

Density of states for gravitational waves

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Outline

- ▶ Underlying physics and motivation: Composite Dark Matter
- ▶ Numerical tool: LLR
 - ▶ LLR explained
 - ▶ Review $U(1)$ case
 - ▶ Bulk phase transitions $SU(N)$
 - ▶ Confinement phase Transition $SU(4)$

Composite Dark Matter

- ▶ Broad class of composite Dark Matter models
 - ▶ Dark mesons
 - ▶ Dark glueballs
 - ▶ Dark baryons

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- ▶ Broad class of composite Dark Matter models
 - ▶ "Stealth dark matter confinement transition and gravitational waves", LSD Collaboration, arXiv:2006.16429
 - ▶ "Testing the Dark Confined Landscape: From Lattice to Gravitational Waves", Huang et al., arXiv:2012.11614
 - ▶ "Dark Confinement-Deconfinement Phase Transition: A Roadmap from Polyakov Loop Models to Gravitational Waves", Kang et al., arXiv:2101.03795
 - ▶ "Review of strongly-coupled composite dark matter models and lattice simulations", Kribs and Neil, arXiv:1604.04627

Baryon Dark Matter

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- ▶ New strongly coupled $SU(N)$ gauge sector coupled to new massive fermions

Baryon Dark Matter

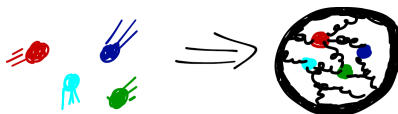
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If confinement transition is first order \rightarrow stochastic background of gravitational waves



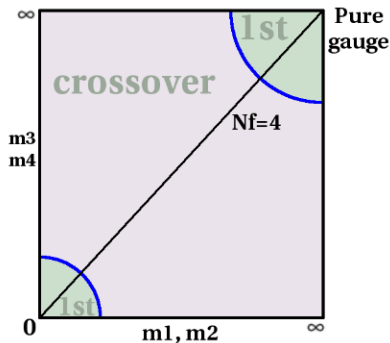
Stealth Dark Matter

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- ▶ $SU(4)$ gauge theory coupled to four flavors of moderately heavy fermions in the fundamental representation

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- ▶ $SU(4)$ gauge theory coupled to four flavors of moderately heavy fermions in the fundamental representation
- ▶ Pure gauge $SU(4)$ theory recovered in the 'quenched' case, fermions infinitely massive



Challenges of the Calculation

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Solution:

Wang-Landau type algorithms

Standard Technique:

$$\langle \mathcal{O} \rangle = \frac{1}{Z(\beta)} \int d\phi \mathcal{O}[\phi] e^{\beta S[\phi]}$$

approximate by

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_c \mathcal{O}[c]$$

Density of States

High dimensional integral:

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Density of States

High dimensional integral:

$$Z(\beta) = \int \mathcal{D}\phi e^{\beta S[\phi]}$$

Density of states:

$$\rho(E) = \int \mathcal{D}\phi \delta(S[\phi] - E)$$

1 dimensional integral:

$$Z(\beta) = \int dE \rho(E) e^{\beta E}$$

Calculate observable:

$$\langle \mathcal{O} \rangle = \frac{1}{Z(\beta)} \int dE \mathcal{O} \rho(E) e^{\beta E}$$

Density of States

Task:

Calculate $\rho(E)$ with near constant relative error over a wide range of E

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Solution:

Linear Logarithmic Relaxation

Re-weighted Expectation Value

Re-weighted expectation value ΔE

$$\begin{aligned}\langle\langle E - E_0 \rangle\rangle_\delta(a) &= \frac{1}{N} \int \mathcal{D}\phi \theta_{E_0, \delta}(E - E_0) e^{-aS} \\ &= \frac{1}{N} \int_{E_0 - \frac{\delta}{2}}^{E_0 + \frac{\delta}{2}} dE (E - E_0) \rho(E) e^{-aE} \\ &= \Delta E\end{aligned}$$

Normalization factor

$$N = \int \mathcal{D}\phi \theta_{E_0, \delta} e^{-aS[\phi]}$$

Set re-weighted expectation value to zero

$$\Delta E(a) = \frac{1}{N} \int_{E_0 - \frac{\delta}{2}}^{E_0 + \frac{\delta}{2}} dE (E - E_0) \rho(E) e^{-aE} = 0$$

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Approximate integral with trapezium rule

Let interval go to zero $\delta \rightarrow 0$

Solve for a

$$a_0 = \frac{1}{\rho(E_0)} \left. \frac{d\rho(E)}{dE} \right|_{E=E_0} = \left. \frac{d \ln(\rho(E))}{dE} \right|_{E=E_0}$$

numerically integrate a and we get ρ with constant relative error!

Integrate the exponent using the Trapezium rule

$$\begin{aligned}\rho(E_N) &= e^{\int_{-\infty}^{E_N} dE' a(E')} \\ &= \rho_c e^{a_0 \frac{\delta E}{2} + \sum_{k=1}^{N-1} a_k \delta + a_N \delta + \mathcal{O}(\delta^2)}\end{aligned}$$

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Get $\rho(E)$ up to a constant with exponential error suppression

$$\rho(E_N) = \rho_c \tilde{\rho}(E_N) e^{\mathcal{O}(\delta^2)}$$

Solve $\Delta E(a) = 0$ via Robbins-Monro iteration

$$a^{(n+1)} = a^{(n)} + \frac{12}{\delta^2(n+1)} \Delta E(a^{(n)})$$

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Evaluate $\Delta E(a^{(n)})$ with standard Monte Carlo

$$\begin{aligned} \Delta E(a^{(n)}) &= \langle\langle E - E_0 \rangle\rangle_{\delta}(a^{(n)}) \\ &= \frac{1}{N} \int \mathcal{D}\phi \theta_{E_0, \delta}(E - E_0) e^{-a^{(n)} S} \end{aligned}$$

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New Boltzmann weight!

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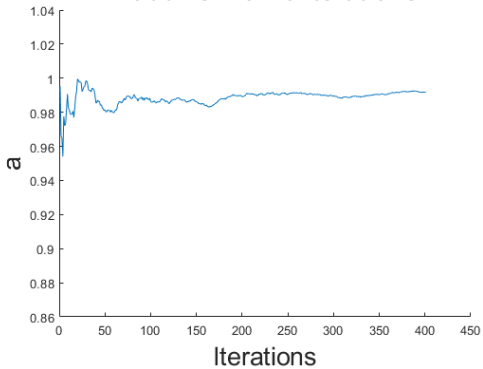
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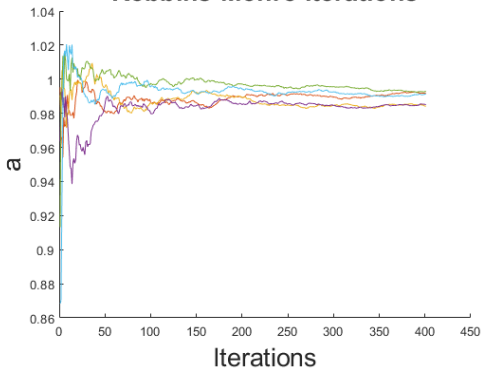
New Boltzmann weight!

Alternatively: $\Delta E(a^{(n)}) = \frac{1}{N} \int \mathcal{D}\phi (E - E_0) e^{-a^{(n)} S - \frac{(S - E_0)}{2\delta^2}}$

Robbins-Monro Iterations



Robbins-Monro Iterations



Pseudocode

loop over energy interval $[E_i \pm \frac{\delta}{2}]$

loop over Robbins-Monro iteration

loop over Monte Carlo sweeps through the lattice

Evaluate $\Delta E_i(a_i^{(n)})$ using standard Monte Carlo

New Boltzmann weight and restricted energy interval!

End loop

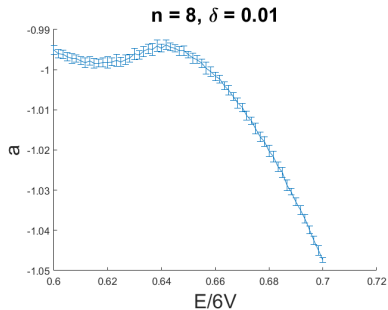
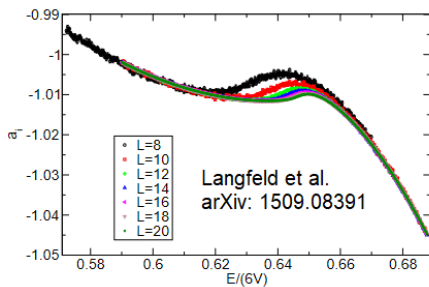
Calculate new $a_i^{(n+1)}$

$$a_i^{(n+1)} = a_i^{(n)} + \frac{12}{\delta^2(n+1)} \Delta E_i(a_i^{(n)})$$

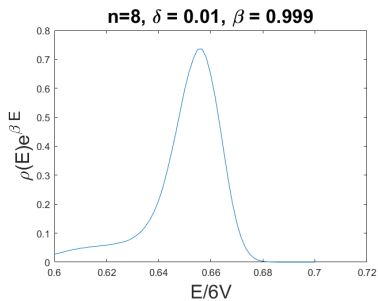
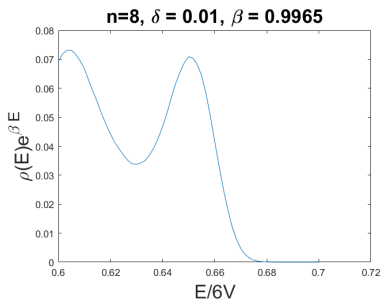
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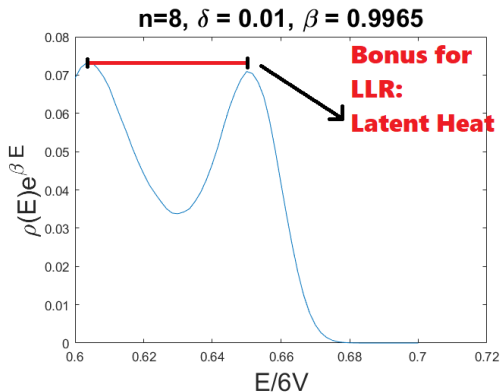
Compact U(1) Lattice Gauge Theory



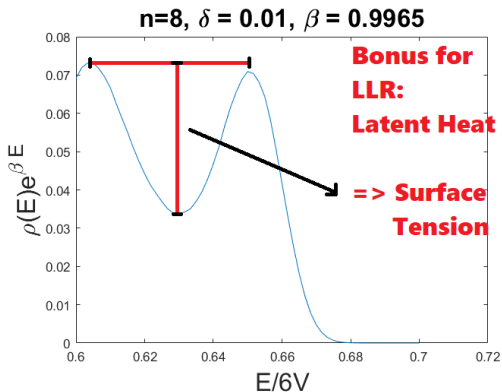
U(1) Density of States



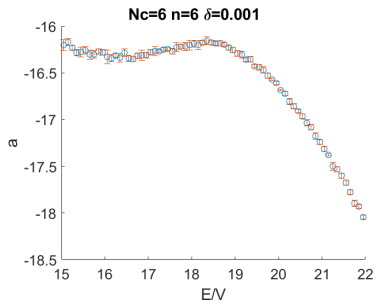
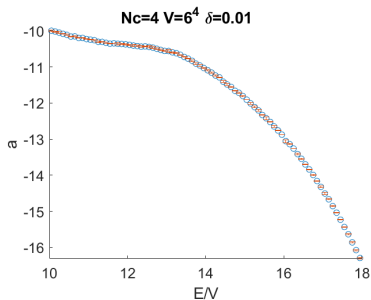
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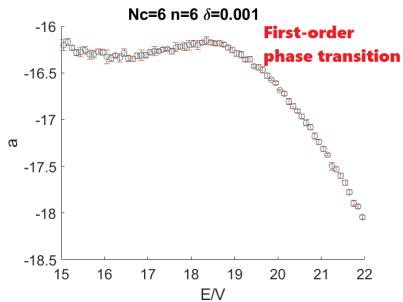
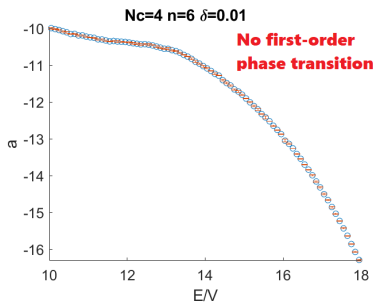
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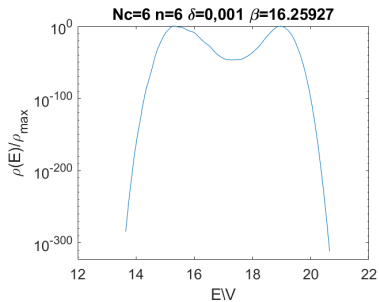
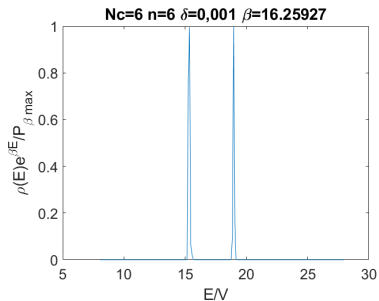
SU(N) Bulk Phase Transition



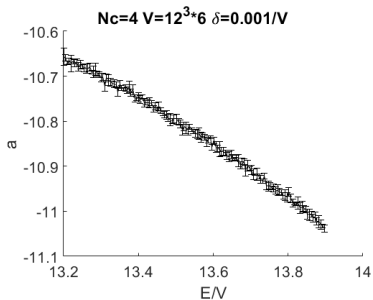
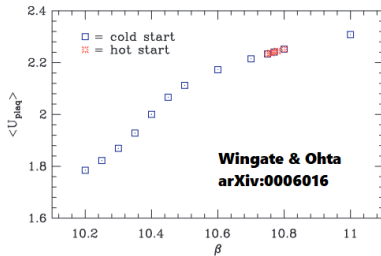
SU(N) Bulk Phase Transition



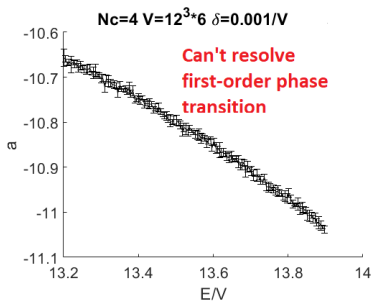
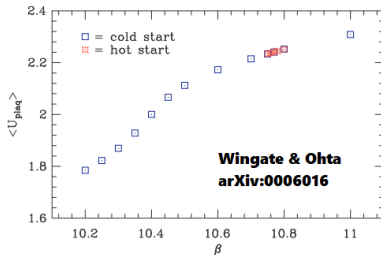
SU(6) Density of States



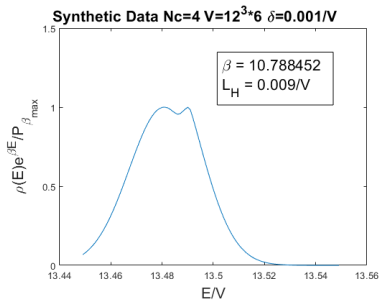
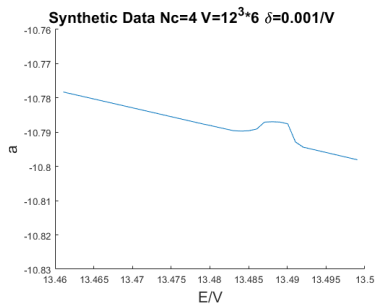
SU(4) Thermal Phase Transition



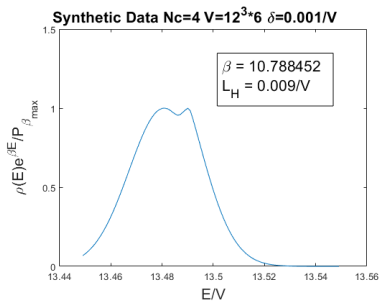
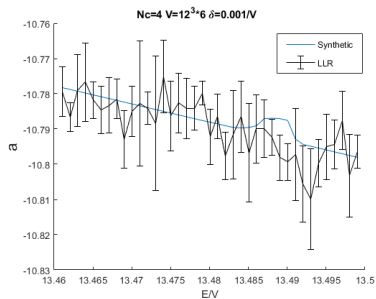
SU(4) Thermal Phase Transition



Synthetic Data



Synthetic Data



Recap

- ▶ Broad class of Composite Dark Matter models
- ▶ Gravitational waves offer new way to probe these models
- ▶ First order phase transition difficult with MCMC → LLR
- ▶ LLR works for strong bulk phase transitions (U(1), SU(N))
- ▶ Weaker confinement Transition not yet resolved
 - ▶ Improve statistics
 - ▶ Go to bigger lattice sizes and increase aspect ratios
 - ▶ Decrease $\delta = 0.001/V \rightarrow \delta = 0.0001/V$
 - ▶ Go to SU(N) with $N \geq 6$

Thank you for your attention!

