Position-Space Renormalisation of the Energy-Momentum Tensor Two-Point Function

Lattice 2022

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Summary

LatCos Collaboration

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- Joseph K.L. Lee (Edinburgh)
- Antonin Portelli (Edinburgh)
- Henrique Bergallo Rocha (Edinburgh)
- Kostas Skenderis (Southampton)

Motivation

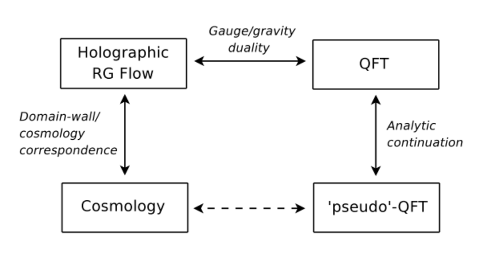
Holographic Cosmology

Interested in a class of conjectured gauge/gravity dualities for cosmology[1]

- Cosmology in $d+1$ dimensions $\Leftrightarrow$ QFT in d dimensions

Dictionary:

$$\langle T_{ij}(\bar{q})T_{kl}(-\bar{q})\rangle = A(\bar{q})\Pi_{ijkl} + B(\bar{q})\pi_{ij}\pi_{kl} \quad (1)$$



[1] McFadden, P. & Skenderis, K. Holography for cosmology. *Physical Review D - Particles, Fields, Gravitation and Cosmology* **81.** arXiv: 0907.5542 (2010)

Holographic Cosmology

Scalar power spectrum:

$$\Delta_R^2(q) = \frac{-q^3}{16\pi^2 \mathrm{Im} B(-iq)} = \frac{\Delta_0^2}{1 + \frac{gq*}{q}\log|\frac{q}{\beta gq*}|} \quad (2)$$

at two loops,

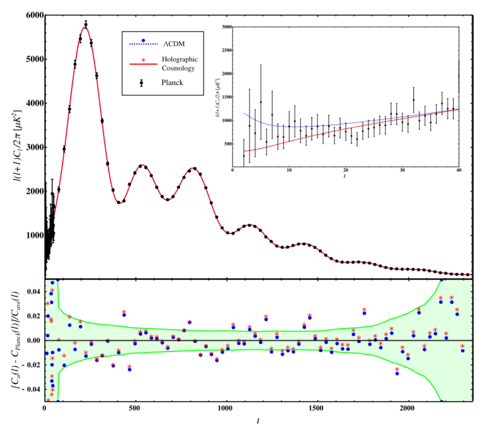
$$\Delta_R^2(q) = \frac{\Delta_0^2}{1 + \frac{gq*}{q}\log|\frac{q}{\beta gq*}|} \quad (3)$$

In ΛCDM, the spectrum follows a power law:

$$\Delta_0^2 \left(\frac{q}{q*}\right)^{n_s - 1} \quad (4)$$

Holographic Cosmology

Perturbative results fit the CMB power spectrum well against ΛCDM, except for low multipoles[2]



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[2] Afshordi, N. *et al.* Constraining holographic cosmology using Planck data. *Physical Review D* **95.** arXiv: 1703.05385 (2017)

$T_{\mu\nu}$ Renormalisation

Discretisation

3D action:

$$S = \frac{a^3 N}{g} \sum_{x \in \Lambda^3} \mathrm{Tr} \left\{ \sum_{\mu} [\Delta_\mu \phi(x)]^2 + (m^2 - m_c^2)\phi^2(x) + \phi^4(x) \right\} \quad (5)$$

ϕ are traceless hermitian $N \times N$ matrices valued in the $\mathfrak{su}(N)$ algebra and Δ_μ is the forward discrete derivative. The theory is perturbatively IR-divergent, but nonperturbatively finite[3]. Bare lattice $T_{\mu\nu}$:

$$T^0_{\mu\nu} = \frac{N}{g} \mathrm{Tr} \Bigg\{ 2(\bar{\Delta}_\mu \phi)(\bar{\Delta}_\nu \phi) - \delta_{\mu\nu} \left[\sum_\rho (\bar{\Delta}_\rho \phi)^2 + (m^2 - m_c^2)\phi^2 + \phi^4 \right] + \xi \left[\delta_{\mu\nu} \sum_\rho (\bar{\Delta}_\rho \phi)^2 - (\bar{\Delta}_\mu \phi)(\bar{\Delta}_\nu \phi) \right] \Bigg\} \quad (6)$$

[3] Cossu, G. *et al.* Nonperturbative Infrared Finiteness in a Superrenormalizable Scalar Quantum Field Theory. *Physical Review Letters* **126.** arXiv: 2009.14768 (June 2021)

Discretisation

However, this naïvely discretised form of $T_{\mu\nu}$ breaks the Ward Identity when the lattice regulator is removed

$$\langle \bar{\Delta}_\mu T^0_{\mu\nu}(x) P(y) \rangle = -\left\langle \frac{\delta P(y)}{\delta \phi(x)} \bar{\Delta}_\nu \phi(x) \right\rangle + \langle X_\nu(x) P(y) \rangle \tag{7}$$

Formally, the expectation value $\langle X_\nu(x) P(y) \rangle$ should go to zero as $a \to 0$. However, radiative corrections induce mixings with lower-dimensional operators and in fact it diverges with a^{-1}. Our renormalisation prescription will be then to counteract this linearly divergent term by subtracting it from $T^0_{\mu\nu}$.

Operator Mixing

$T^0_{\mu\nu}$ contains divergent mixings with all lower-dimensional operators the theory allows for. In 4D there are 5 of those[4]. In 3D, there is only one:

$$\tilde{O} = \delta_{\mu\nu} \mathrm{Tr}\phi^2 \tag{8}$$

Therefore, the renormalisation condition we impose is that the WI be recovered as we remove the lattice regulator:

$$T^R_{\mu\nu} = T^0_{\mu\nu} - \frac{N c_3}{a} \delta_{\mu\nu} \mathrm{Tr}\phi^2 \tag{9}$$

Perturbatively,

$$c_3^{\text{1-loop}} = \left(2 - \frac{3}{N^2}\right)\left(\frac{6Z_0 - 1}{12}\right) \tag{10}$$

[4] Caracciolo, S. *et al.* The energy-momentum tensor on the lattice: The scalar case. *Nuclear Physics, Section B* **309** (1988)

Extracting c_3

We can find c_3 with the aid of the following lattice correlator:

$$C^0_{\mu\nu}(q) = \frac{N}{g} a^3 \sum_x e^{-iq\cdot x} \langle T^0_{\mu\nu}(x) \mathrm{Tr}\phi^2(0)\rangle = C_{\mu\nu}(q) + \frac{g}{a} c_3 \delta_{\mu\nu} C_2(q) + \frac{\kappa}{a}\delta_{\mu\nu}, \tag{11}$$

where the κ/a factor is a contact term, and

$$C_2(q) = \left(\frac{N}{g}\right)^2 a^3 \sum_x e^{-iq\cdot x} \langle \mathrm{Tr}\phi^2(x) \mathrm{Tr}\phi^2(0)\rangle \tag{12}$$

Both the renormalisation and the contact terms diverge as $1/a$. Is there a way to filter out the contact term so we are left only with the c_3 term?

The Window Function

Consider the following function, defined in the interval $[0, \infty)$:

$$\Gamma(x) = \begin{cases} 0, & 0 \leq x \leq r_0 \\ \bar{\Gamma}_{r_0,\epsilon}(x), & r_0 < x < r_0 + \epsilon \\ 1, & r_0 + \epsilon \leq x < \infty \end{cases} \tag{13}$$

where, between r_0 and $r_0 + \epsilon$, the function is defined as

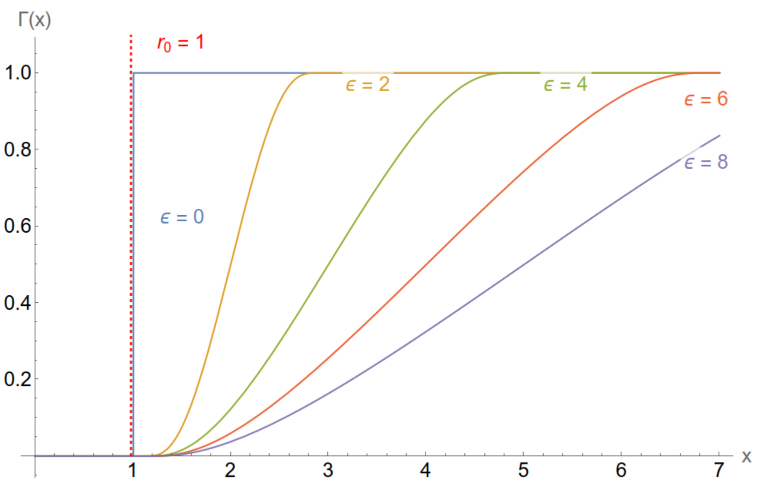
$$\bar{\Gamma}_{r_0,\epsilon}(x) = 1 - \frac{\int_x^{r_0+\epsilon} du\, \beta(u, r_0, \epsilon)}{\int_{r_0}^{r_0+\epsilon} du\, \beta(u, r_0, \epsilon)} \tag{14}$$

where here we have

$$\beta(x, r_0, \epsilon) = \exp\left[-\frac{\epsilon}{(x - r_0)(r_0 + \epsilon - x)}\right] \tag{15}$$

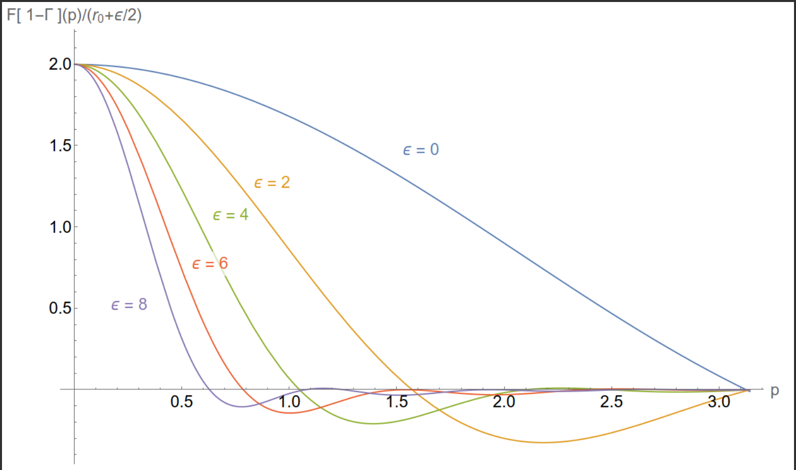
The Window Function

- Γ is C^∞ and interpolates smoothly between 0 and 1.



The Window Function

- $\mathcal{F}[\Gamma_{r_0,\epsilon}](p)$ decays faster than any power of $\frac{1}{|p|}$, and the decay is faster the larger the value of ϵ.



The Position-Space Method

Let us define "windowing" a lattice operator $\mathcal{O}(q)$ as the following operation:

$$W_{r_0,\epsilon}[\mathcal{O}](q) = \left(\frac{a}{L}\right)^3 \sum_x e^{-iq\cdot x}\, \Gamma_{r_0,\epsilon}(|x|) \sum_{q'} e^{iq'\cdot x}\, \mathcal{O}(q') \qquad (16)$$

- This operation "kills" all contact terms by removing their compact support.
- It mixes high and low momenta of the original operator.

The Position-Space Method

If we apply the window function to the full expression of the lattice correlator, the contact term is removed:

$$W_{r_0,\epsilon}[C^0_{\mu\nu}](q) = W_{r_0,\epsilon}[C_{\mu\nu}](q) + \frac{g}{a} c_3 \delta_{\mu\nu} W_{r_0,\epsilon}[C_2](q) + \cancel{W_{r_0,\epsilon}\left[\frac{\kappa}{a}\delta_{\mu\nu}\right]}. \quad (17)$$

Dividing through by $W[C_2]$ and rearranging,

$$c_3 = \frac{a}{g}\left(\frac{W_{r_0,\epsilon}[C^0_{\mu\nu}](q) - W_{r_0,\epsilon}[C_{\mu\nu}](q)}{W_{r_0,\epsilon}[C_2](q)}\right). \quad (18)$$

The Position-Space Method

Now if we consider the zero mode in the large-ϵ limit, one can show that:

$$c_3 \sim \frac{a}{g}\left(\frac{C^0_{\mu\nu}(0)}{C_2(0)} - \frac{b_2}{\epsilon}\right) \tag{19}$$

where b_2 is some constant. This suggests that we can vary ϵ while measuring the ratio between the bare correlators and fit these results to the form

$$\frac{aW_{r_0,\epsilon}[C^0_{22}](q_l = 0)}{W_{r_0,\epsilon}[C_2](q_l = 0)} = \bar{c_3} + \frac{b}{\epsilon}, \tag{20}$$

The Position-Space Method

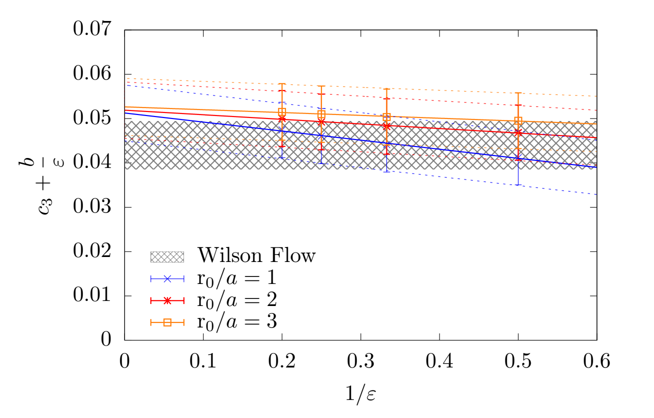


Figure: Position-Space method against results obtained using the Wilson Flow method[5]. $ag = 0.1$, $(am)^2 = -0.0313$

[5] Del Debbio, L. *et al.* Renormalization of the energy-momentum tensor in three-dimensional scalar SU(N) theories using the Wilson flow. *Physical Review D* **103.** arXiv: 2009.14767 (June 2021)

The Position-Space Method

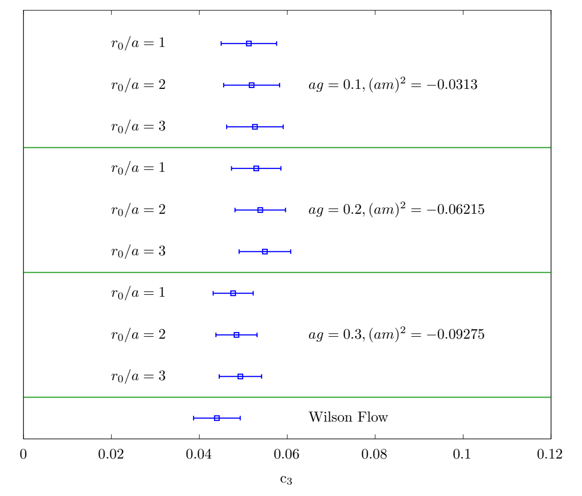


Figure: Summary of results for value of c_3 using the Position-Space method.

Two-Point Function Renormalisation

Perturbation Theory

In the continuum,

$$C_{\mu\nu\rho\sigma} = \langle T^R_{\mu\nu}(q) T^R_{\rho\sigma}(-q)\rangle = A(q)\Pi_{\mu\nu\rho\sigma} + B(q)\pi_{\mu\nu}\pi_{\rho\sigma} \tag{21}$$

where

$$\pi_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \tag{22}$$

is the transverse projector and

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho} - \pi_{\mu\nu}\pi_{\rho\sigma}) \tag{23}$$

is the transverse-traceless projector.

Windowing $C_{\mu\nu\rho\sigma}$ - Exploratory Results

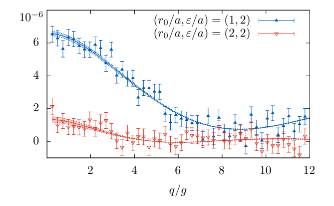
$C^0_{\mu\nu\rho\sigma}$ will have contact terms of the form $\gamma_0/(ag)^3$ and $(\hat{q}/g)^2\beta_0/(ag)$.
On synthetic data: generate distributions of the form

$$\frac{C(q)}{g^3} = \alpha_0 \left(\frac{\hat{q}}{g}\right)^3 + \frac{\beta_0}{ag}\left(\frac{\hat{q}}{g}\right)^2 + \frac{\gamma_0}{(ag)^3} \tag{24}$$

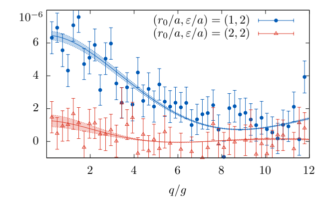
with Gaussian noise added (with a tunable s.d. σ), then apply the window and fit results against the windowed pure $\hat{p}^3$ continuum result, thus restoring the value of α_0, which gives us the renormalised function.

Windowing $C_{\mu\nu\rho\sigma}$ - Exploratory Results

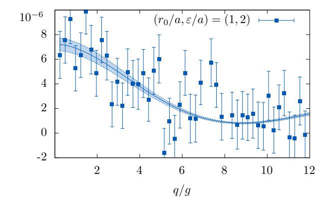
Synthetic data generated at various levels of noise σ:



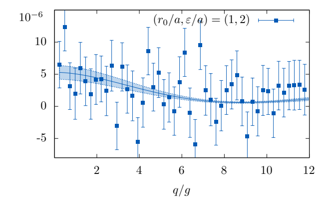
(a) $\sigma = 1$



(b) $\sigma = 2$



(c) $\sigma = 4$



(d) $\sigma = 8$

Outlook and Conclusion

Next Steps

- Position-Space renormalisation can readily help us renormalise the lattice $T_{\mu\nu}$ operator by removing the compact support of contact terms.
- It shows promise for renormalising the two-point function of $T_{\mu\nu}$ as well, at least on synthetic data.
- Applying this to the lattice two-point function requires better control of discretisation effects and noise.
- The same strategy can in principle be used to renormalise $T_{\mu\nu}$ and its two-point function in more complicated theories, for instance coupled with gauge fields. This is one of our objectives for the near future.

Questions

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