

Exploring conformality in lattice $\mathcal{N} = 4$ super-Yang–Mills

David Schaich (U. Liverpool)



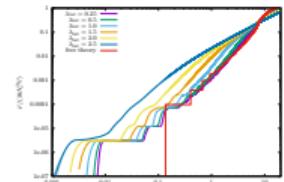
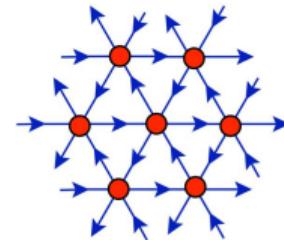
Lattice 2022, August 8

[arXiv:2102.06775](https://arxiv.org/abs/2102.06775) with Georg Bergner, and more to come

Overview

Lattice studies of $\mathcal{N} = 4$ super-Yang–Mills (SYM)
→ opportunities to explore conformality

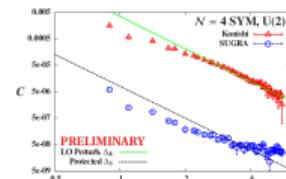
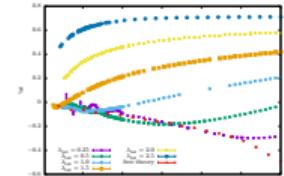
Reproduce reliable analytic results
then access new regimes from first principles



Lattice $\mathcal{N} = 4$ SYM motivational review

Mass anomalous dimension from eigenmode number

Konishi and SUGRA anomalous dimensions



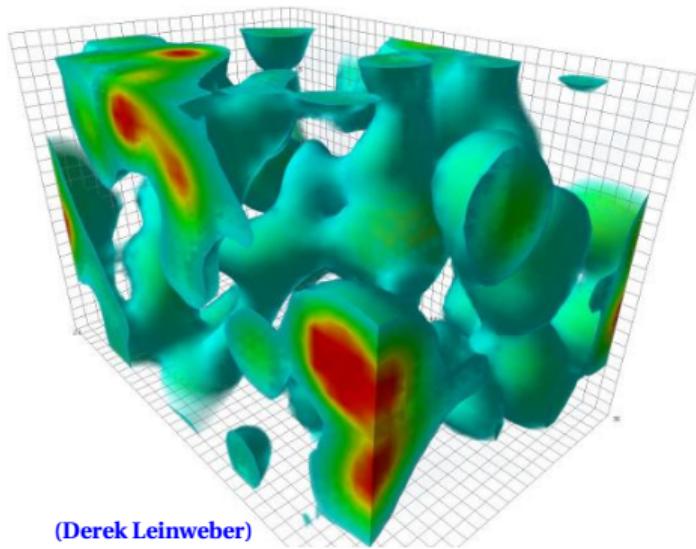
Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

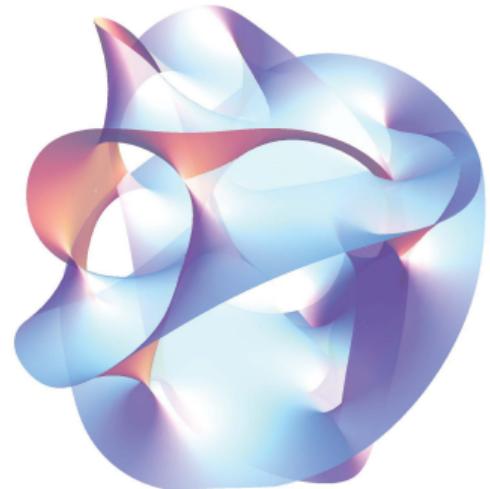
BSM



QFT



Holography



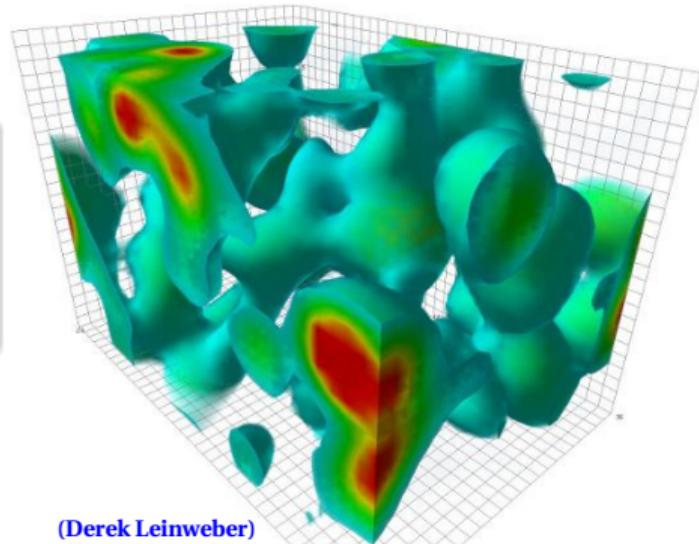
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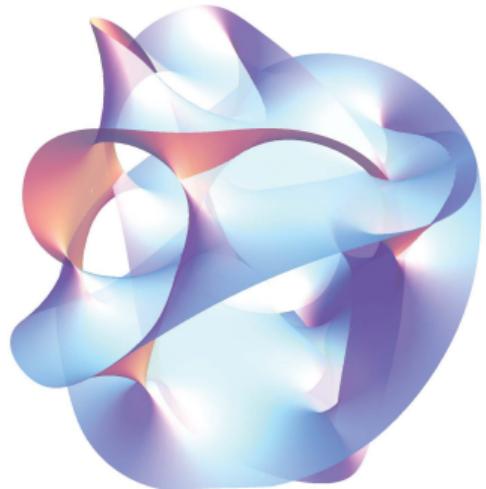
$\mathcal{N} = 4$ SYM

Cornerstone of
S-duality, AdS/CFT
and more

QFT



Holography



$\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT \rightarrow dualities, amplitudes, ...

$SU(N)$ gauge theory with $\mathcal{N} = 4$ fermions ψ^I and 6 scalars ϕ^{IJ} ,
all massless and in adjoint rep.

Symmetries relate coefficients of kinetic, Yukawa and ϕ^4 terms

Maximal 16 supersymmetries Q_α^I and $\bar{Q}_{\dot{\alpha}}^I$ $I = 1, \dots, 4$
transform under global $SU(4) \sim SO(6)$ **R symmetry**

Conformal \rightarrow β function is zero for all values of $\lambda = g^2 N$

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Conformal \rightarrow β function is zero for all values of $\lambda = g^2 N$
Exact, perturbative, holographic & bootstrap results
for spectrum of scaling dimensions $\Delta(\lambda)$

$\mathcal{N} = 4$ SYM on the lattice

$$\left\{ Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J \right\} = 2\delta^{IJ} \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad \text{broken in discrete space-time}$$

→ relevant susy-violating operators



Preserve susy sub-algebra in discrete lattice space-time

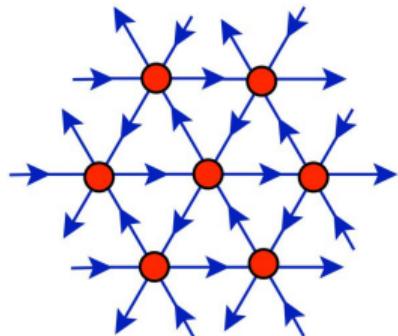
⇒ correct continuum limit with little or no fine tuning

$\mathcal{N} = 4$ SYM on the lattice

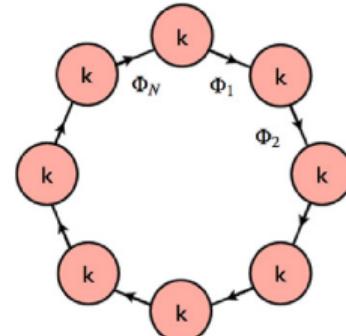
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⇒ correct continuum limit with little or no fine tuning

Equivalent constructions from ‘topological’ twisting and dim'l deconstruction



Review:
Catterall–Kaplan–Ünsal,
[arXiv:0903.4881](https://arxiv.org/abs/0903.4881)



Twisted lattice $\mathcal{N} = 4$ SYM

Change of variables

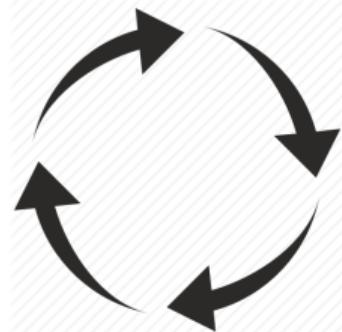
p -form supersymmetries $\{\mathcal{Q}, \mathcal{Q}_\mu, \mathcal{Q}_{\mu\nu}, \overline{\mathcal{Q}}_\mu, \overline{\mathcal{Q}}\}$ transform with integer ‘spin’

under ‘twisted rotation group’ $\text{SO}(4)_{\text{tw}} \equiv \text{diag} \left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right]$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



Twisted lattice $\mathcal{N} = 4$ SYM

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The cost of twisted lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \quad (18)$$

$$\begin{aligned} S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[-\bar{\mathcal{F}}_{ab}(n) \mathcal{F}_{ab}(n) - \chi_{ab}(n) \mathcal{D}_{[a}^{(+)} \psi_{b]}(n) - \eta(n) \bar{\mathcal{D}}_a^{(-)} \psi_a(n) \right. \\ &\quad \left. + \frac{1}{2} \left(\bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}} \end{aligned}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_b^{-1}(n) \psi_b(n) + \mathcal{U}_a^{-1}(n + \hat{\mu}_b) \psi_a(n + \hat{\mu}_b)]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[\epsilon_{abcde} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \bar{\mathcal{D}}_c^{(-)} \chi_{ab}(n) \right],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left(\frac{1}{N} \text{Tr} [\mathcal{U}_a(n) \bar{\mathcal{U}}_a(n)] - 1 \right)^2$$

Computationally challenging, e.g. $\gtrsim 100$ gathers per fermion matrix–vector op.

Public parallel code github.com/daschaich/susy [arXiv:1410.6971]

actively developed for improved performance and new applications

Conformality broken by finite volume and non-zero lattice spacing

Consider analogue of mass anomalous dimension,

$$\gamma_*(\lambda) = 0 \text{ for continuum } \mathcal{N} = 4 \text{ SYM}$$

Antisymmetric fermion operator \rightarrow paired eigenvalues $\pm \lambda_k$

$$\Psi^T D \Psi = \chi_{ab} \mathcal{D}_{[a}^{(+)} \psi_{b]} + \eta \mathcal{D}_a^{\dagger(-)} \psi_a + \frac{1}{2} \epsilon_{abcde} \chi_{ab} \mathcal{D}_c^{\dagger(-)} \chi_{de}$$

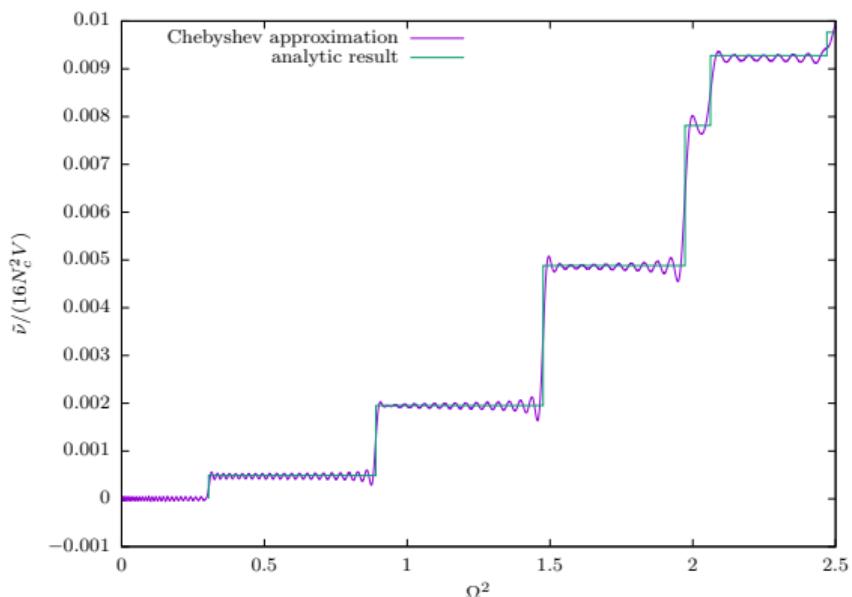
Anomalous dimension related to mode number of $D^\dagger D$

$$\nu(\Omega^2) = \int_0^{\Omega^2} \rho(\omega^2) d\omega^2 \propto (\Omega^2)^{2/(1+\gamma_*)} \quad \rho(\omega^2) = \frac{1}{V} \sum_k \langle \delta(\omega^2 - \lambda_k^2) \rangle$$

Chebyshev expansion for mode number

Stochastically estimate Chebyshev expansion
[Fodor et al., arXiv:1605.08091]

$$\rho_r(x) \approx \sum_{n=0}^P \frac{2 - \delta_{n0}}{\pi \sqrt{1 - x^2}} c_n T_n(x)$$



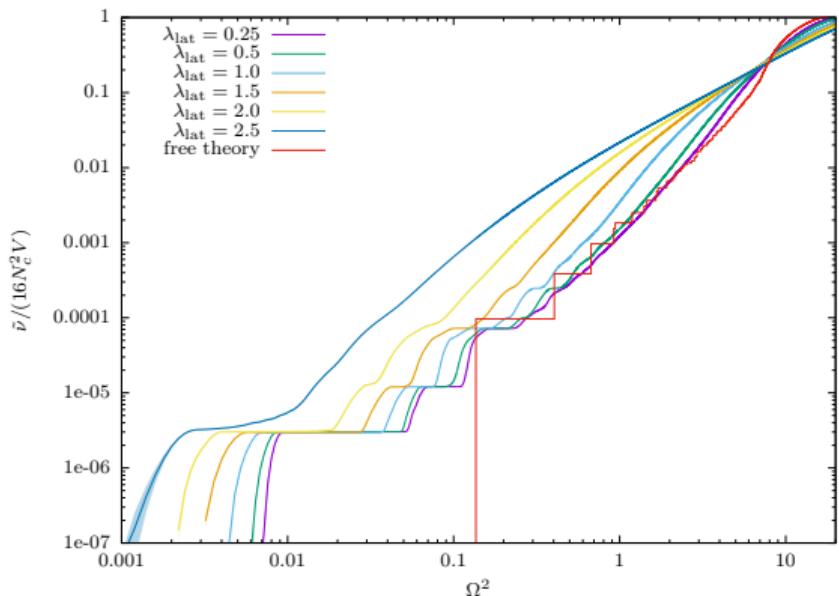
← Example mode number
for U(2) 8^4 free theory, $P = 1000$

$5000 \leq P \leq 10000$ for $N = 2, 3, 4$
volumes up to 16^4

Checked vs. direct eigensolver
and stochastic projection

Mode number scale dependence

Anomalous dimension from $D^\dagger D$ mode number $\nu(\Omega^2) \propto (\Omega^2)^{2/(1+\gamma_*)}$



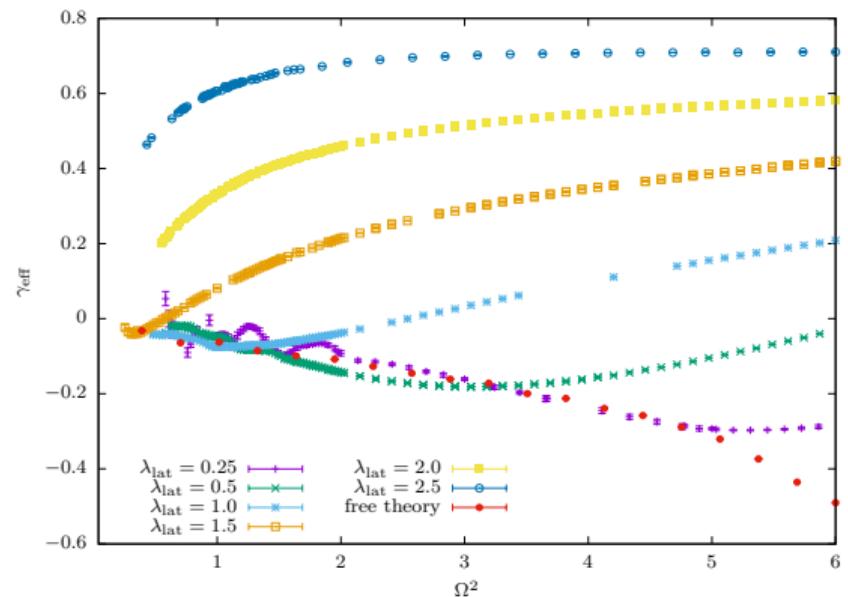
U(2) 16^4 lattices with $0.25 \leq \lambda_{\text{lat}} \leq 2.5$
Free theory also shows lattice effects

Power law varies with scale Ω^2
→ scale-dependent effective $\gamma_{\text{eff}}(\Omega^2)$

Extract by fitting in windows $[\Omega^2, \Omega^2 + \ell]$
with fixed $\ell \in [0.03, 1]$

Convergence to continuum $\gamma_* = 0$

Broken conformality \rightarrow scale-dependent effective anomalous dim. $\gamma_{\text{eff}}(\Omega^2)$



U(2) 16^4 lattices with $0.25 \leq \lambda_{\text{lat}} \leq 2.5$
Free theory also shows lattice effects

Recover true $\gamma_* = 0$ in IR, $\Omega^2 \ll 1$

Stronger couplings \rightarrow larger artifacts

Konishi operator scaling dimension Δ_K

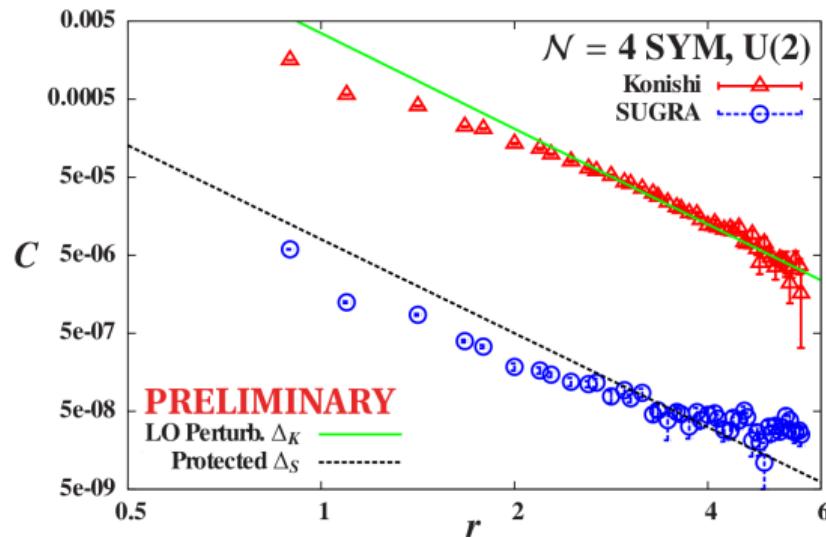
$\mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x)\Phi^I(x)]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through
perturbation theory (& S duality), holography, conformal bootstrap

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

20' 'SUGRA' op. has $\Delta_S = 2$

Work in progress to compare:
Direct power-law decay
Finite-size scaling
Monte Carlo RG



Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Conformal fixed point $\rightarrow H^* = R_b H^*$ with couplings c_i^*

Linear expansion around fixed point \rightarrow **stability matrix** T_{ik}^*

$$c_i^{(n)} - c_i^* = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^*} (c_k^{(n-1)} - c_k^*) \equiv \sum_k T_{ik}^* (c_k^{(n-1)} - c_k^*)$$

Correlators of $\mathcal{O}_i, \mathcal{O}_k \rightarrow$ elements of stability matrix

[Swendsen, 1979]

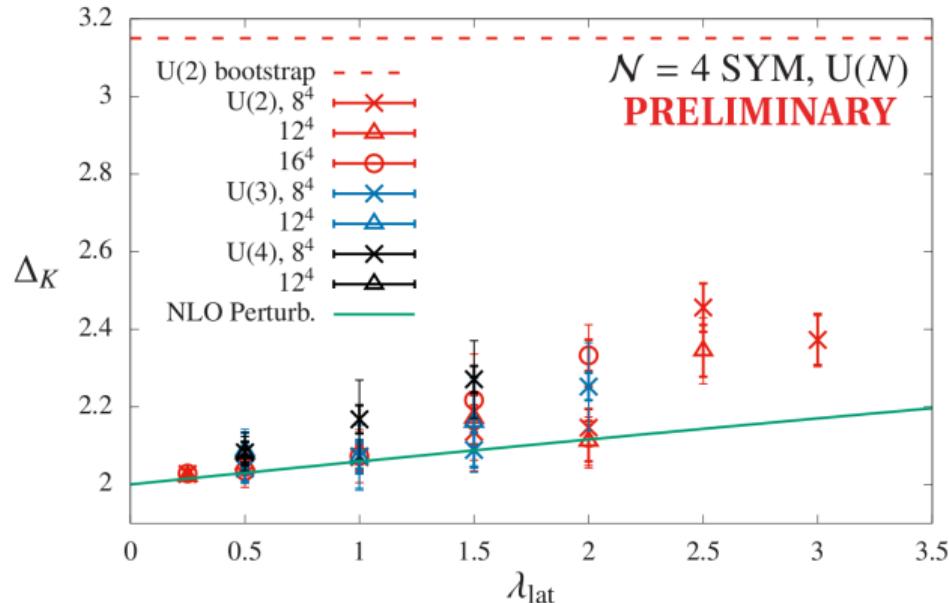
Eigenvalues of T_{ik}^* \rightarrow scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

Both Konishi and SUGRA in T_{ik}^*

Impose protected $\Delta_S = 2$
→ Δ_K consistent with pert. theory

Systematic uncertainties from
different amounts of smearing



Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $\text{SO}(4)_R$ -singlet part of $\text{SO}(6)_R$ -nonsinglet \mathcal{O}_S
→ disentangle via variational analyses

Recap and outlook

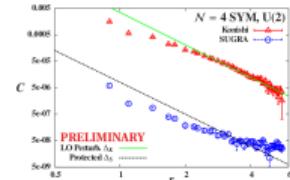
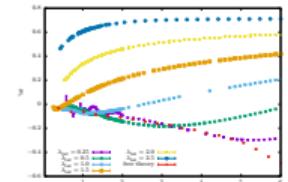
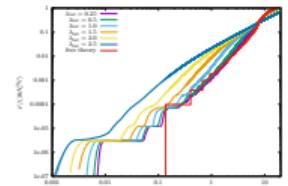
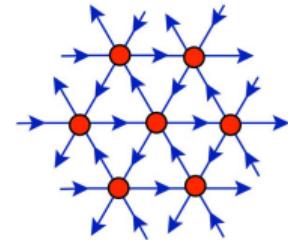
Lattice studies of $\mathcal{N} = 4$ super-Yang–Mills (SYM)
→ opportunities to explore conformality

Reproduce reliable analytic results
then access new regimes from first principles

Public code available for lattice $\mathcal{N} = 4$ SYM

Mass anomalous dimension from eigenmode number
tests discretization and finite-volume artifacts

Non-trivial Konishi anomalous dimension
being analyzed with several techniques



Thanks for your attention!

Collaborators

Georg Bergner, Simon Catterall, Joel Giedt
also Raghav Jha, Anosh Joseph, Angel Sherletov

Funding and computing resources

UK Research
and Innovation



Backup: Naively regulating U(1) flat directions

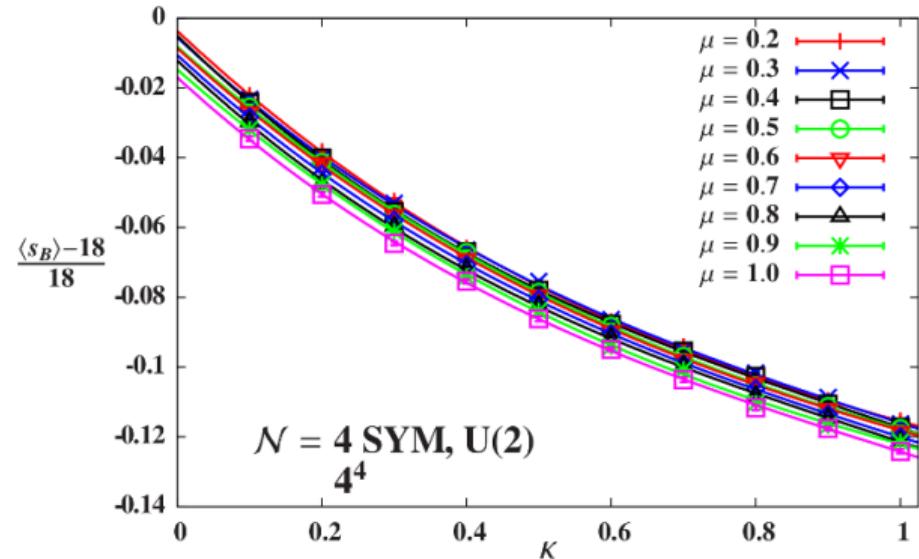
In earlier work we added another soft \mathcal{Q} -breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left(\frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2 + \kappa \sum_{a < b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to κ than to μ^2

Showing \mathcal{Q} Ward identity
from bosonic action

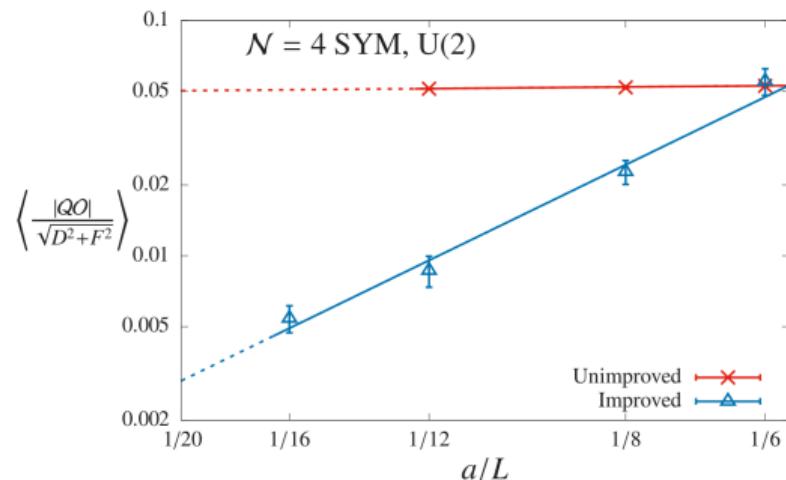
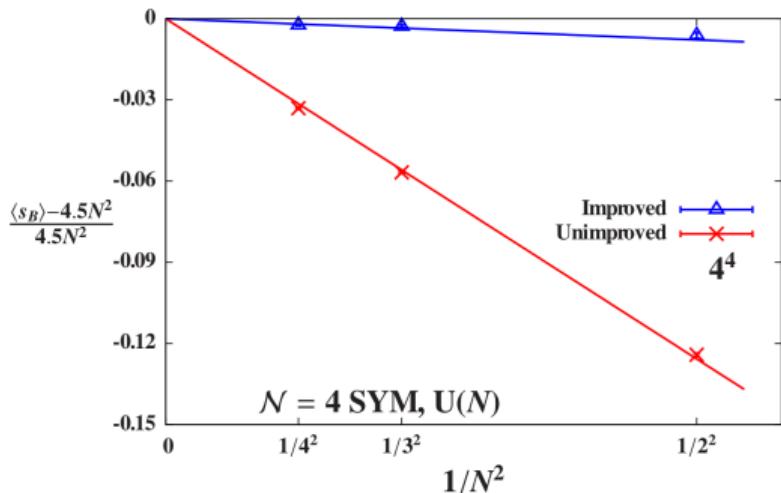
$$\langle s_B \rangle = 9N^2/2$$



Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \left\{ \bar{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a < b} [\det \mathcal{P}_{ab} - 1] \mathbb{I}_N \right\} - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

\mathcal{Q} Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)
 ~ effective ‘ $\mathcal{O}(a)$ improvement’ since \mathcal{Q} forbids all dim-5 operators



Backup: Supersymmetric moduli space modification

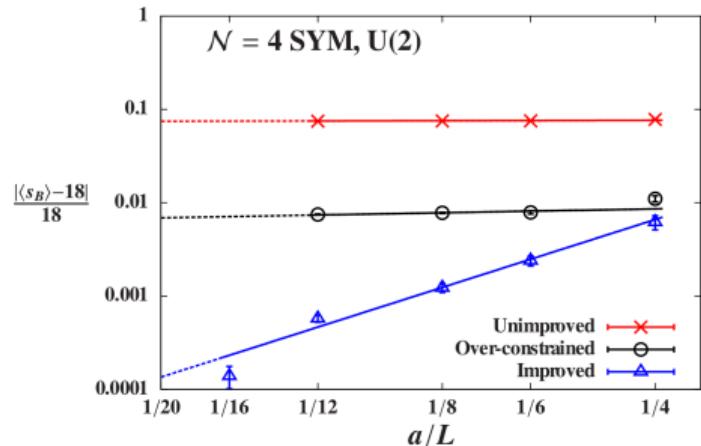
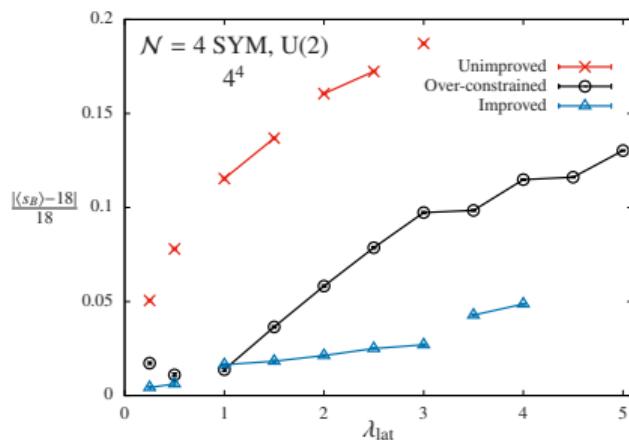
[arXiv:1505.03135]

Method to impose \mathcal{Q} -invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion \rightarrow moduli space

$$d(n) = \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) \quad \rightarrow \quad d(n) = \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both $U(1)$ and $SU(N) \in \mathcal{O}(n)$ over-constraints system



Backup: Lattice scalars for Konishi operator

Combining A_μ and $\Phi^I \rightarrow A_a$ and \bar{A}_a

produces $U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q} A_a = \psi_a$

Further motivation: Under $SO(d)_{\text{tw}} = \text{diag}[SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$ $\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

\Rightarrow lattice scalars $\varphi(n)$ from polar decomposition $\mathcal{U}_a(n) = e^{\varphi_a(n)} U_a(n)$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev} \quad \mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]$$

Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve \mathcal{Q} and S_5 symmetries \longleftrightarrow geometric structure

Simple transformation constructed in [arXiv:1408.7067](#)

$$\begin{aligned}\mathcal{U}'_a(n') &= \xi \mathcal{U}_a(n) \mathcal{U}_a(n + \hat{\mu}_a) & \eta'(n') &= \eta(n) \\ \psi'_a(n') &= \xi [\psi_a(n) \mathcal{U}_a(n + \hat{\mu}_a) + \mathcal{U}_a(n) \psi_a(n + \hat{\mu}_a)] & \text{etc.}\end{aligned}$$

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

\Rightarrow shift $\varphi \rightarrow \varphi + \log \xi$ to keep blocked U unitary

\mathcal{Q} -preserving RG transformation

\rightarrow only one log. tuning to recover continuum \mathcal{Q}_a and \mathcal{Q}_{ab}

Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: $\underline{\quad}$ \rightarrow $(1 - \alpha)\underline{\quad} + \frac{\alpha}{8} \sum \square,$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)

