

COMPLEX LANGEVIN STUDY OF SPONTANEOUS SYMMETRY BREAKING IN IKKT MATRIX MODEL



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LATTICE 2022
10TH AUG, 4:50 CET

WITH ANOSH JOSEPH AND PIYUSH KUMAR AT IISER MOHALI

PLAN OF TALK

COMPLEX LANGEVIN STUDY OF SPONTANEOUS SYMMETRY BREAKING IN IKKT MATRIX MODEL

1. IKKT Matrix Model
2. SSB and Sign-Problem
3. Complex Langevin to Euclidean IKKT Model
4. Mass Deformations
5. Preliminary Results
6. Summary and Future Work

- Promising candidate for non-perturbative formulation of superstring theory

IKKT: $0d$ matrix model in large- N limit

IIB: $10d$ superstring theory

- Originally from Green-Schwarz action of type IIB superstring with Schild gauge
- Can also be derived from Kaluza-Klein compactification of $10d$ $\mathcal{N} = 1$ SYM theory

Euclidean IKKT Model Wick rotated: $X_0 \rightarrow iX_{10}, \Gamma^0 \rightarrow -i\Gamma_{10}$

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Mitsuaki Hirasawa
11th Aug, 11:30 CET

$$Z = \int \mathcal{D}X \mathcal{D}\psi e^{-(S_b + S_f)}$$

$$S_b = -\frac{1}{4}N \operatorname{tr} \left([X_\mu, X_\nu]^2 \right)$$

$$S_f = -\frac{1}{2}N \operatorname{tr} \left(\psi_\alpha \Gamma_{\alpha\beta}^\mu [X_\mu, \psi_\beta] \right)$$

Vectors X_μ

Majorana-Weyl spinors ψ_α

$X_\mu, \psi_\alpha : N \times N$ Hermitian traceless matrices

$\Gamma^\mu : 2^4 \times 2^4$ gamma matrices

$\mu, \nu = 1, 2, \dots, 10$

$\alpha, \beta = 1, 2, \dots, 16$

Z finite Krauth, Nicolai and Staudacher (1998)
Austing and Wheeler (2001)

- Euclidean action is invariant under following symmetries

A. $SU(N)$ gauge symmetry

B. $\mathcal{N} = 2$ supersymmetry

C. $SO(10)$ rotational symmetry

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A. Gauge Symmetry: $SU(N)$

- Inherited from $D=10$ SYM, for $U \in SU(N)$, model invariant under

$$X_\mu \rightarrow X'_\mu = U^\dagger X_\mu U \quad \psi_\alpha \rightarrow \psi'_\alpha = U^\dagger \psi_\alpha U$$

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B. Supersymmetry: $\mathcal{N} = 2$

- Two set of supersymmetries: $\delta^{(1)}S = \delta^{(2)}S = 0$

$$\begin{aligned} \delta^{(1)}X_\mu &= i\bar{\epsilon}_1 \Gamma_\mu \psi & \delta^{(2)}X_\mu &= 0 \\ \delta^{(1)}\psi &= -\frac{i}{2} [X_\mu, X_\nu] \Gamma^{\mu\nu} \epsilon_1 & \delta^{(2)}\psi &= \epsilon_2 \end{aligned}$$

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C. Rotational Symmetry: $SO(10)$

$$X_\mu \rightarrow X'_\mu = \Lambda_\mu^\rho X_\rho$$

- Spontaneously broken:
 $SO(10) \rightarrow SO(d) \times SO(10 - d)$
- Realized as $SO(d)$ symmetric vacuum

Nishimura, Okubo and Sugino (2011)

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*integrating out
fermions*

$$Z = \int \mathcal{D}X \, \mathbf{Pf} \mathcal{M} \, e^{-S_b} = \int \mathcal{D}X \, e^{-S_{\text{eff}}}$$

$$S_{\text{eff}} = S_b - \ln(\mathbf{Pf} \mathcal{M})$$

expected cause of SSB

IKKT MODEL: SSB AND SIGN PROBLEM

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- After integrating out fermions, we obtain Fermion matrix $\mathcal{M}$:

anti-symmetric matrix: $16(N^2 - 1) \times 16(N^2 - 1)$

complex nature: $\text{Pf}\mathcal{M} = |\text{Pf}\mathcal{M}| e^{i\theta}$

θ fluctuates wildly ~~$\theta \approx 0$~~

$$Z = \int \mathcal{D}X \text{Pf}\mathcal{M} e^{-S_b} = \int \mathcal{D}X e^{-S_{\text{eff}}}$$

$$S_{\text{eff}} = S_b - \ln(\text{Pf}\mathcal{M})$$

$$S_b = -\frac{1}{4}N \text{tr} \left([X_\mu, X_\nu]^2 \right)$$

- MC and $1/D$ expansion to bosonic IKKT model - No SSB
Hotta, Nishimura and Tsuchiya (1998)

- MC to phase-quenched IKKT model with $|\text{Pf}\mathcal{M}|$ - No SSB *why phase quenched?*
Ambjorn, Anagnostopoulos, Bietenholz, Hotta and Nishimura (2000)
Anagnostopoulos, Azuma and Nishimura (2013)

~~$e^{-S_{\text{eff}}} \rightarrow \text{probability weight}$~~

sign-problem in Monte Carlo!

- Phase $e^{i\theta}$ responsible for SSB! *how to incorporate complex phase?*

complex Langevin

COMPLEX LANGEVIN DYNAMICS IN A NUTSHELL

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Stochastic Quantization

expectation values $\leftrightarrow$ equilibrium values

Klauder (1983), Parisi (1983)
Damgaard and Huffel (1987)

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Daniel Alvestad
8th Aug, 14:40 CET

Paul Hotzy
9th Aug, 11:50 CET

Felix Ziegler
11th Aug, 11:30 CET

- Consider 0-d theory: action $S[\Phi]$

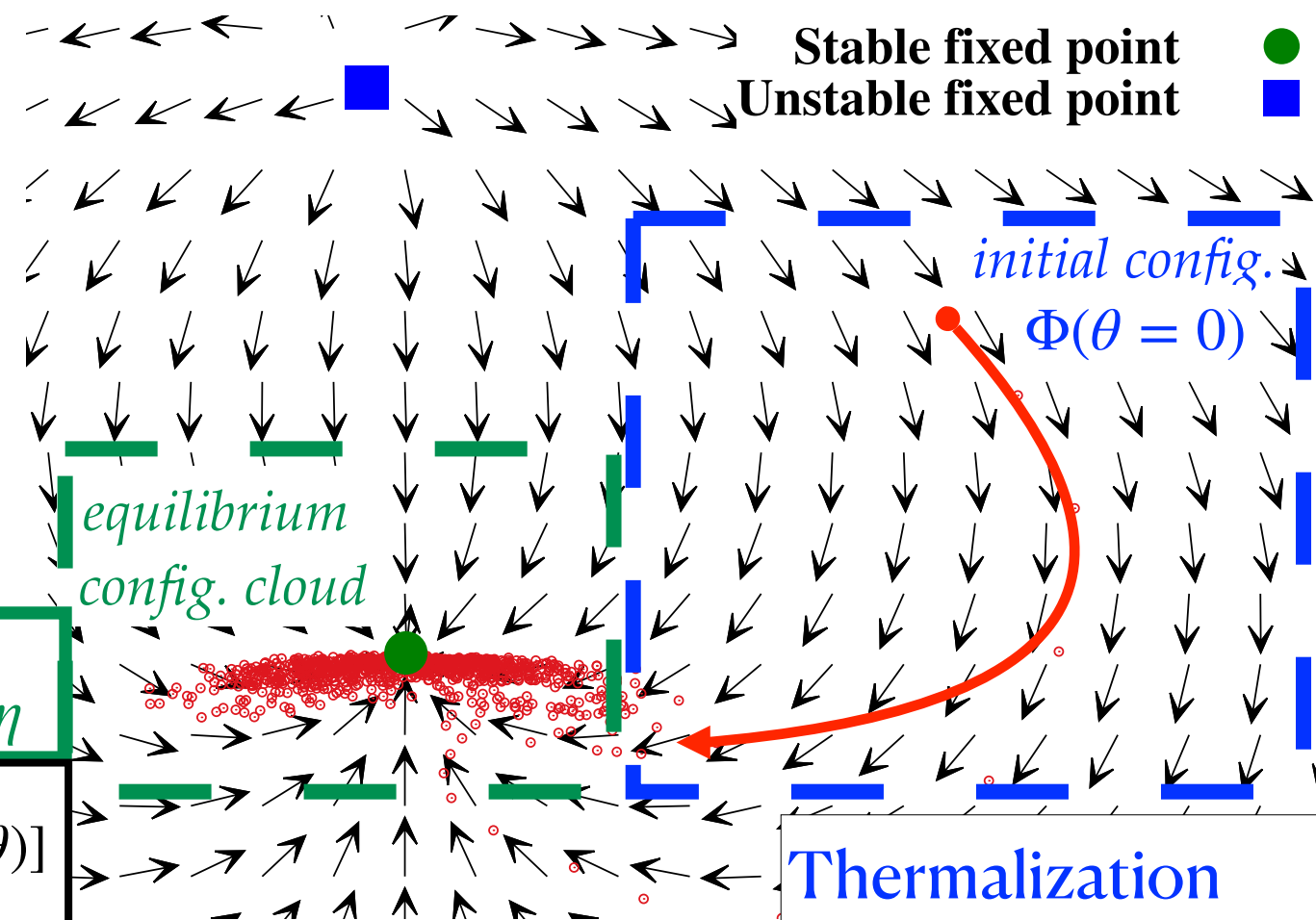
$$\partial_\theta \Phi(\theta) = v(\Phi, \theta) + \eta(\theta)$$

$$-\frac{\delta S[\Phi]}{\delta \Phi(\theta)} = v(\Phi, \theta) \quad \langle \eta(\theta) \rangle = 0$$

$$\langle \eta(\theta) \eta(\theta') \rangle = 2\delta(\theta - \theta')$$

$$\langle \mathcal{O}[\Phi(\theta)] \rangle_\eta$$

$$\langle \mathcal{O}[\Phi(\theta)] \rangle_\eta = \int \mathcal{D}\Phi P[\Phi, \theta] \mathcal{O}[\Phi(\theta)]$$



- At large Langevin time

$$\theta \rightarrow \infty : P[\Phi, \theta] \sim e^{-S[\Phi]}?$$

- When action is complex, no such proof of convergence exists!
need correctness criteria

Aarts, James, Seiler, and
Stamatescu (2010)

$$\langle L\mathcal{O} \rangle = 0$$

Fokker-Planck Eq.

$$\frac{\partial P[\Phi, \theta]}{\partial \theta} = LP[\Phi, \theta]$$

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Michael Westh Hansen
11th Aug, 11:50 CET

Nagata, Nishimura, and
Shimasaki (2016)

magnitude of drift : $u = \left| \frac{\partial S[\Phi]}{\partial \Phi} \right|$

Probability distribution $P(u)$ falls off exponentially or faster

- Langevin evolution of X_μ at Langevin time τ

$$\left. \begin{aligned} \frac{d(X_\mu)_{ji}}{d\tau} &= -\frac{\partial S_{\text{eff}}}{\partial (X_\mu)_{ji}} + (\eta_\mu)_{ij}(\tau) \\ \frac{\partial S_{\text{eff}}}{\partial (X_\mu)_{ji}} &= \frac{\partial S_b}{\partial (X_\mu)_{ji}} - \frac{1}{2} \text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (X_\mu)_{ji}} \mathcal{M}^{-1} \right) \end{aligned} \right| \begin{aligned} &\eta_{\mu,ij}(\tau) \text{ Hermitian noise obeying} \\ &\exp \left(-\frac{1}{4} \int \text{Tr} \left(\eta_\mu^2(\tau) \right) d\tau \right) \end{aligned}$$

- Violation of correctness criteria when

- A. Excursion problem
 - B. Singular drift problem

Excursion problem:

- Langevin evolution $X_\mu : \text{SU}(N) \rightarrow \text{SL}(N, \mathbb{C})$
- X_μ too far away from Hermitian: results are unreliable

GAUGE COOLING

Seiler, Sexty and Stamatescu (2012)

- Hermiticity Norm $\mathcal{N}_H = -\frac{1}{10N} \sum_{\mu=1}^{10} \text{Tr} \left[\left(X_\mu - X_\mu^\dagger \right)^2 \right]$ Nagata, Nishimura and Shimasaki (2016)

- X_μ is invariant under $X_\mu \rightarrow X'_\mu = g X_\mu g^{-1}$ where $g \in \text{SL}(N, \mathbb{C})$ $\mathcal{N}_H$ not invariant under the extra symmetry

$$g = e^{\alpha \delta \mathcal{N}_H}, \quad \delta \mathcal{N}_H = -\frac{1}{N} \sum_{\mu=1}^{10} \left[X_\mu, X_\mu^\dagger \right].$$

α : step size (constant/adaptive)

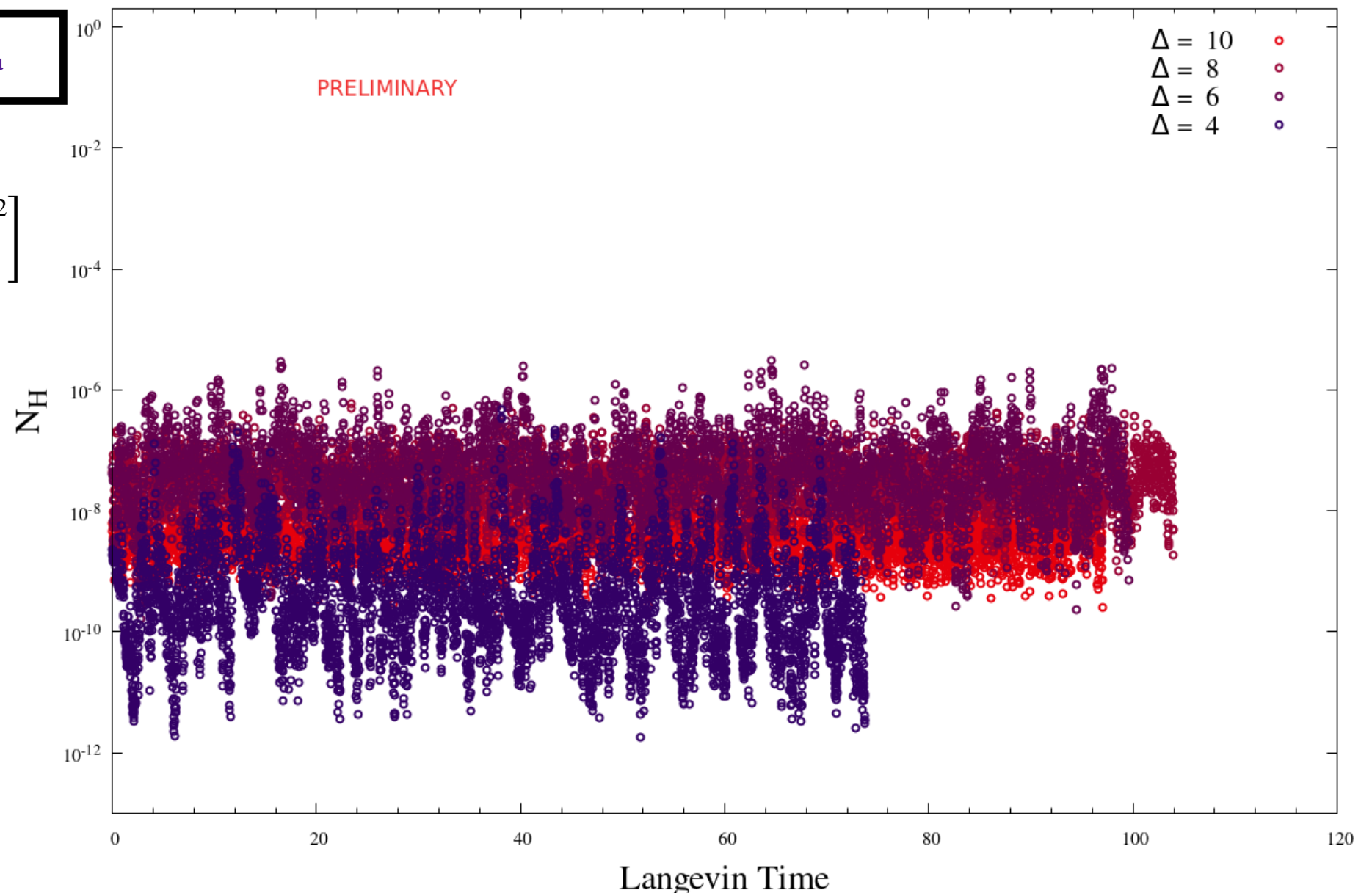
- Each Langevin step: repeat above extra symmetry $\rightarrow$ minimized $\mathcal{N}_H$

Excursion problem:

- Langevin evolution $X_\mu : \text{SU}(N) \rightarrow \text{SL}(N, \mathbb{C})$
- X_μ too far away from Hermitian: results are unreliable

GAUGE COOLING

$$\mathcal{N}_H = -\frac{1}{10N} \sum_{\mu=1}^{10} \text{Tr} \left[\left(X_\mu - X_\mu^\dagger \right)^2 \right]$$



Singular drift problem:

- Fermion operator $\mathcal{M}$ has near-zero eigenvalues
 - The drift term diverges: results unreliable
- $$\mathcal{M}_{a\alpha,b\beta} = \frac{N}{2} \Gamma_{\alpha\beta}^{\mu} \text{Tr} \left(X_{\mu} [t^a, t^b] \right)$$
- $$\frac{\partial S_f}{\partial (X_{\mu})_{ji}} = -\frac{1}{2} \text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (X_{\mu})_{ji}} \mathcal{M}^{-1} \right)$$

MASS DEFORMATIONS: $S \rightarrow S = S_{\text{IKKT}} + \Delta S$

Ito and Nishimura (2016)

- In general, $\Delta S = \Delta S_b + \Delta S_f$

$$\Delta S_b \propto \text{Tr} \left(M_{\mu\nu} X_{\mu} X_{\nu} \right)$$

$$\Delta S_f \propto \text{Tr} \left(\psi_{\alpha} \gamma_{\alpha\beta} \psi_{\beta} \right) \text{ shifts eigenvalue distribution of } \mathcal{M} \quad \mathcal{M}_{a\alpha,b\beta} \rightarrow \tilde{\mathcal{M}}_{a\alpha,b\beta} = \frac{N}{2} \Gamma_{\alpha\beta}^{\mu} \text{tr} \left(X_{\mu} [t^a, t^b] \right) + \gamma_{\alpha\beta} \delta_{ab}$$

- Explicitly break the SO(10) rotational symmetry *supersymmetry?*
- Recover original model: $\lim \Delta S \rightarrow 0$

MASS DEFORMATIONS: $S \rightarrow S = S_{\text{IKKT}} + \Delta S$

- Recently, [Anagnostopoulos, Azuma, Ito, Nishimura, Okubo, Papadoudis \(2020\)](#)

breaks SO(10) symmetry:
$$\Delta S_b = \frac{1}{2} N \epsilon \sum_{\mu} m_{\mu} \text{Tr} \left(X_{\mu} \right)^2$$

shifts eigenvalue distribution of $\mathcal{M}$:
$$\Delta S_f = -\frac{N}{2} m_f \text{Tr} \left(\psi_{\alpha} \gamma_{\alpha\beta} \psi_{\beta} \right); \quad \gamma = i\Gamma^8 \Gamma^9 \Gamma^{10}$$

supersymmetry explicitly broken

- Probe SSB in original IKKT model by extrapolations:

$$\begin{array}{ll} \text{A. } N \rightarrow \infty \\ \text{B. } \epsilon \rightarrow 0 \\ \text{C. } m_f \rightarrow 0 \end{array} \implies \begin{array}{l} \text{SO}(10) \rightarrow \text{SO}(d) \times \text{SO}(10-d) \\ \text{SO}(d) \text{ symmetric vacua} \end{array}$$

- Results:

$$\begin{array}{ll} m_f = 3.0 & \implies \text{SO}(7) \\ m_f = 1.4 & \implies \text{SO}(4) \\ m_f = 1.0, 0.9, 0.7 & \implies \text{SO}(3) \end{array}$$

consistent with GEM study
[Nishimura, Okubo, Sugino \(2011\)](#)

MASS DEFORMATIONS: $S \rightarrow S = S_{\text{IKKT}} + \Delta S$

- We introduce SUSY preserving mass deformations: Bonelli and Natuurkunde (2002)

$$\Delta S = N \operatorname{tr} \left(\underbrace{-M^{\mu\nu} X_\mu X_\nu}_{M_{\mu\nu}: \text{bosonic mass matrix}} + \underbrace{\frac{i}{8} \bar{\psi} N_3 \psi}_{N_3: \text{fermion mass matrix}} + \underbrace{i N^{\mu\nu\rho} X_\mu [X_\nu, X_\rho]}_{\text{Myers term}} \right)$$

- SUSY preserving mass deformation (similar to BMN)

$$\delta X^\mu = -\frac{1}{2} \bar{\epsilon} \Gamma^\mu \psi \quad \delta \psi = \frac{1}{4} [X^\mu, X^\nu] \Gamma_{\mu\nu} \epsilon - \frac{i}{16} X^\mu \left(\Gamma_\mu N_3 + 2 N_3 \Gamma_\mu \right) \epsilon$$

provided flux constraint: $\left[N_3 (\Gamma^\mu N_3 + 2 N_3 \Gamma^\mu) + 4^3 M^{\mu\nu} \Gamma_\nu \right] \epsilon = 0$

- Consider:

$$N_3 = -\Delta \Gamma^8 \Gamma^{9\dagger} \Gamma^{10}, \quad N^{\mu\nu\rho} = \frac{\Delta}{3!} \sum_{\mu\nu\rho=8}^{10} \epsilon^{\mu\nu\rho} \quad \text{and} \quad M = -\frac{\Delta^2}{4^3} (\mathbb{I}_7 \oplus 3\mathbb{I}_3)$$

$$\text{MASS DEFORMATIONS: } S \rightarrow S = S_{\text{IKKT}} + \Delta S$$

- We introduce SUSY preserving mass deformations:

$$\Delta S = N \text{ Tr} \left(\frac{\Delta^2}{4^3} \sum_{i=1}^7 X_i^2 + \frac{3\Delta^2}{4^3} \sum_{a=8}^{10} X_a^2 + \frac{i\Delta}{3!} \epsilon^{abc} X_a [X_b, X_c] - \frac{N\Delta}{8} \psi_\alpha \gamma_{\alpha\beta} \psi_\beta \right)$$

$$\gamma_{\alpha\beta} = i(\Gamma^8 \Gamma^9 \Gamma^{10})_{\alpha\beta}$$

- Avoids singular drift problem

shifts eigenvalue distribution of $\mathcal{M}$

$$\mathcal{M}_{a\alpha,b\beta} \rightarrow \tilde{\mathcal{M}}_{a\alpha,b\beta} = \frac{N}{2} \Gamma_{\alpha\beta}^\mu \text{tr} \left(X_\mu [t^a, t^b] \right) - \frac{N\Delta}{8} \gamma_{\alpha\beta} \delta_{ab}$$

- Probe SSB in IKKT model

A. $N \rightarrow \infty$

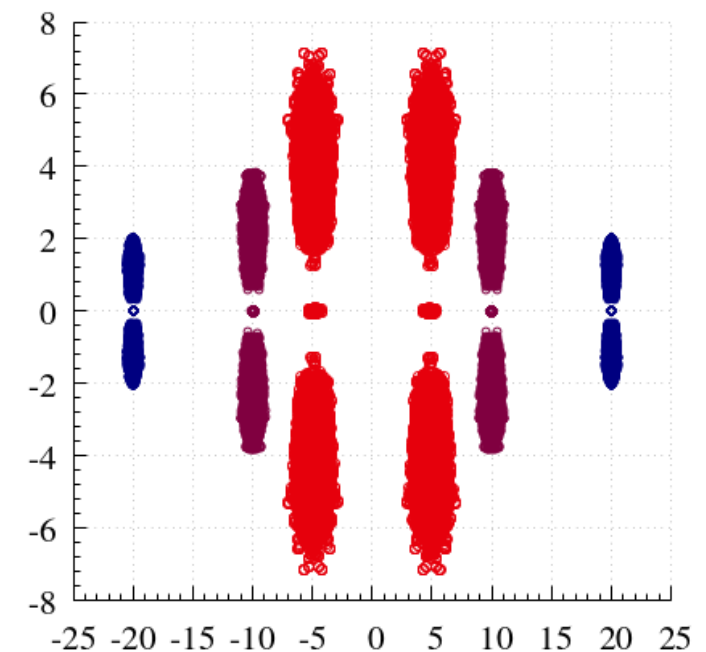
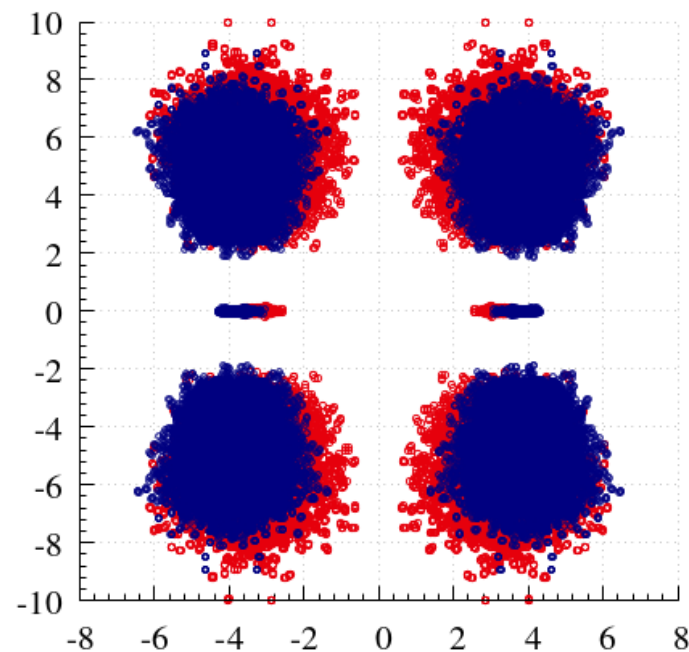
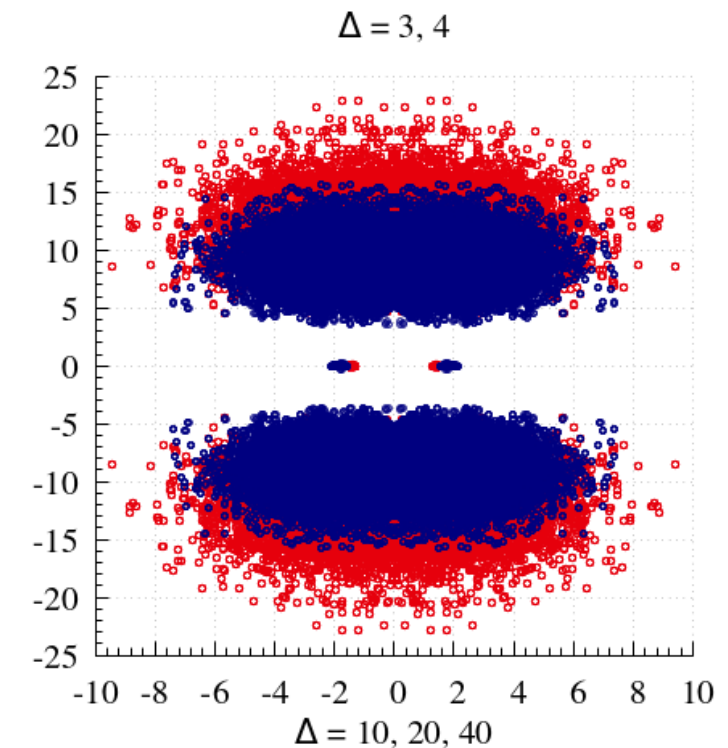
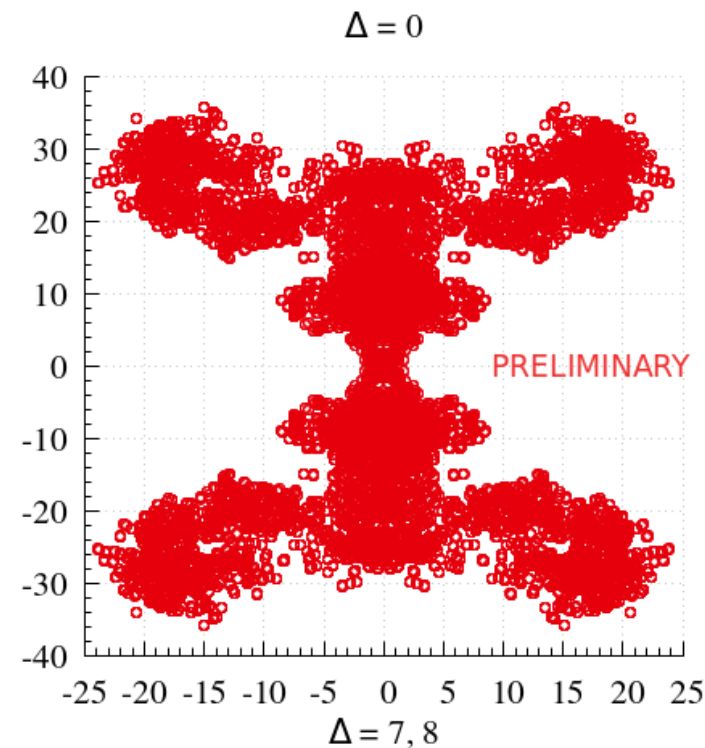
B. $\Delta \rightarrow 0$

MASS DEFORMATIONS: $S \rightarrow S = S_{\text{IKKT}} + \Delta S$

Eigenvalues distribution $\mathcal{M}(\Delta)$

$$\mathcal{M}_{a\alpha,b\beta} \rightarrow \mathcal{M}_{a\alpha,b\beta}(\Delta) = \frac{N}{2} \Gamma_{\alpha\beta}^{\mu} \text{tr} \left(X_{\mu} [t^a, t^b] \right) - \frac{N\Delta}{8} \gamma_{\alpha\beta} \delta_{ab}$$

$$\gamma_{\alpha\beta} = i(\Gamma^8 \Gamma^9 \Gamma^{10})_{\alpha\beta}$$



MASS DEFORMATIONS: $S \rightarrow S = S_{\text{IKKT}} + \Delta S$

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 $\text{SO}(10) \rightarrow \text{SO}(d) \times \text{SO}(10 - d)$

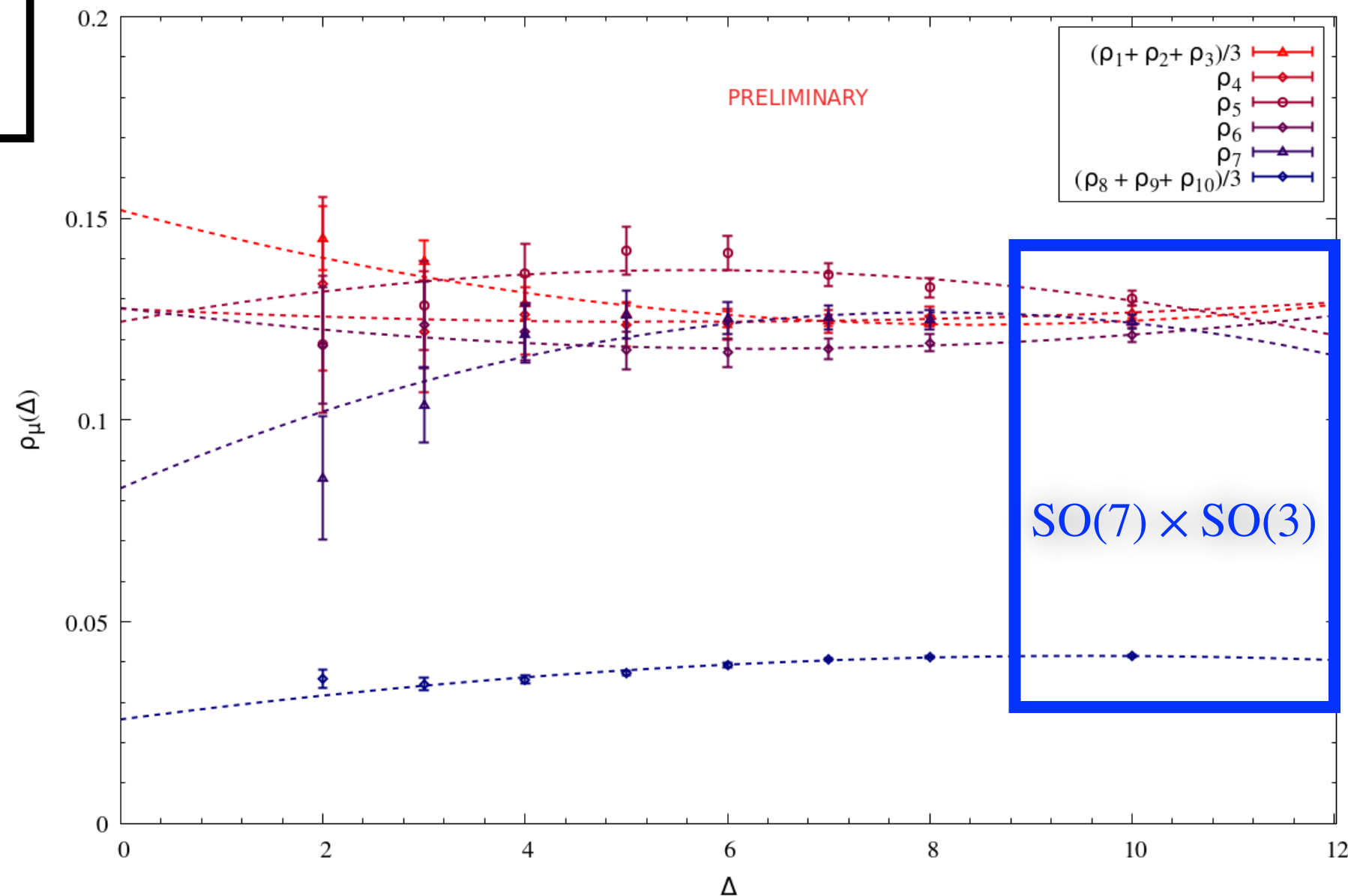
Extent of space-time as
an order parameter

$$\lambda_\mu(\Delta) = \frac{1}{N} \text{tr} (X_\mu)^2$$

$$\rho_\mu(\Delta) = \frac{\lambda_\mu(\Delta)}{\sum_\mu \lambda_\mu(\Delta)}$$

- Even for $N = 4$ extents
are not same

$\Rightarrow$ hints $\text{SO}(d) : d < 7$



- Complex Langevin is a powerful tool to study theories with complex actions.
- Complex Langevin method can be used to detect spontaneous symmetry breaking.
- SUSY-preserving mass deformations successfully evade singular drift problem
- Large- N extrapolations for SUSY-preserving mass deformations and investigate $SO(d)$ symmetric vacuum

Thank You!

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