 & 2 & \cdots & T
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$$\rho(\omega) \underset{\text{ill-posed problem}}{\overset{\text{trivial}}{\rightleftarrows}} C(t)$$

Extracting the hadronic tensor is an ill-posed problem (**inverse problem**)

lattice data for inclusive

$$\boxed{C_{\mu\nu}(t)} = \int_0^\infty \mathrm{d}\omega \, \boxed{W_{\mu\nu}(\boldsymbol{q}, \omega)} \, e^{-\omega t}$$

$$\sum_{X_c} \langle B_s| J_\mu^\dagger |X_c\rangle \langle X_c| J_\nu |B_s\rangle \, \delta(\omega - E_{X_c})$$

# Decay rate from lattice data

kinematics factors

$$
\begin{aligned}
\bar{X} &= \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)} \\
&= \int_{\boxed{\omega_0}}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)} \longrightarrow \text{kernel operator} \\
&= \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)
\end{aligned}
$$

$0 \leq \omega_0 \leq \omega_{\min} \longleftarrow \omega_0$

# Decay rate from lattice data

kinematics factors

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} d\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)}$$

$$= \int_{\boxed{\omega_0}}^{\infty} d\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)} \rightarrow \text{kernel operator}$$

$$0 \le \omega_0 \le \omega_{\min}$$

$$= \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)$$

Can we trade

$$\boxed{C^{\mu\nu}(t)} = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu} e^{-\omega t} \quad \leftarrow \textbf{?} \rightarrow \quad \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)$$

lattice data

# Decay rate from lattice data

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega\, W^{\mu\nu} \underbrace{\boxed{k_{\mu\nu}(\boldsymbol{q},\omega)}}_{\text{kinematics factors}}$$

$$= \int_{\boxed{\omega_0}}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)} \rightarrow \text{kernel operator}$$

$$0 \le \omega_0 \le \omega_{\min}$$

$$= \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)$$

We can approximate in $K_{\mu\nu}$ in power of $e^{-a\omega}$ (here $a = 1$ in lattice units)

$$K_{\mu\nu} \simeq c_{\mu\nu,0} + c_{\mu\nu,1} e^{-\omega} + \cdots + c_{\mu\nu,N} e^{-\omega N},$$

$$\Rightarrow \bar{X} \simeq c_{\mu\nu,0} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu}}_{C^{\mu\nu}(0)} + c_{\mu\nu,1} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} e^{-\omega}}_{C^{\mu\nu}(1)} + \cdots + c_{\mu\nu,N} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} e^{-\omega N}}_{C^{\mu\nu}(N)}$$

# Polynomial approximation strategies

$$K(\omega):\ [\boxed{\omega_0},\infty)\to\mathbb{R},\qquad K(\omega)\simeq\sum_j^N c_j\,\boxed{p_j(\omega)}\,.$$

$\omega_0 \in [0, \omega_{\min})$

family of polynomials in $e^{-\omega}$

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## Chebyshev approach

Standard Chebyshev polynomials:

$$T_j(\omega) : [-1, 1] \to [-1, 1] ,$$

generic shifted Chebyshev

$$K(\omega) \simeq \sum_{j=0}^N \tilde{c}_j \boxed{\tilde{T}_j(\omega)} ,$$

$$\tilde{c}_j = \langle K, \tilde{T}_j \rangle .$$

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$$\tilde{c}_j = \langle K, \tilde{T}_j \rangle .$$

## Backus-Gilbert approach

We minimize the functional ($L_2$-norm)

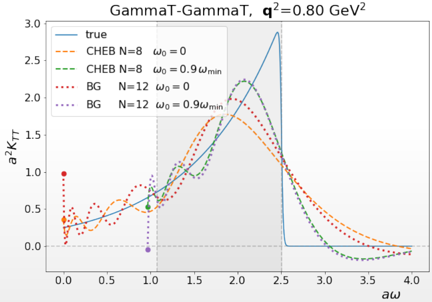
$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[ K(\omega) - \sum_{j=1}^N g_j e^{-j\omega} \right]^2 ,$$

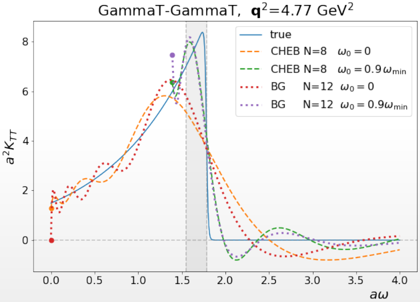
$$g_j \quad \leftrightarrow \quad \frac{\delta A}{\delta g_j} = 0 .$$

# Kernel: polynomial approximation

$$K_{\mu\nu}(\boldsymbol{q},\omega) = e^{2\omega t_0}\, k_{\mu\nu}(\boldsymbol{q},\omega)\, \boxed{\theta_\sigma\left(\omega_{\max}-\omega\right)} \rightarrow$$

smooth step-function (sigmoid): cut the unphysical states above $\omega_{\max}$

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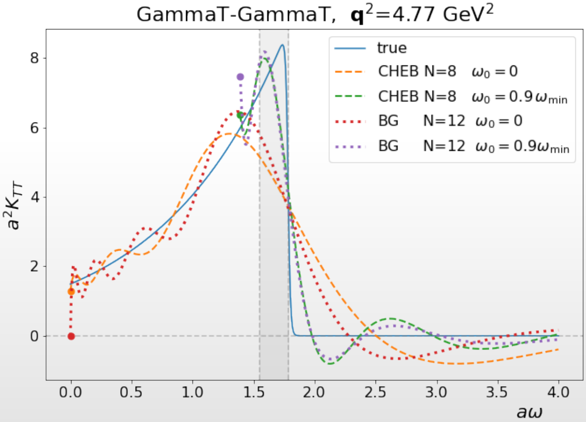

# Kernel: polynomial approximation

$$K_{\mu\nu}(\boldsymbol{q},\omega) = e^{2\omega t_0}\, k_{\mu\nu}(\boldsymbol{q},\omega)\, \boxed{\theta_\sigma\left(\omega_{\max} - \omega\right)} \rightarrow$$

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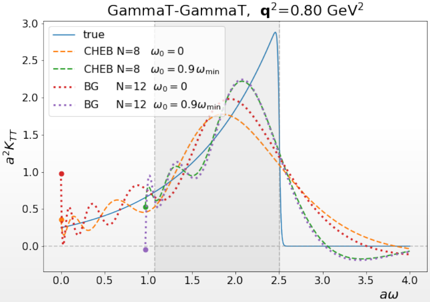


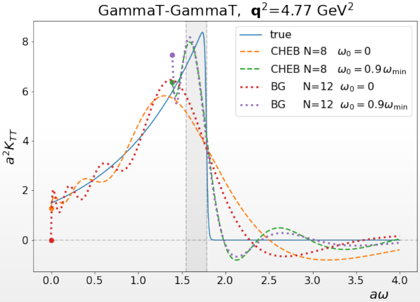


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# Analysis strategy

**Problem**: data too noisy, statistical errors add up!

$$\bar{X} \simeq c_{\mu\nu,0} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} p_0(\omega) + \cdots + c_{\mu\nu,N} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} p_N(\omega)$$

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Chebyshev approach
[Bailas et al. (2020)[4]]

$$\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \boxed{p_j(\omega)} \to \tilde{T}_j(\omega)$$

$|\tilde{T}_k| \leq 1$ so we normalize

$\Downarrow$

We can extract the Chebyshev components through a **Bayesian fit with constraints**!

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## Backus-Gilbert approach[5]

[Hansen et al. (2019)[6], Bulava et al. (2021)[7]]

$$\boxed{W_\lambda[g] = (1-\lambda)\frac{A[g]}{A[0]} + \lambda \frac{B[g]}{C_{\mu\nu}(0)^2}} \rightarrow \text{We minimise } W_\lambda[g]$$

systematic error

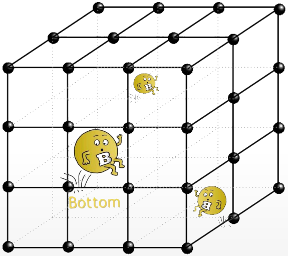
$$\boxed{A[g]} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[ K_{\mu\nu}(\omega, \boldsymbol{q}) - \sum_{j=1}^{N} g_{\mu\nu,j} e^{-j\omega} \right]^2$$

$$\boxed{B[g]} = \sum_{i,j=1}^{N} g_{\mu\nu,i} \mathrm{Cov}[C_{\mu\nu}(i), C_{\mu\nu}(j)] g_{\mu\nu,j}\,.$$

statistical error

# Inclusive decays on the lattice: setup

Simulations carried out on the DiRAC Extreme Scaling service at the University of Edinburgh using the **Grid**[Boyle et al.[8]] and **Hadrons**[Portelli et al.[9]] software packages




**Pilot study** with RBC/UKQCD 2+1 flavour ensembles [Allton et al. (2008)[10]]:

- lattice size: $24^3 \times 64$;
- lattice spacing $a \simeq 0.11\,\mathrm{fm}$;
- $M_\pi \simeq 330\,\mathrm{MeV}$.

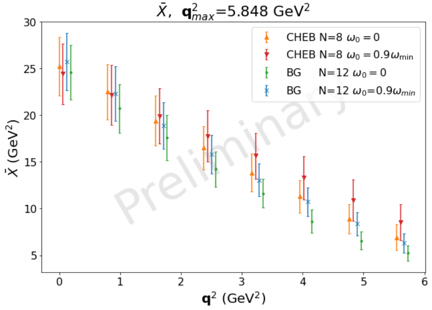
$\Downarrow$

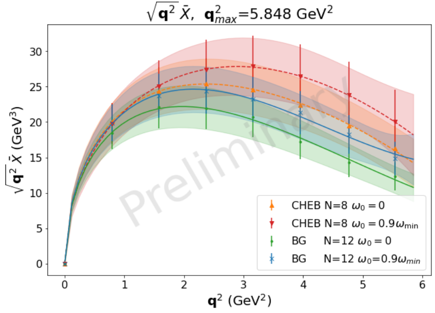
Limited statistics/qualitative results!

**Simulation**:

- $b$ quark simulated at its **physical** mass (RHQ action [El-Khadra et al. (1997)[11], Christ et al. (2007)[12], Lin and Christ (2007)[13]] );
- $s, c$ quarks simulated at **near-to-physical** mass (DWF action [Shamir (1993)[14], Furman and Shamir (1994)[15]]).

# Preliminary results and comparison




Key points:

- Chebyshev and Backus-Gilbert approaches are fully compatible;
- pilot study:
  - values are in the right ballpark (compared to $B$ decay rate, based on $SU(3)$ flavour symmetry);
  - low statistics, roughly $10\%$ error.

# Summary and outlook

**Summary**:

- promising prospects for inclusive decays on the lattice;
- double approach for the analysis: Chebyshev and Backus-Gilbert approaches compatible within error.

**Coming next**:

- continue to work towards understanding the systematics involved in solving the inverse problem;
- dedicated simulations to address the systematics for polynomial approximation ($\rightarrow$ see **Ryan Kellermann**'s talk), finite volume effects, continuum limit,...;
- prepare for a full study $B_s/B$.

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THANK YOU!

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[2] S. Hashimoto, "Inclusive semi-leptonic B meson decay structure functions from lattice QCD", Progress of Theoretical and Experimental Physics **2017**, 53–56 (2017), arXiv:1703.01881.

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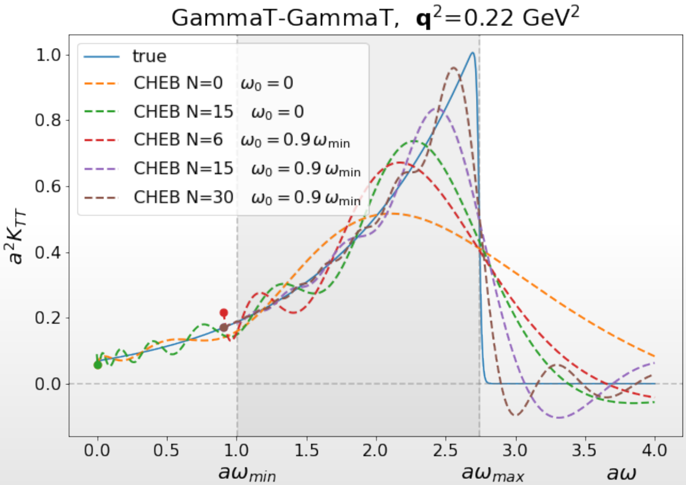
# Backup

BACKUP

# Chebyshev polynomial approximation: more



# Analysis strategy: Chebyshev

[Bailas et al. (2020)[4]]

$$\bar{X} \simeq \tilde{c}_{\mu\nu,0} \int_0^\infty \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_0(\omega) + \cdots + \tilde{c}_{\mu\nu,N} \int_0^\infty \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_N(\omega)$$

*Problem*: data too noisy, statistical errors add up!
Chebyshev polynomial are bounded $|\tilde{T}_k| \leq 1$, so we normalize

$$\int_0^\infty \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_j(\omega) \quad \rightarrow \quad \underbrace{\boxed{N_{\mu\nu}}}_{\text{normalization}} \; \underbrace{\boxed{\frac{\int_0^\infty \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_j(\omega)}{N_{\mu\nu}}}}_{\equiv \; \tilde{T}_j^{\mu\nu}}$$

$$|\tilde{T}_j^{\mu\nu}| \leq 1$$

We can extract $\tilde{T}_j^{\mu\nu}$ through a **Bayesian fit with constraints**

$$\Rightarrow \bar{X} \simeq \sum_{j=0}^{N} \tilde{c}_{\mu\nu,j} \boxed{N_{\mu\nu}} \boxed{\tilde{T}_j^{\mu\nu}}$$

# Analysis strategy: Backus-Gilbert

[Backus and Gilbert (1968)[5], Hansen et al. (2019)[6], Bulava et al. (2021)[7]]

Aside from the functional $A[g]$, which approximates the target function (kernel), we include some information on the data

$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[ K_{\mu\nu}(\omega, \boldsymbol{q}) - \sum_{j=1}^{N} g_j e^{-j\omega} \right]^2 ,$$

$$B[g] = \sum_{i,j=1}^{N} g_i \mathsf{Cov}[C_{\mu\nu}(i), C_{\mu\nu}(j)] g_j .$$

We minimise

$$\boxed{W_\lambda[g] = (1-\lambda)\frac{A[g]}{A[0]} + \lambda \frac{B[g]}{C_{\mu\nu}(0)^2}} .$$

The parameter $\lambda$ control the interplay between the 2 functionals, i.e. between **statistical** and **systematic** errors. It is chosen in a way to balance the to errors.

# Analysis strategy: more (1)

$$\bar{X} = \int_{\omega_0}^{\infty} W^{\mu\nu} K_{\mu\nu} \simeq \sum_{j}^{N} c_{\mu\nu,j} C^{\mu\nu}(j)$$

We can rewrite

$$C^{\mu\nu}(j) = \int_{\omega_0}^{\infty} W^{\mu\nu} e^{-\omega j} = \langle B_s | \, J^{\mu\dagger} \, e^{-t\hat{H}} \, J^{\nu} \, | B_s \rangle$$

$$\bar{X} \simeq \langle B_s | \, J^{\mu\dagger} \, K_{\mu\nu}(\boldsymbol{q}, \hat{H}) \, J^{\nu} \, | B_s \rangle = \langle \psi^{\mu} | \, K_{\mu\nu}(\boldsymbol{q}, \hat{H}) \, | \psi^{\nu} \rangle$$

# Analysis strategy: more (2)

$$\overset{\text{contains } e^{+2t_0\hat{H}}}{\phantom{x}}$$

$$\underbrace{\langle B_s|\,\tilde{J}^{\mu\dagger}(-\boldsymbol{q},0)}_{\langle\psi^\mu(\boldsymbol{q})|e^{-t_0\hat{H}}}\boxed{K_{\mu\nu}(\hat{H},\boldsymbol{q};t_0)}\underbrace{\tilde{J}^{\nu}(\boldsymbol{q},0)\,|B_s\rangle}_{e^{-t_0\hat{H}}|\psi^\nu(\boldsymbol{q})\rangle} \equiv \langle\psi_\mu|\,K_{\mu\nu}(\hat{H},\boldsymbol{q})\,|\psi_\nu\rangle\ ,$$

where we introduced $\boxed{e^{-t_0\hat{H}}}$ to avoid contact terms.

$$\boxed{\langle\psi^\mu|\,K_{\mu\nu}(\hat{H},\boldsymbol{q};t_0)\,|\psi^\nu\rangle \simeq \sum_{j=0}^{N} c_{\mu\nu,j}\,\langle\psi^\mu|\,e^{-\hat{H}j}\,|\psi^\nu\rangle = \sum_{j=0}^{N} c_{\mu\nu,j}\,C^{\mu\nu}(j+2t_0)}$$

$$\boxed{\bar{X} \simeq \langle\psi^\mu|\,K_{\mu\nu}(\hat{H},\boldsymbol{q};t_0)\,|\psi^\nu\rangle}$$

- $j \leftrightarrow t$: degree corresponds to a certain time-slice;
- $N$ is limited by the available data (choice of $t_2$) and the noise of the signal.

# Analysis strategy: Chebyshev (revisited)

For the Chebyshev polynomials we need to enforce the bounds $|\tilde{T}_k| \leq 1$.
We then normalize the previous quantity using $C^{\mu\nu}(2t_0) = \langle \psi^\mu | \psi^\nu \rangle$

$$\frac{\langle \psi^\mu | \, K_{\mu\nu}(\hat{H}, \boldsymbol{q}) \, | \psi^\nu \rangle}{\langle \psi^\mu | \psi^\nu \rangle} \simeq \frac{\tilde{c}_0}{2} + \sum_{j=1}^{N} \tilde{c}_j \boxed{\frac{\langle \psi^\mu | \, \tilde{T}_j(\hat{H}) \, | \psi^\nu \rangle}{\langle \psi^\mu | \psi^\nu \rangle}} \,, \qquad \tilde{T}_j^{\mu\nu} \equiv \left| \frac{\langle \psi^\mu | \, \tilde{T}_j(\hat{H}) \, | \psi^\nu \rangle}{\langle \psi^\mu | \psi^\nu \rangle} \right| \leq 1 \,.$$

Very noisy if built from data!
Extracted with a Bayesian fit with constraints

The value of $\bar{X}$ can be obtained as

$$\bar{X} \simeq \langle \psi^\mu | \psi^\nu \rangle \, \frac{\langle \psi^\mu | \, K_{\mu\nu}(\hat{H}, \boldsymbol{q}) \, | \psi^\nu \rangle}{\langle \psi^\mu | \psi^\nu \rangle} \,.$$