# Inclusive semi-leptonic decays of charmed mesons with Möbius domain wall fermions


**Ryan Kellermann**

in collaboration with

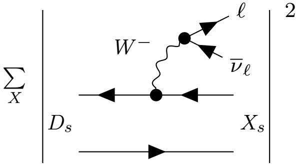
Alessandro Barone, Shoji Hashimoto, Andreas Jüttner and Takashi Kaneko

Graduate University for Advanced Studies SOKENDAI

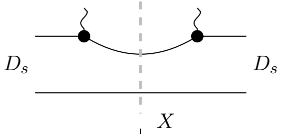

Lattice2022, August 12th, 2022

# Introduction and Motivation

Inclusive semileptonic decay rate




Lattice: **4pt functions**




The weight over $X$ has to be adjusted by

contributions from all possible states

$$\int d\omega_X \; K(\omega_X) \; \left[ \; \right]_{\text{Lattice}}$$

kinematics; requires some approximation → source of systematic error

**This work**

- Understand systematic errors
- Validate using experimental data (BESIII) [BESIII, arxiv.2203.04938]

See also

- Comparison of Chebyshev to Backus-Gilbert (A. Barone's talk)
- Inclusive $B$-meson decay [Gambino et al., arXiv:2203.11762] (A. Smecca's talk)

# Hadronic Tensor

contains all non-perturbative information

$$W^{\mu\nu}(p,q) = \sum_{X_s} (2\pi)^3 \delta^{(4)}(p-q-r) \times \frac{1}{2E_{D_s}} \langle D_s(\boldsymbol{p}) | \tilde{J}^{\mu\dagger}(-\boldsymbol{q}) | X_s(\boldsymbol{r}) \rangle \langle X_s(\boldsymbol{r}) | \tilde{J}^{\nu}(\boldsymbol{q}) | D_s(\boldsymbol{p}) \rangle$$

# Total Decay Rate [Gambino, Hashimoto,arXiv:2005.13730]

$$\Gamma = \frac{G_F^2 |V_{cs}|^2}{24\pi^3} \int_0^{\boldsymbol{q}^2_{\mathbf{max}}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\ \bar{X}$$

$$\begin{aligned} \bar{X} &= \int_{\omega_{\mathbf{min}}}^{\omega_{\mathbf{max}}} d\omega\ K_{\mu\nu}(\boldsymbol{q},\omega) \times W^{\mu\nu} \\ &= \int_0^\infty d\omega K_{\mu\nu}(\omega,\boldsymbol{q}) \langle D_s(0) | \tilde{J}^\dagger_\mu(-\boldsymbol{q}) \delta(\hat{H}-\omega) \tilde{J}_\nu(\boldsymbol{q}) | D_s(0) \rangle \\ &= \boxed{\langle D_s(\mathbf{0}) | \tilde{J}^\dagger_\mu(-\boldsymbol{q}) K_{\mu\nu}(\boldsymbol{q},\hat{H}) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\mathbf{0}) \rangle} \end{aligned}$$

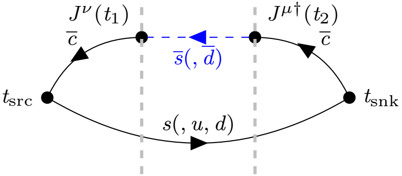
# Reconstruction from lattice data

lattice data

$$C^{JJ}_{\mu\nu}(t) \sim \langle D_s|\tilde{J}^\dagger_\mu(-\boldsymbol{q})e^{-\hat{H}t}\tilde{J}_\nu(\boldsymbol{q})|D_s\rangle$$

energy integral in incl. rate

$$\bar{X} = \langle D_s(\boldsymbol{0})|J^\dagger_\mu K_{\mu\nu}(\boldsymbol{q},\hat{H})J_\nu|D_s(\boldsymbol{0})\rangle$$




One can construct an approximation of $K(\boldsymbol{q},\hat{H})$ in terms of $e^{-\hat{H}}$

$$K(\boldsymbol{q},\hat{H}) = k_0 + k_1 e^{-\hat{H}} + \ldots + k_N e^{-N\hat{H}},$$

which then allows us to write

$$\bar{X} \sim k_0 \underbrace{\langle D_s|\tilde{J}^\dagger_\mu(-\boldsymbol{q})\tilde{J}_\nu(\boldsymbol{q})|D_s\rangle}_{C^{JJ}_{\mu\nu}(0)} + k_1 \underbrace{\langle D_s|\tilde{J}^\dagger_\mu(-\boldsymbol{q})e^{-\hat{H}}\tilde{J}_\nu(\boldsymbol{q})|D_s\rangle}_{C^{JJ}_{\mu\nu}(1)} + \ldots$$
$$+ k_N \underbrace{\langle D_s|\tilde{J}^\dagger_\mu(-\boldsymbol{q})e^{-\hat{H}N}\tilde{J}_\nu(\boldsymbol{q})|D_s\rangle}_{C^{JJ}_{\mu\nu}(N)}$$

# Chebyshev Approximation

Modified Chebyshev polynomials $T_j^*(e^{-\omega})$ to approximate $K(\omega)$ in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{\rm min}$

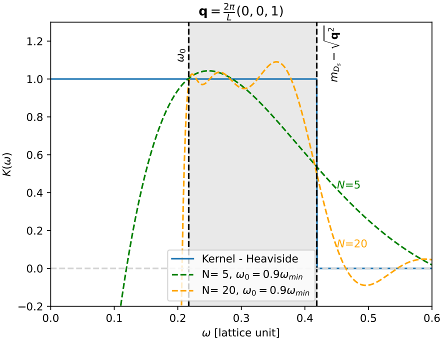
$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

$$T_1^*(x) = 2x - 1,\ T_2^*(x) = 8x^2 - 8x + 1,\ T_3^*(x) = 32x^3 - 48x^2 + 18x - 1,\ \ldots$$

Kernel function

$$K(\omega) = \boxed{\theta(m_{D_s} - \sqrt{\boldsymbol{q}^2} - \omega)}$$

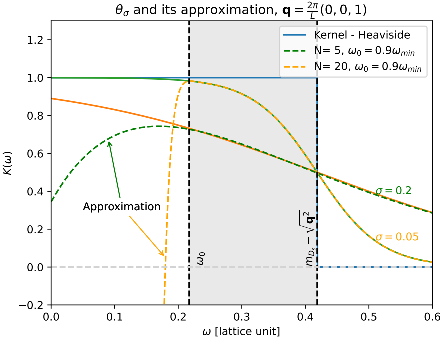
step function to implement the upper limit of the $\omega$ integral; $\omega_{\rm max} = m_{D_s} - \sqrt{\boldsymbol{q}^2}$



# Chebyshev Approximation – Kernel Function

Stabilize approximation by applying a smearing with a width $\sigma$

$$K_\sigma(\omega) = \boxed{\theta_{\sigma}\left(m_{D_s} - \sqrt{\boldsymbol{q}^2} - \omega\right)}$$




Limits to be taken

- smearing of the Kernel, $\sigma \to 0$
- Order of Chebyshev polynomials, $N \to \infty$

# Constructing the Approximation

The $\omega$-integral can be approximated as

$$\frac{\langle\psi_\mu|K(\hat{H})|\psi_\nu\rangle}{\langle\psi_\mu|\psi_\nu\rangle} = \frac{c_0^*}{2} + \sum_{j=1}^{N} c_j^* \underbrace{\frac{\langle\psi_\mu|\,T_j^*(e^{-\hat{H}})|\psi_\nu\rangle}{\langle\psi_\mu|\psi_\nu\rangle}}_{C(t+2t_0)/C(2t_0)},$$

with $|\psi_\nu\rangle \equiv e^{-\hat{H}t_0}\,\tilde{J}_\nu(\boldsymbol{q})\,|D_s(\mathbf{0})\rangle$. Coefficients are given by

$$c_j^* = \frac{2}{\pi}\int_0^\pi d\theta K\left(-\ln\frac{1+\cos\theta}{2}\right)\cos(j\theta) \quad , \quad \text{when } \omega_0 = 0$$

Property of the Chebyshev polynomials

$$\left|\frac{\langle\psi_\mu|\,T_j^*(e^{-\hat{H}})|\psi_\nu\rangle}{\langle\psi_\mu|\psi_\nu\rangle}\right| \leq 1\,.$$

- Use as a constraint: suppress statistical noise for large $j$, for which huge cancellations are expected
- Use as an upper limit of the error: bound by $0 \pm 1$, for all $j$

# Lattice setup

A first study with JLQCD ensembles

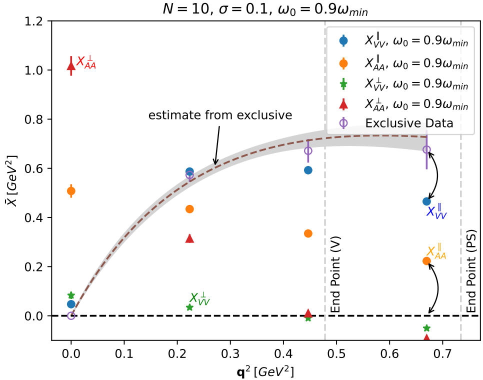
- Lattice size: $48^3 \times 96$
- Lattice spacing: $a = 0.055\,\text{fm}$
- $2+1$ Möbius domain-wall fermions
- $u, d$ quarks at $m_\pi \simeq 300\,\text{MeV}$
- $s, c$ quark simulated at near-physical values
- 4 choices of momentum insertion corresponding to $\boldsymbol{q} = (0,0,0) \to (1,1,1)$
- Numerical computation on Fugaku
- Used Grid/Hadrons

[Colquhoun et al., arXiv:2203.04938]

| ID | $a$ (fm) | $\beta$ | $L^3 \times N_T \times L_s$ | $N_{\text{cfg}}$ | $am_l$ | $am_s$ | $am_Q$ |
|---|---|---|---|---|---|---|---|
| C-$ud$5-$s$a | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.019 | 0.04 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$5-$s$b | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.019 | 0.03 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$4-$s$a | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.012 | 0.04 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$4-$s$b | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.012 | 0.03 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$3-$s$a | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.007 | 0.04 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$3-$s$b | 0.080 | 4.17 | $32^3 \times 64 \times 12$ | 100 | 0.007 | 0.03 | 0.44037 |
| | | | | | | | 0.68808 |
| C-$ud$2-$s$a-L | 0.080 | 4.17 | $48^3 \times 96 \times 12$ | 100 | 0.0035 | 0.04 | 0.44037 |
| | | | | | | | 0.68808 |
| M-$ud$5-$s$a | 0.055 | 4.35 | $48^3 \times 96 \times 8$ | 50 | 0.012 | 0.025 | 0.27287 |
| | | | | | | | 0.42636 |
| | | | | | | | 0.68808 |
| M-$ud$4-$s$a | 0.055 | 4.35 | $48^3 \times 96 \times 8$ | 50 | 0.008 | 0.025 | 0.27287 |
| | | | | | | | 0.42636 |
| | | | | | | | 0.68808 |
| M-$ud$3-$s$a | 0.055 | 4.35 | $48^3 \times 96 \times 8$ | 42 | 0.0042 | 0.025 | 0.27287 |
| | | | | | | | 0.42636 |
| | | | | | | | 0.68808 |
| F-$ud$3-$s$a | 0.044 | 4.47 | $64^3 \times 128 \times 8$ | 50 | 0.003 | 0.015 | 0.210476 |
| | | | | | | | 0.328869 |
| | | | | | | | 0.5138574 |

# First Numerical Results

$$\bar{X} = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K_{\mu\nu}(\boldsymbol{q}, \hat{H}) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\mathbf{0}) \rangle$$



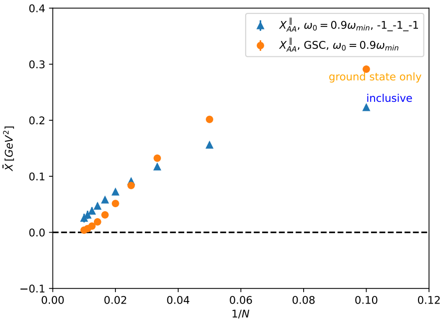

fixed $\sigma = 1/N = 0.1$

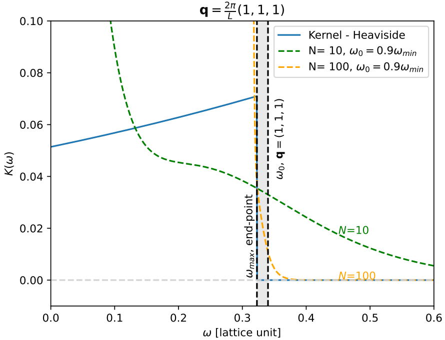
- Decompose $\bar{X}$ into different channels of $V$ and $A$; $\|$ and $\perp$
- Comparison of $VV \parallel$ with exclusive $D \to K$ decay data indicates that values are in the right ballpark
- further analysis on *dangerous region*, i.e close to the end of the phase space

# Above the end-point, $AA \parallel$

At $\boldsymbol{q} = (1, 1, 1)$, where $X^{\parallel}_{AA}$ receives contributions only from the vector state. $E_{V,\mathrm{min}}$ is above the threshold $\rightarrow \bar{X}$ expected to be zero.

- set $\sigma = \frac{1}{N}$; enables us to take both limits simultaneously
- due to statistical uncertainties higher orders in the Chebyshev approximation are basically $0 \pm 1$; only add errors to $\bar{X}$ by $\sum_j |c_j^*|$




Approaches zero for sufficiently large $N$

# Close to the end-point, $VV \parallel$

For $X^{\parallel}_{VV}$ at $\boldsymbol{q} = (1,1,1)$

- Contributions from the pseudoscalar and vector mesons:

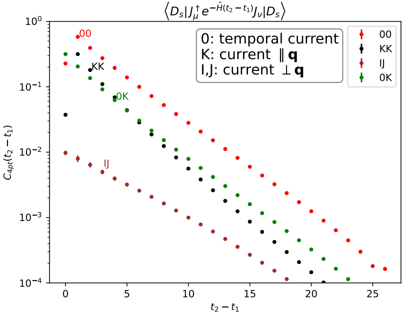
$$\langle \eta(p_\eta) | V^\mu | D_s(p_D) \rangle = f_+(q^2)(p_D + p_\eta)^\mu + f_-(q^2)(p_D - p_\eta)^\mu$$
$$\langle \varphi(p_\Phi) | V^\mu | D_s(p_D) \rangle = 2g(q^2)\varepsilon^{\mu\nu\varrho\chi} p_D^\nu p_\varphi^\varrho \epsilon^{*,\chi}$$

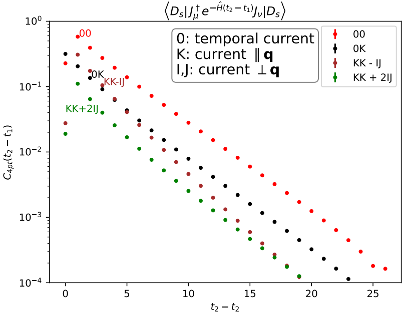
Contributions to each channel

- $KK$: $A^2_{\mathrm{PS}} e^{-E_{\mathrm{PS}} t} + 2/3 A^2_{\mathrm{V}} e^{-E_{\mathrm{V}} t}$
- $IJ$: $A^2_{\mathrm{PS}} e^{-E_{\mathrm{PS}} t} - 1/3 A^2_{\mathrm{V}} e^{-E_{\mathrm{V}} t}$

Extract the ground state contribution from

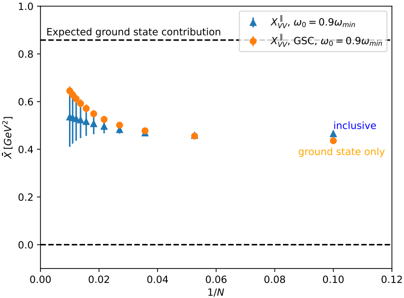
- $KK \ - \ IJ = A^2_{\mathrm{V}} e^{-E_{\mathrm{V}} t}$
- $KK \ + 2IJ = 3 A^2_{\mathrm{PS}} e^{-E_{\mathrm{PS}} t}$



# Close to the end-point, $VV \parallel$

Higher orders of the Chebyshev approximation are dominated by statistical uncertainties and are basically $0 \pm 1$; only add errors to $\bar{X}$ by $\sum_j |c_j^*|$

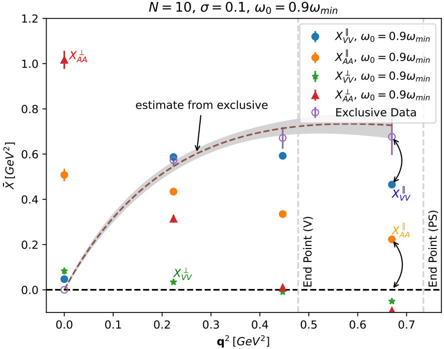



- expected behavior: $\boldsymbol{q} = (1, 1, 1)$ is dominated by the ground state
  $\rightarrow$ require $\sigma \rightarrow 0$ to obtain a reliable estimate
- Ground state contribution is covered by the error of the inclusive analysis; as it should

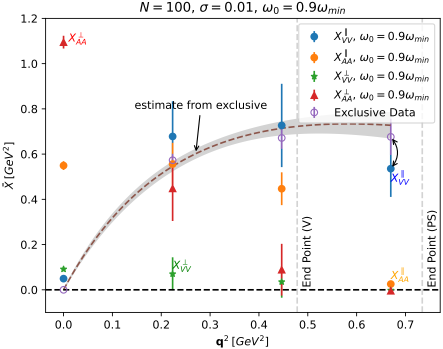
# Taking the $N \to \infty$ limit

Conservative error estimate from the $N$ (or $\sigma$) dependence

$N = 10$




$N = 100$




- central values of the points remain stable
- error bars show the mathematical upper limit, when the $N \to \infty$ limit is taken
  $\to$ likely an overestimation; real error expected to be smaller
- proper estimate requires knowledge on the spectrum
  - With a flat spectrum, errors cancel around the threshold
  - Real problem might occur when the spectrum rapidly changes; expected only near the ground state

# Summary and Outlook

- Inclusive $D \to X\ell\nu$: towards a lattice computation with fully controlled systematics
- Important systematic error due to the approximation of the kernel function
  $\to$ Conservative error estimate employing the mathematical properties of the Chebyshev polynomials; needs more study
- Extend to different ensembles (two more lattice spacings)
- Extend to different inclusive channels $D \to X_s$, $D_s \to X_d$ and $D \to X_d$
- Estimates for the total decay rate and compare the results with the experiment