The spectral reconstruction of inclusive rates

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Lattice `22
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Inclusive rates: the R-ratio

Simplest hadronic inclusive rate:

$$\rho(s) \propto \frac{\sigma[e^+e^- \to \mathrm{hadrons}](s)}{4\pi\alpha_{\mathrm{em}}(s)^2/(3s)}$$

Can we compute it in lattice QCD?

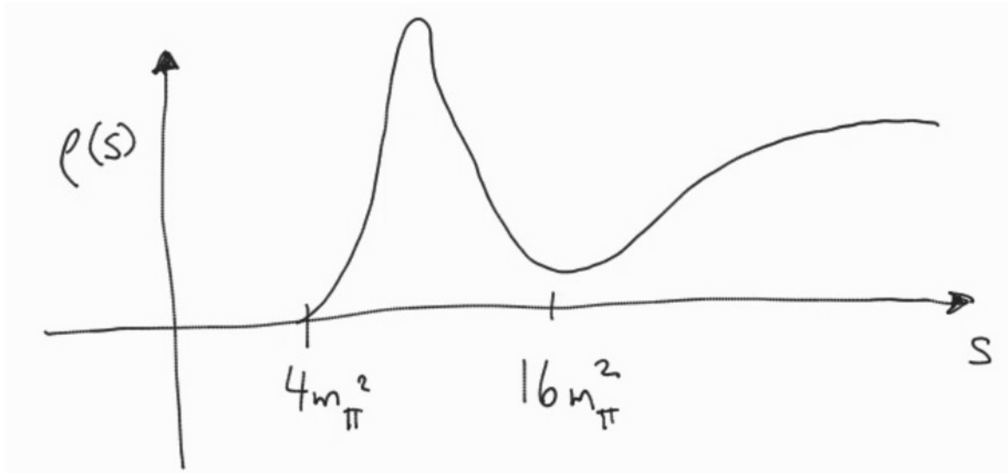
Time-momentum representation (TMR) of the vector-vector correlator:

$$C(t) = \int d^3\boldsymbol{x}\,\langle\Omega|\hat{j}_z^{\mathrm{em}}(\boldsymbol{x})\,\mathrm{e}^{-\hat{H}t}\,\hat{j}_z^{\mathrm{em}}(0)^\dagger|\Omega\rangle$$

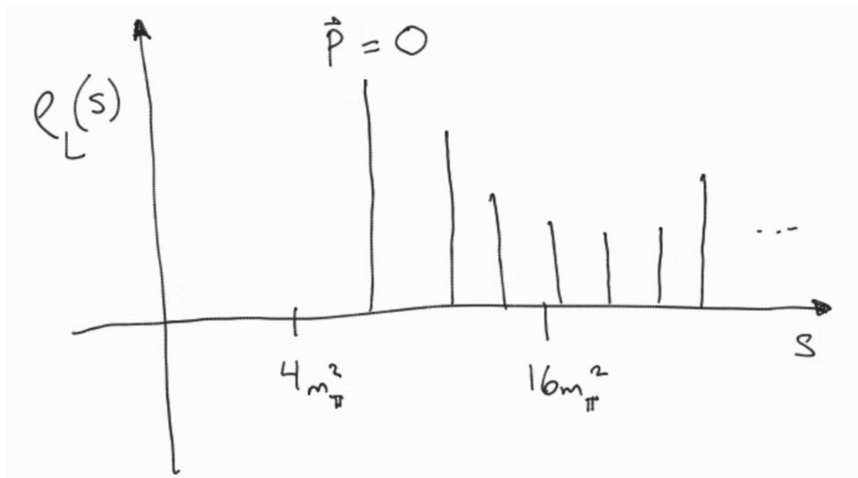
$$= \int_0^\infty d\omega\,\omega^2\rho(\omega^2)\,\mathrm{e}^{-\omega t}$$

Finite vs. infinite volume

Infinite volume: continuous



Finite volume: sum of Dirac-delta peaks.



Not 'close' to infinite volume at finite L!

(Finite-volume formalism applicable to elastic region)

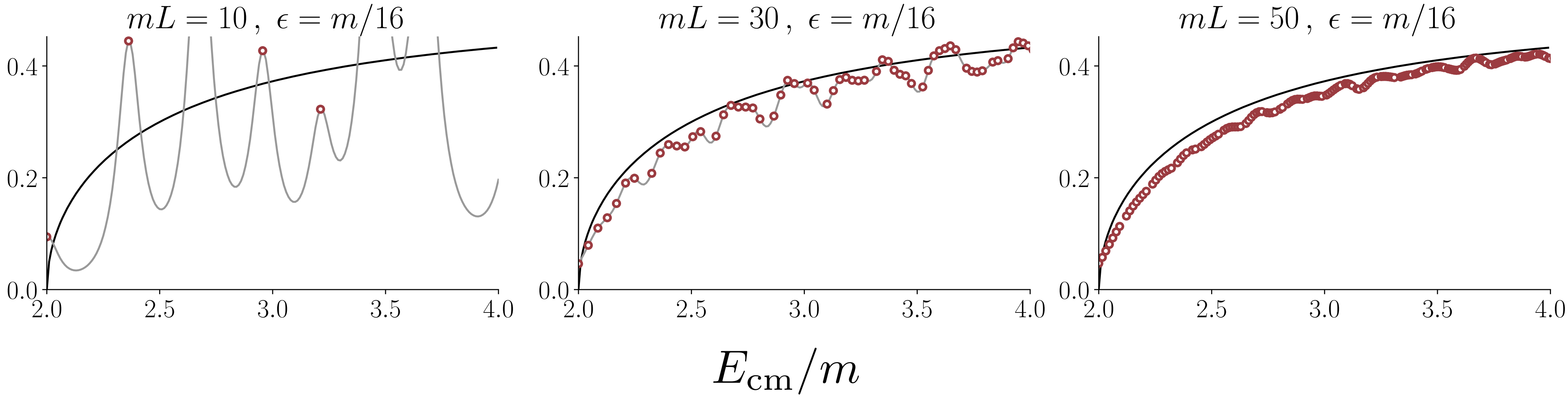
L. Lellouch, M. Lüscher `01; H. Meyer `11;
R. Briceño, M. T. Hansen `15; …

Finite vs. infinite volume

Smeared spectral densities:

$$\rho_\epsilon(E) = \int_0^\infty d\omega\, \delta_\epsilon(E-\omega)\,\rho(\omega), \qquad \lim_{\epsilon\to 0}\delta_\epsilon(x) = \delta(x)$$

Bridge between finite- and infinite-volume:



Toy example from JB, M. T. Hansen `19. See also M. T.Hansen, H. Meyer, D. Robaina `17

Spectral Reconstruction

Backus, Gilbert ‘68, ‘70
F. Pijpers, M. Thompson ‘92
M. R. Hansen, A. Lupo, N. Tantalo, PRD99 (2019)

Linear ansatz:

$$\hat{\rho}_\epsilon(E) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, C(t), \qquad \hat{\delta}_\epsilon(E, \omega) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, \mathrm{e}^{-\omega t}$$

Two criteria when choosing $\{q_t(\epsilon, E)\}$

- Accuracy: $$A[q] = \int_{E_0}^{\infty} \mathrm{d}\omega \left\{ \delta_\epsilon(E-\omega) - \hat{\delta}_\epsilon(E, \omega) \right\}^2$$

- Precision: $$B[q] = \mathrm{Var}\left\{ \hat{\rho}_\epsilon(E) \right\}$$

Optimal coeffs minimize:

$$G_\lambda[q] = (1-\lambda) A[q] + \lambda B[q]$$

Controlled Test

JB, M. W. Hansen, M. T. Hansen, A. Patella, N. Tantalo, JHEP `22

2d O(3)-model: …, M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990),...

$$S[\sigma] = -\beta \sum_{x,\mu} \sigma(x) \cdot \sigma(x+\hat{\mu}), \qquad \sigma(x) \in \mathbb{R}^3, |\sigma(x)| = 1$$

Conserved (global) current:

$$j_\mu^a = \beta \epsilon^{abc} \sigma^b(x) \hat{\partial}_\mu \sigma^c(x)$$

Massive single-particle states. Target process: inclusive rate for $j \to X$

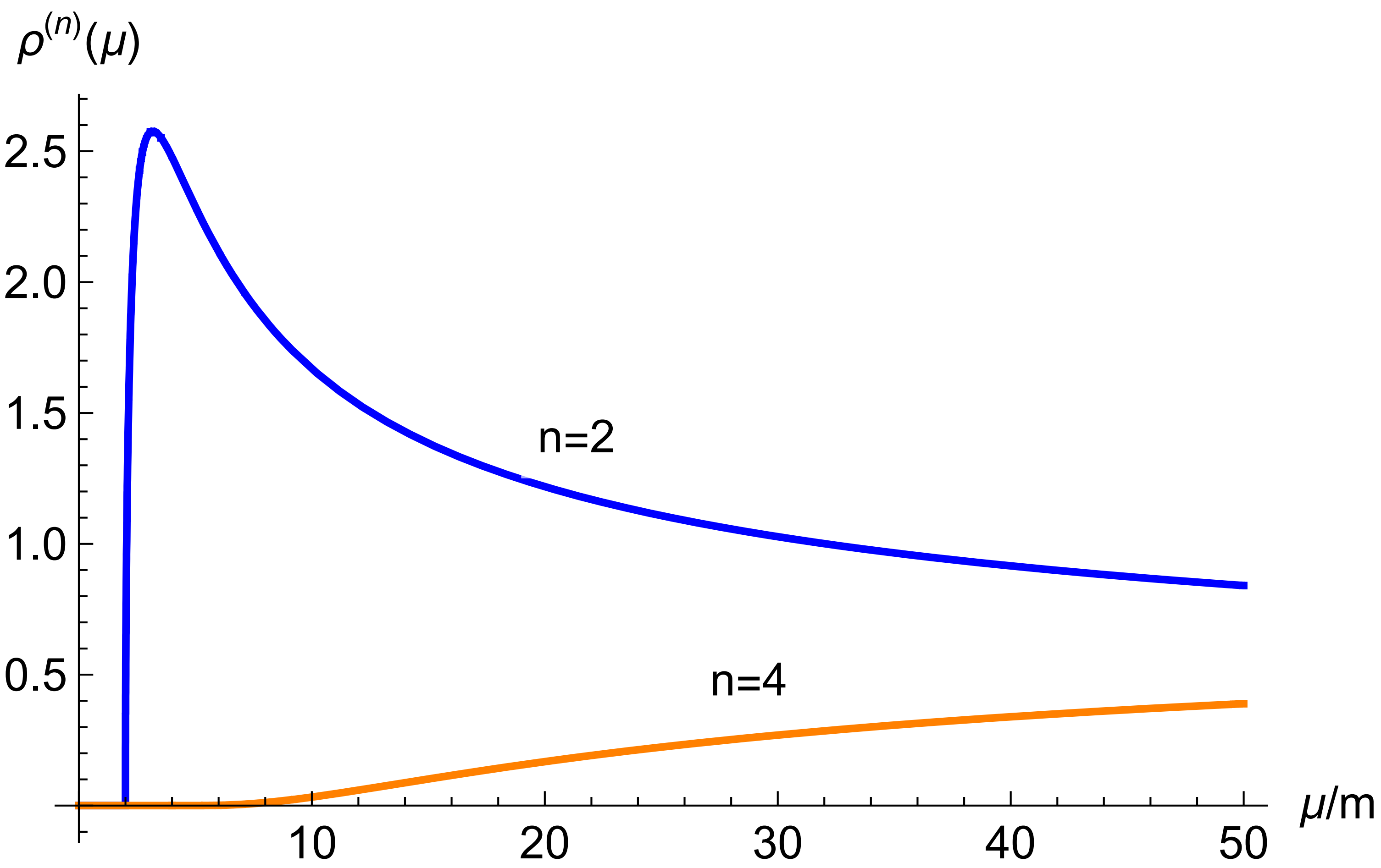
$$\rho(E) = \sum_\alpha \delta(\boldsymbol{P}_\alpha)\delta(E - E_\alpha) \left| {}_{\text{out}}\langle \alpha | \hat{j}(0) | 0 \rangle \right|^2$$

$$= \sum_{n=2,4,6,\ldots} \rho^{(n)}(E)$$

Controlled Test

Integrable model => spectral function known exactly:

M. Karowski, P. Weisz, Nucl. Phys. B139 (1978)
A. B. Zamolodchikov, A. B. Zamolodchikov, Nucl. Phys. B133 (1978)
J. Balog, M. Niedermaier, Nucl. Phys. B500 (1997)



Two-particle contribution dominant, four-particle ~2% near E = 10m

Controlled Test

Four smearing kernels $\delta_\epsilon^{\mathrm{x}}(E-\omega)$:

$$\delta_\epsilon^{\mathsf{g}}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right],$$

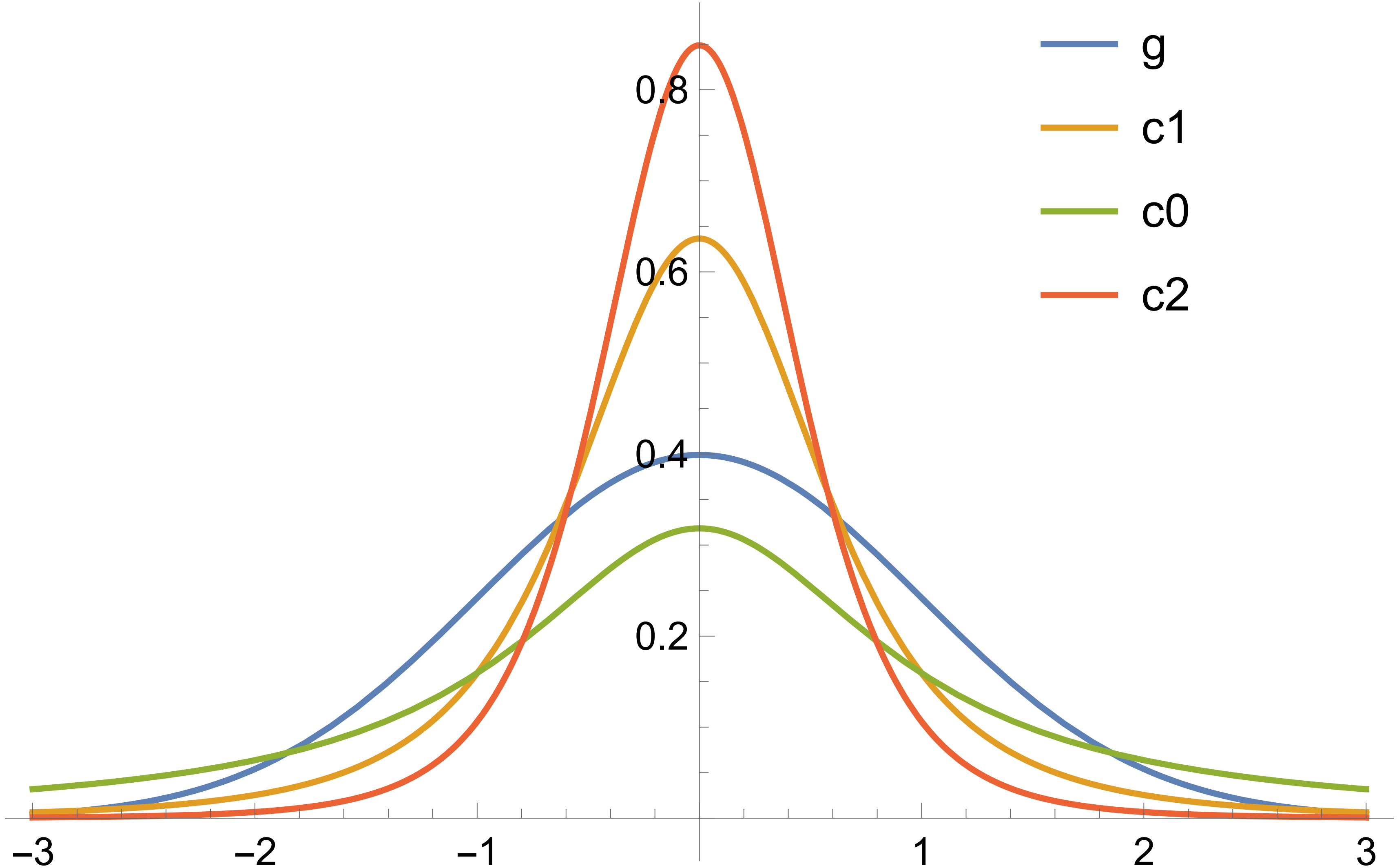
$$\delta_\epsilon^{\mathsf{c}0}(x) = \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2},$$

$$\delta_\epsilon^{\mathsf{c}1}(x) = \frac{2}{\pi} \frac{\epsilon^3}{(x^2+\epsilon^2)^2},$$

$$\delta_\epsilon^{\mathsf{c}2}(x) = \frac{8}{3\pi} \frac{\epsilon^5}{(x^2+\epsilon^2)^3}$$

Controlled Test

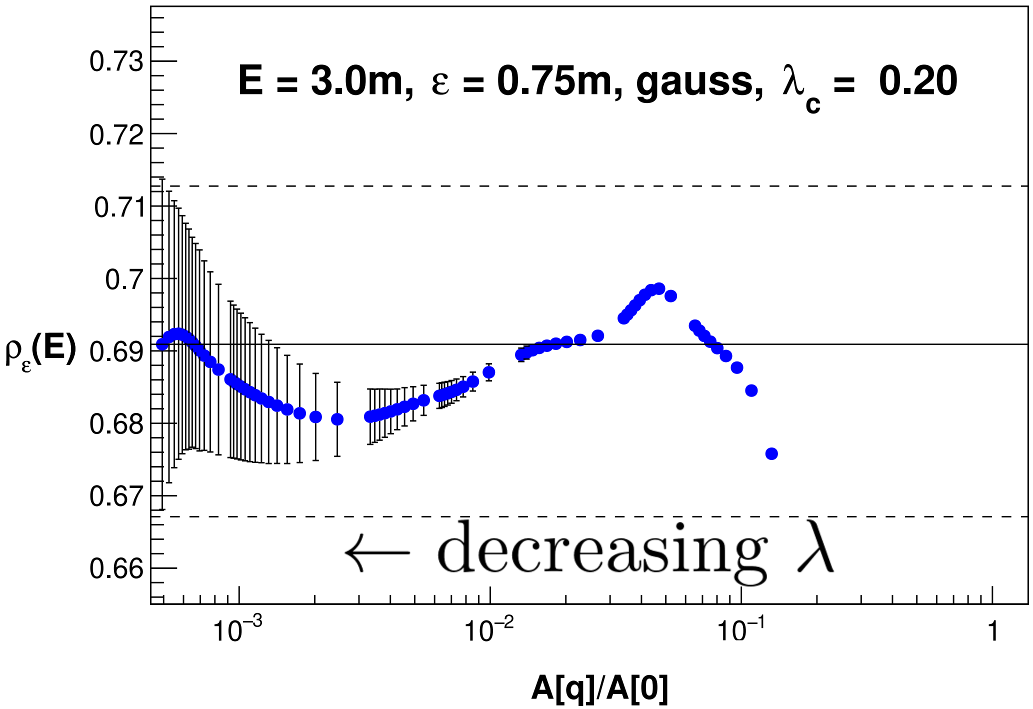
Four smearing kernels $\delta_\epsilon^{\mathrm{x}}(E-\omega)$:

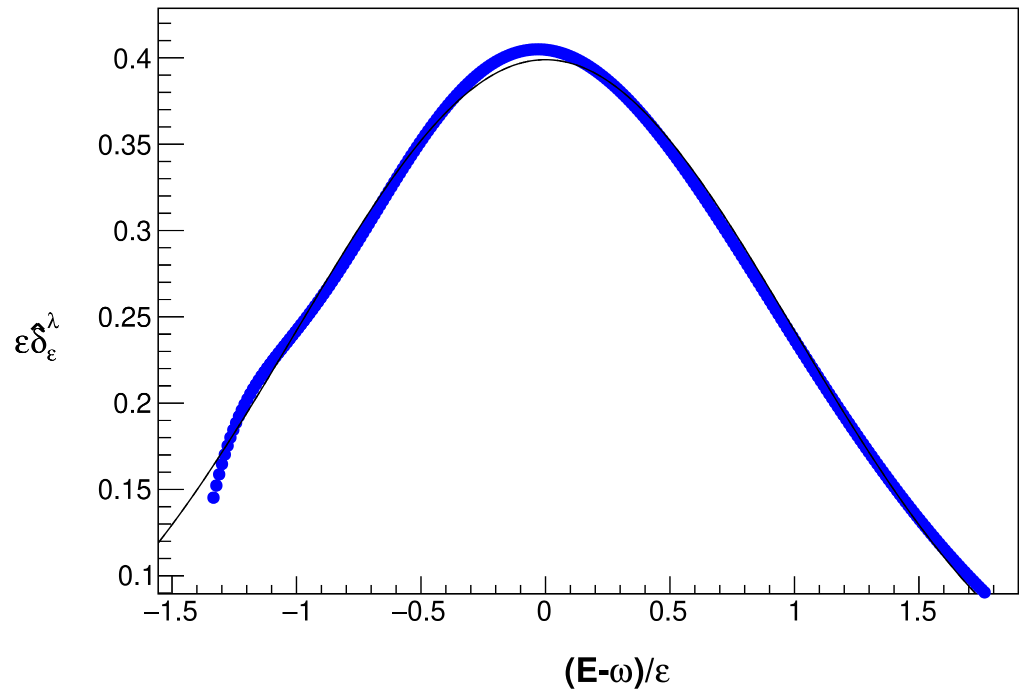


Spectral Reconstruction

$$G_\lambda[q] = (1-\lambda)A[q] + \lambda B[q]$$

Trade off parameter (λ) balances systematic (A) and statistical (B) error



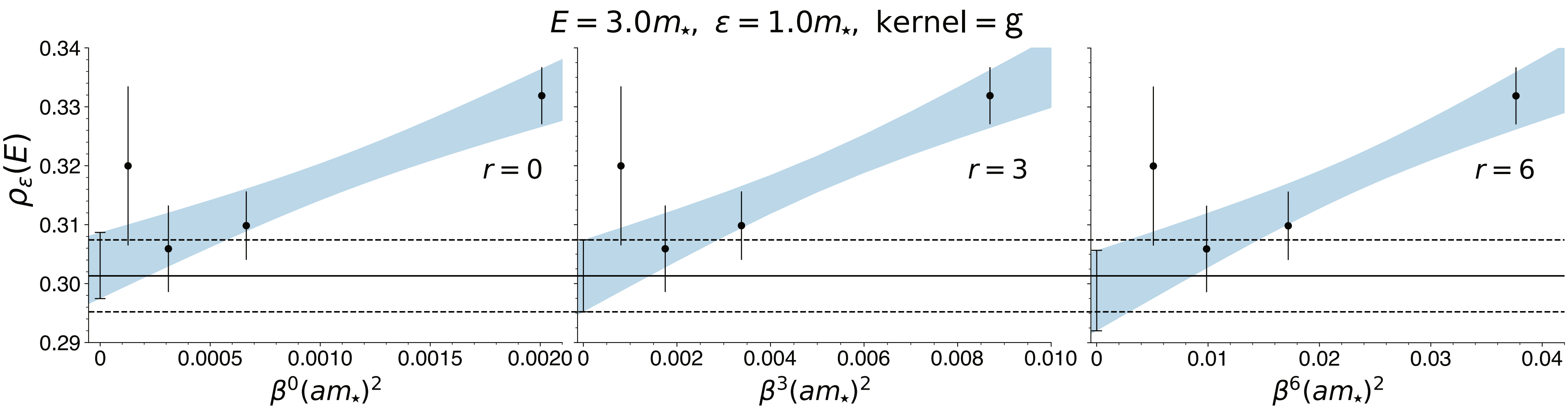


Plateau indicates statistics-limited regime, automatically selected.

Continuum Limit

Long History! For spectral quantities: … , J. Balog, F. Niedermayer, P. Weisz, Nucl. Phys. B824 (2010)

$$\lim_{a\to 0} E(a) = E^{\text{phys}} + A\beta^3 a^2 (1 + \frac{r}{\beta} + \frac{c}{\beta^2} + \dots)$$



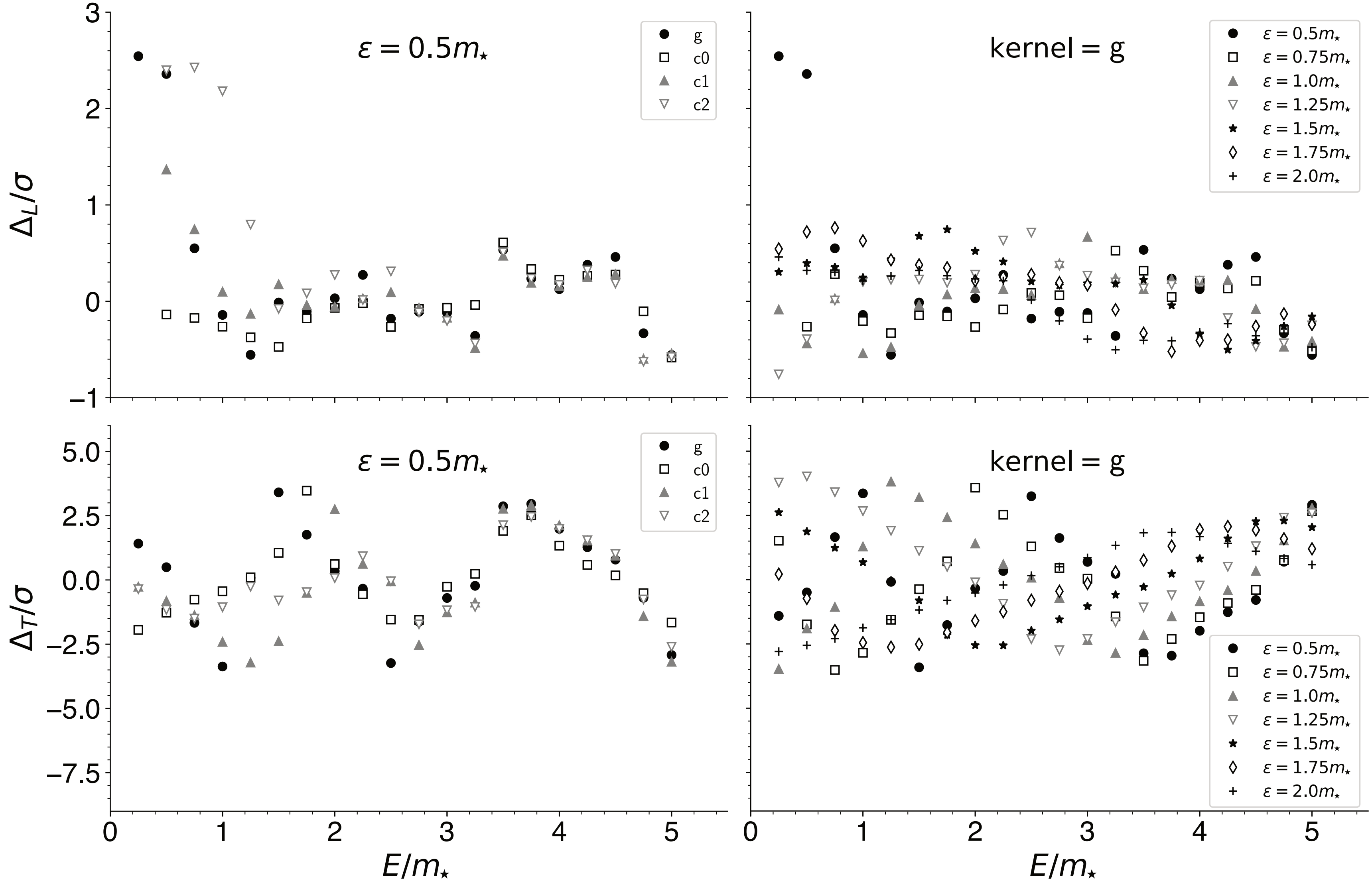
Sys. error estimate from three (arbitrary) fit forms:

$$\beta^p (a/m)^2, \quad p = 0,\ 3,\ 6$$

Continuum limit possible at fixed eps and E!

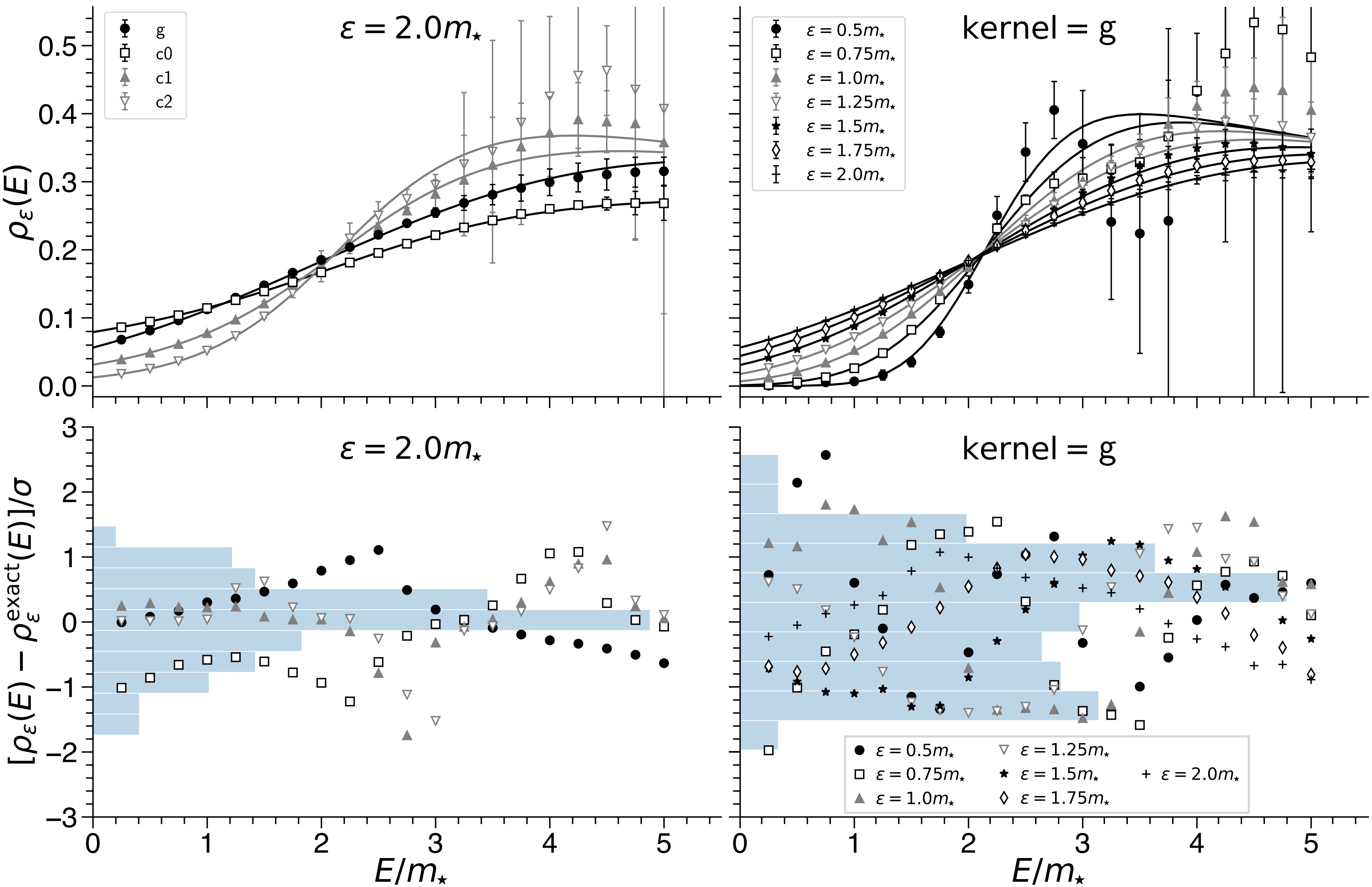
Finite-volume effects

$$\Delta_{\mathrm{L}}(\epsilon, E) = \rho_{T,L,\epsilon}(E) - \rho_{T,2L,\epsilon}(E)$$



Differences taken as additional systematic error.

Results: fixed smearing width



Solid lines: exact smeared spectral function, using N=2, 4, 6 particle contributions.

Results: extrapolation to zero width

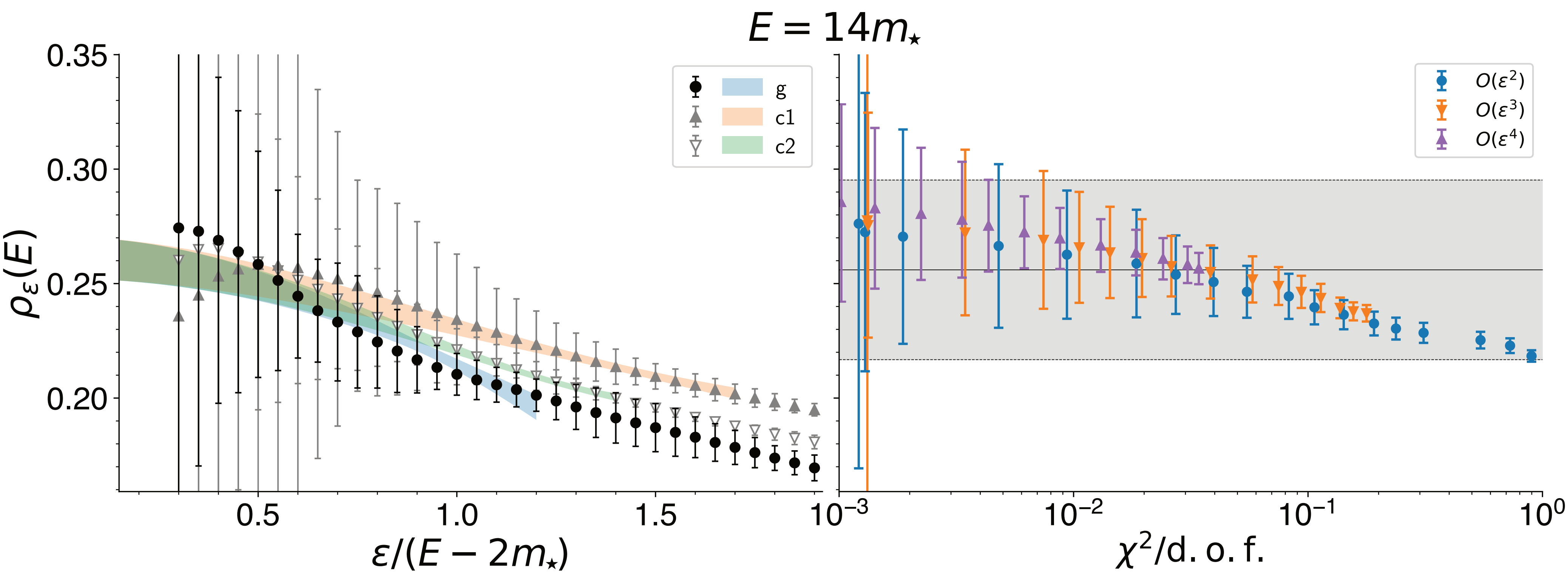
All kernels have the same $\mathrm{O}(\epsilon^2)$ coefficient (up to a sign):

$$\rho_\epsilon^{\mathsf{x}}(E) = \rho(E) + \sum_{k=1}^{\infty} w_k^{(\mathsf{x})} a_k(E)\epsilon^k \ ,$$

x	w_k^{x}, even k	w_k^{x}, odd k	w_1^{x}	w_2^{x}	w_3^{x}	w_4^{x}
g	$\frac{k!}{(-2)^{k/2}(k/2)!}$	0	0	-1	0	3
c0	1	1	1	1	1	1
c1	$(1-k)$	$(1-k)$	0	-1	-2	-3
c2	$\frac{1}{3}(k-3)(k-1)$	$\frac{1}{3}(k-3)(k-1)$	0	$-1/3$	0	1

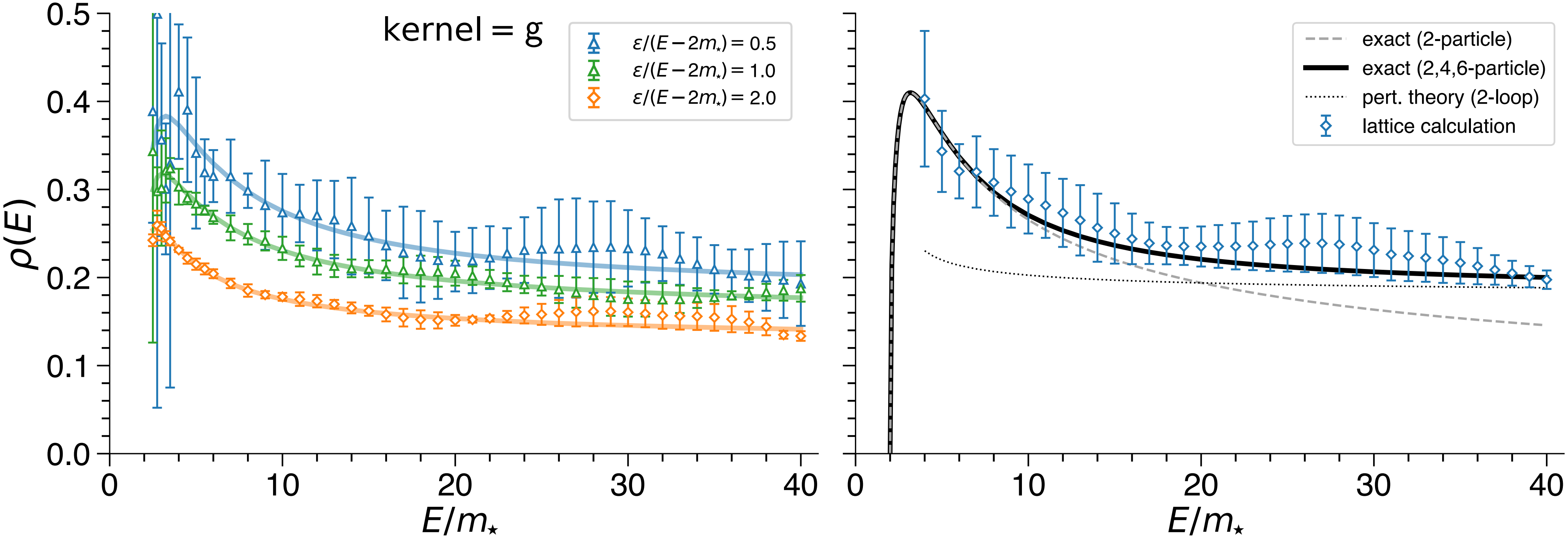
Known coefficients enable constrained extrapolation of all kernels

Results: extrapolation to zero width



Key insight: larger smearing width sufficient at larger energy

Results: extrapolation to zero width



Some sensitivity to four-particle contribution at higher energies

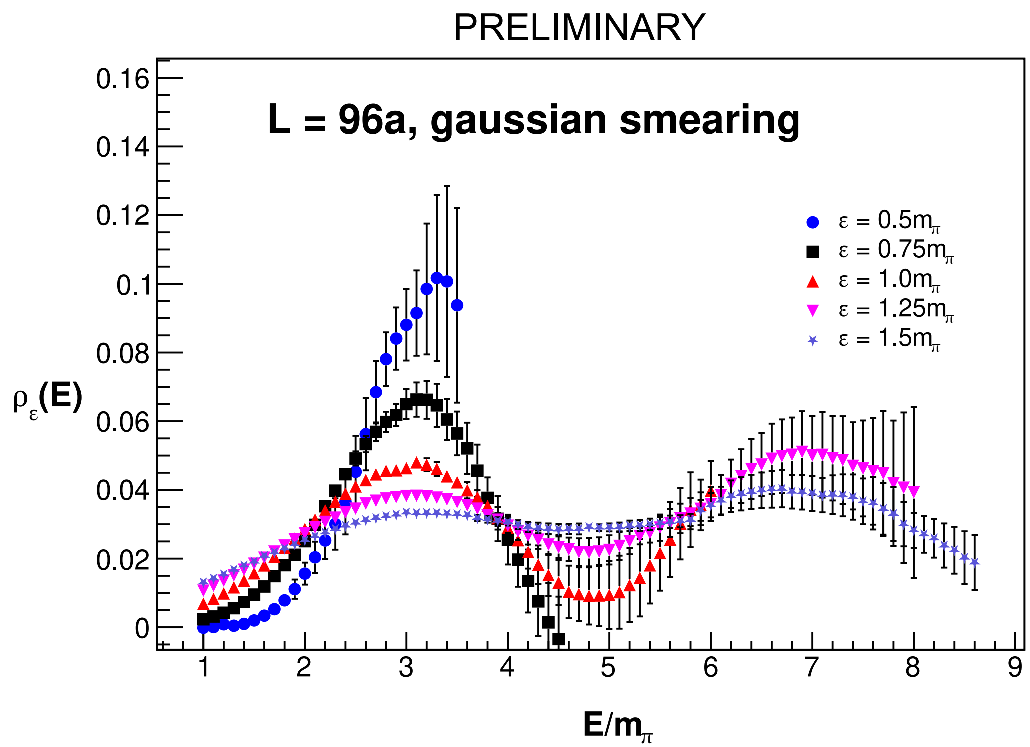
What about the *R*-ratio in QCD?

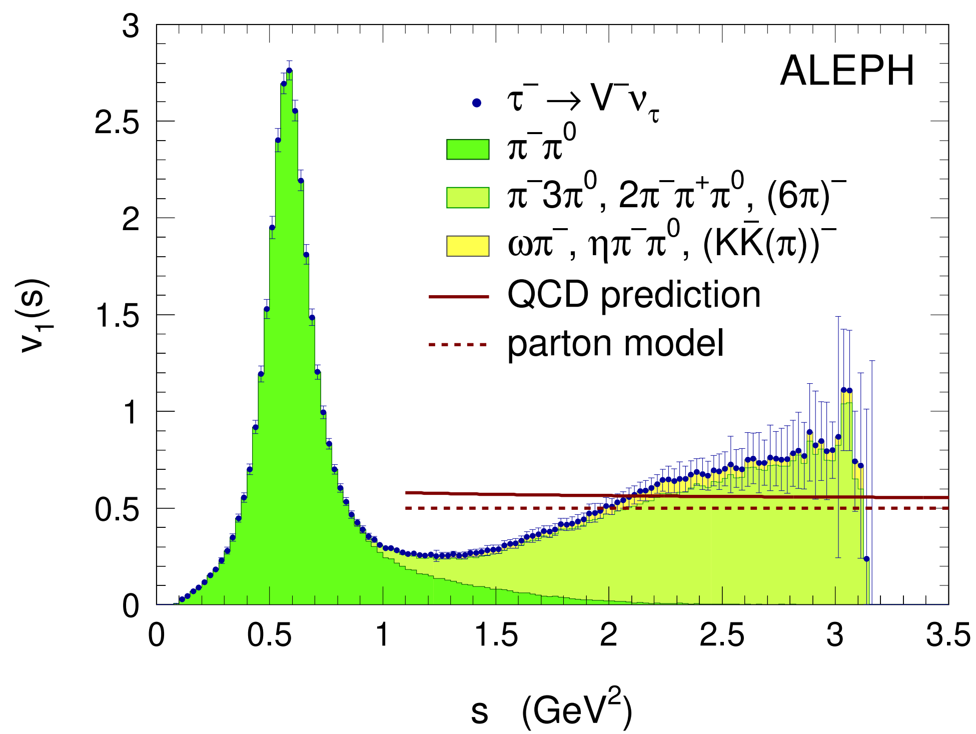
- Large volumes needed: in O(3)-model $mL \approx 30$
- Relevant idea: masterfield simulation paradigm
  - ➔ Plenary talk by P. Fritzsch, Sat. 9:20am
  - ➔ M. Cè, Thurs. 11:50am, Hadron Specroscopy + Interactions
- Preliminary application to isovector-vector correlator at

$$N_{\rm f} = 2+1, \quad L = 9\text{fm and } 18\text{fm},$$
$$a = 0.09\text{fm}, \quad m_\pi = 270\text{MeV}$$

M. Bruno, JB, A. Francis, P. Fritzsch, J. Green, M. T. Hansen, M. Lüscher, A. Patella, A. Rago, *in prep.*

What about the *R*-ratio in QCD?





- Comparison to hadronic tau-decay (right) ALEPH collaboration `05
- No extrapolation to zero-width yet
- Mild indication of four-particle effects.

M. Bruno, JB, A. Francis, P. Fritzsch, J. Green, M. T. Hansen, M. Lüscher, A. Patella, A. Rago, *in prep.*

Conclusions

- The HLT approach to spectral reconstruction produces results with controlled systematic errors.

- In 2-D O(3)-model, smoothness enables extrapolation to the zero-width limit.

- Application to QCD seems promising, although no zero-width extrapolation yet.

- An alternative approach: Chebyshev polynomials G. Bailas, S. Hashimoto, T. Ishikawa, PTEP `22
  - ➔ R. Kellermann, Fri. 2:30pm, Weak Decays + Matrix Elements
  - ➔ A. Barone, Fri. 2:50pm, Weak Decays + Matrix Elements
  - ➔ A. Smecca, Fri. 3:10pm, Weak Decays + Matrix Elements

- Related work:
  - ➔ A. De Santis, Thurs. 10:00am, QCD in Searches for BSM Physics
  - ➔ A. Evangelista, Poster